SUFFICIENTLY GENERIC ORTHOGONAL GRASSMANNIANS

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Abstract. We prove the following conjecture due to Bryant Mathews (2008). Let Q_i be the orthogonal grassmannian of totally isotropic i-planes of a non-degenerate quadratic form q over an arbitrary field (where i is an integer satisfying $1 \leq i \leq m := [(\dim q)/2]$). Assume that for a given i, the form q has the following property (possessed by the generic quadratic form): the degree of each closed point on Q_i is divisible by 2^i and the Witt index of q over the function field of Q_i is equal to i. Then the variety Q_i is 2-*incompressible*.

Assuming that the form q is sufficiently close to the generic one in a different sense, we prove a stronger property of Q_i saying that its Chow motive with coefficients in $\mathbb{F}_2$ (the finite field of 2 elements) is indecomposable. This result contrasts with recent results of Zhykhovich (2010) [21] on decomposability of the motives of incompressible *twisted grassmannians*.

The above two main results of the paper were known for the quadric Q_1 and the maximal grassmannian Q_m due to the works of A. Vishik.

The proofs are based on the theory of upper motives. The results allow one to compute the canonical 2-dimension of any projective homogeneous variety (i.e., orthogonal flag variety) associated to the generic quadratic form.

This paper is an extended version of Karpenko (2011) [10] including the results of Karpenko (2010) [5].

1. Introduction

The classical question [13, Question 4.13(i)] of M. Knebusch asks about the minimum of the transcendence degree of the generic zero fields of the generic quadratic form (over a field) of a fixed dimension. This question being implicitly answered in [16], in the present paper we answer its extended version where, for a given i, the zero should be at least i-fold in the sense of the Witt index (the original question corresponds to $i = 1$). Moreover, we provide conditions on an arbitrary quadratic form which ensure that the form is close enough to the generic one in the sense that the above question for that form has the same answer.

This paper deals with motives of certain smooth projective varieties associated to quadratic forms over fields of arbitrary characteristic. We refer to [4] for notation and basic results concerning quadratic forms. By motives, we mean the Grothendieck Chow motives with coefficients in the finite field $\mathbb{F}_2$ as introduced in [4]. We are using the theory of upper motives conceived in [11] and [9].

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Let q be a non-zero non-degenerate quadratic form over a field F (which may have characteristic 2). For any integer i with $0 \leq i \leq m := [(\dim q)/2]$ we write Q_i for the variety of i-dimensional totally isotropic subspaces of q.

For any i, the variety Q_i is smooth and projective. It is geometrically connected if and only if $i \neq m$. In particular, Q_i is connected for any i if $\dim q$ is odd. For even-dimensional q and $i = m$, the variety Q_i is connected if and only if the discriminant of q is non-trivial.

If a variety Q_i is not connected, it has two connected components and they are isomorphic. In particular, the dimension of Q_i is always the dimension of any connected component of Q_i. Here is a formula for the dimension, where $n := \dim q$ (see, e.g., [2]):

$$\dim Q_i = i(i-1)/2 + i(n-2i).$$

In the case where the quadratic form q is "sufficiently generic" (the precise condition is formulated in terms of the *J-invariant* of q introduced in [18], its definition and meaning are recalled in the beginnings of Sections 3, 4, and 5), we are going to show (see Theorems 3.1, 4.1, and 5.1) that the motives of all Q_i are indecomposable, if we stay away from the two exceptional cases described below (where the motive evidently decomposes).

Each of the both exceptional cases arises only if the dimension of q is even and the discriminant of q is trivial. The first case is the case of $i = m$, where the variety $Q_i = Q_m$ has two connected components. Our assumption on q ensures that the motive of each component of Q_m is indecomposable.

The second case is the case of $i = m - 1$, where the variety $Q_i = Q_{m-1}$ is a rank i projective bundle over a component of $Q_{i+1} = Q_m$ (this statement is proved in the proof of Theorem 4.1). Therefore, the motive of Q_{m-1} is a sum of shifts of m copies of the motive of a component of Q_m, and this is a *complete* motivic decomposition of Q_{m-1} (where *complete* means that the summands of this decomposition are indecomposable).

We recall that a connected smooth projective variety X is called *2-incompressible*, if its *canonical 2-dimension*, as defined in [4, §90] (see also Section 6 here), takes its maximal value $\dim X$. This in particular implies that any rational map $X \dashrightarrow X$ is dominant, i.e., that X is *incompressible*.

Any projective homogeneous variety X having indecomposable motive, is 2-incompressible, [11, §2e]. Therefore our indecomposability results imply 2-incompressibility of the corresponding varieties.

Let us point out that our incompressibility results compute the canonical 2-dimension of any projective homogeneous variety (i.e., orthogonal flag variety) associated to a sufficiently generic quadratic form. This is so indeed because for an arbitrary non-degenerate quadratic form q and an arbitrary sequence of integers $i_1, \dots, i_k$ with $0 \leq i_1 < \dots < i_k \leq m$ we have an orthogonal flag variety $Q_{i_1,\dots,i_k}$, the variety of flags of totally isotropic subspaces of q of dimensions $i_1, \dots, i_k$, and the canonical 2-dimension (of a component) of this variety coincides with the canonical 2-dimension of (a component of) Q_{i_k}.

The motivic indecomposability of the varieties Q_i contrasts with a recent result of M. Zhykhovich [21] (see [22] for an extended version) saying that for any prime p, any central division F-algebra D of degree p^n for some n, and any i with $0 < i < n$ (and $i \neq 1$ if $p = 2$), the motive with coefficients in $\mathbb{F}_p$ of the variety of the right ideals of reduced dimension p^i in D (this variety is known to be p-incompressible and is a twisted form of the grassmannian of p^i-dimensional subspaces in a p^n-dimensional vector space) is decomposable.

The paper is organized as follows. In Section 2 we recall (and partially develop) the necessary aspects of the theory of upper motives (with an arbitrary prime integer p in place of 2). In the next three sections we establish our main result: in Section 3 for odd-dimensional forms (Theorem 3.1), in Section 4 for even-dimensional forms of trivial discriminant (Theorem 4.1), and finally in Section 5 for even-dimensional forms of non-trivial discriminant (Theorem 5.1).

In Section 7, we prove a conjecture due to Bryant Mathews (see Theorem 7.1), another main result of the paper. It gives 2-incompressibility of the orthogonal grassmannian Q_i for a given i under the assumption that the quadratic form is sufficiently close to the generic one in a different from the above sense (expressed in terms of Q_i). In the preceding Section 6 we recall (and develop) the necessary aspects of the theory of canonical dimension (with an arbitrary prime integer p in place of 2).

None of the two conditions on the quadratic form appearing in the two main results of the paper is weaker (or stronger) than the other. According to this, none of the two main results implies the other. Of course, a generic quadratic form (constructed in the beginning of Sections 3, 4, and 5) is sufficiently generic in both senses so that both main results apply to it.

We have to point out that both main results of the paper were known for the quadric Q_1 and the maximal grassmannian Q_m due to the works of A. Vishik [16], [17], and [18] (at least in characteristic $\neq 2$).

2. Upper motives

Let us fix a prime integer p and consider Chow motives with coefficients in the prime field $\mathbb{F}_p$. We write Ch for the Chow groups with coefficients in $\mathbb{F}_p$.

Let F be a field and K/F a finite separable field extension. Given a projective homogeneous (under an action of a semisimple affine algebraic group over K) K-variety X, we consider X as an F-variety via the composition $X \to \operatorname{Spec} K \to \operatorname{Spec} F$.

We recall that the following *Krull-Schmidt principle* holds: any summand of the motive $M(X)$ of X decomposes and in a unique way in a finite direct sum of indecomposable motives, [3] or [9, Corollary 2.2].

We define the *upper motive* $U(X)$ of X as an indecomposable summand of the motive $M(X)$ of X with the property that the Chow group $\operatorname{Ch}^0 U(X)$ is non-zero (or, equivalently, the property that $U(X)$ over an algebraic closure of F contains the Tate summand $\mathbb{F}_p$). Since the Chow group $\operatorname{Ch}^0 X$ is a 1-dimensional vector space (over $\mathbb{F}_2$), any given complete motivic decomposition of X contains precisely one upper summand. It follows by the Krull-Schmidt principle that the isomorphism class of $U(X)$ is uniquely determined by X. (See [11, Corollary 2.15] for a more direct proof not relying on the Krull-Schmidt principle.)

Given smooth complete irreducible F-varieties X_1 and X_2, we say that X_1 *dominates* X_2, if there exists multiplicity 1 correspondence $X_1 \rightsquigarrow X_2$. (Our correspondences are with coefficients in $\mathbb{F}_p$, their multiplicities are elements of $\mathbb{F}_p$.) We say that X_1 and X_2 are *equivalent*, if they dominate each other.

Given projective homogeneous varieties X_1 and X_2 (under possibly different algebraic groups) over finite separable field extensions K_1 and K_2 of F, the upper motives of the F-varieties X_1 and X_2 satisfy the following isomorphism criterion:

Lemma 2.1 ([11, Corollary 2.15])**.** *The upper motives $U(X_1)$ and $U(X_2)$ are isomorphic if and only if the F-varieties X_1 and X_2 are equivalent.*

Now let G be a semisimple affine algebraic group over F. The minimal (in a fixed separable closure of F) field extension E/F such that G_E is of inner type, is finite and galois (it corresponds to the kernel of the action of the absolute galois group of F on the Dynkin diagram of G). Assuming that $[E : F]$ is a power of p (the possibility $E = F$ is included), we have the following basic result of the upper motive theory:

Theorem 2.2 ([9, Theorem 1.1])**.** *Let X be a projective G-homogeneous F-variety. Any indecomposable summand of the motive of X is a shift of the upper motive $U(Y)$ of some projective G_K-homogeneous variety Y dominating X, where K is some intermediate field of the extension E/F.*

Apparently, the above theorem, showing that the upper motives are important, does not say anything about their structure. However, it makes it possible to prove the following structure result, which means that the upper motive possesses the same kind of symmetry as the whole motive of a variety. Let us define *dimension* $\dim U(X)$ of $U(X)$ as the biggest integer i such that the Tate motive $\mathbb{F}_p(i)$ is a summand of the motive $U(X)$ over an algebraic closure of F. (Putting $M(X)$ in place of $U(X)$ in the definition given, we will get the usual dimension of the variety X.) We write $U(X)^*$ for the *dual* of the motive $U(X)$, [4, §65].

Proposition 2.3 ([7, Proposition 5.2])**.** $U(X) \simeq U(X)^*\big(\dim U(X)\big)$.

Another drawback of Theorem 2.2 is absence of a precise indication concerning the varieties Y whose upper motives do really appear in the complete motivic decomposition of X. Although this drawback can be recovered in many particular cases, there is no recipe to recover it in general. But the information on possible Y given in Theorem 2.2 can be made more precise using the following argument.

Let X and Y (and K) be as in Theorem 2.2. We assume that $U(Y)$ is, up to a shift, a motivic summand of X and we want to say something more on Y besides the fact that it dominates X claimed in Theorem 2.2. Let X' be a projective G-homogeneous F-variety dominated by X (for instance, $X' = X$). Let us consider the complete motivic decomposition of $X_{F(X')}$. (If X' is non-trivial, the algebraic group acting on X is isotropic over the field $F(X')$; using the motivic decompositions of projective homogeneous varieties under *isotropic* groups constructed in [1], we may know the complete motivic decomposition of $X_{F(X')}$ by induction on the rank of the group.) The complete decomposition of $U(Y)_{F(X')}$ is (up to a shift) a part of this decomposition. It follows that there exist an intermediate field K' of the extension E/F and a projective $G_{K'(X')}$-homogeneous variety Y' such that $U(Y')$ (without shift) is a summand of $U(Y)_{F(X')}$.

Proposition 2.4. *The $F(X')$-varieties $Y_{F(X')}$ and Y' are equivalent. This property determines the equivalence class of the F-variety Y. The motive $U(Y')\big(\dim U(Y) - \dim U(Y')\big)$ is also a summand of $U(Y)_{F(X')}$.*

Proof. The upper motives of the $F(X')$-varieties $Y_{F(X')}$ and Y' are isomorphic, therefore the varieties are equivalent by Lemma 2.1.

To prove the second statement, let us take one more intermediate field L of the extension E/F and a dominating X projective G_L-homogeneous variety Z such that the $F(X')$-varieties $Y_{F(X')}$ and $Z_{F(X')}$ are equivalent. Since $Y_{F(X')}$ dominates $Z_{F(X')}$, the $F(X')(Y) = F(X' \times Y)$-variety $Z_{F(X' \times Y)}$ has a closed point of coprime with p degree (a *p-coprime closed point* for short). Since Y dominates X which dominates X', Y dominates X' and it follows that $X'_{F(Y)}$ has a p-coprime closed point. Let F' be the residue field of such a point. The tensor product $F'' := F' \otimes_{F(Y)} F(X' \times Y)$ is a field containing $F(X' \times Y)$. Moreover, the field extension F''/F' is isomorphic to the function field of the projective homogeneous F'-variety $X'_{F'}$ with rational point. Therefore the field extension F''/F' is purely transcendental. It follows by [4, Lemma 75.2] that $Z_{F'}$ has a p-coprime closed point. Since p does not divide $[F' : F(Y)]$, $Z_{F(Y)}$ has a p-coprime closed point and therefore Y dominates Z.

Exchanging the roles of Y and Z, we get that Z dominates Y as well so that Y and Z are equivalent. This finishes the proof of the second statement of Proposition 2.4.

Finally, since $U(Y')$ is a summand of $U(Y)_{F(X')}$, $U(Y')^*$ is a summand of $U(Y)^*_{F(X')}$. Applying Proposition 2.3, we get the third statement of Proposition 2.4. $\square$

Example 2.5. Let $p = 2$ and $G = O^+(q)$ (in notation of [14, §23]) for a non-degenerate quadratic form q over F. We have $E = F$ if $\dim q$ is odd or $\operatorname{disc} q$ is trivial. Otherwise E/F is the quadratic galois field extension given by the discriminant of q. We set $n := \dim q$. For any integer i with $0 \leq i < n/2$, let Q_i be the variety (orthogonal grassmannian) of i-dimensional totally isotropic subspaces in q. (Note that we do not consider the variety $Q_{n/2}$ here.) In particular, $Q_0 = \operatorname{Spec} F$ and Q_1 is the projective quadric of q. The varieties Q_i are projective G-homogeneous (while the projective variety $Q_{n/2}$ is never homogeneous) and form a complete system of representatives of the equivalence classes of all projective G-homogeneous varieties. (If n is even and $\operatorname{disc} q$ trivial, a component of $Q_{n/2}$ is a projective G-homogeneous variety equivalent to $Q_{n/2-1}$.) Moreover, for $i \geq j$, Q_i dominates Q_j so that if Q_j also dominates Q_i then Q_j is equivalent to Q_i. Therefore, by Theorem 2.2, any indecomposable summand of the motive of Q_i is a shift of the upper motive $U(Q_j)$ or $U(Q_{j\,E})$ for some $j \geq i$.

In order to apply Proposition 2.4, we will use the following motivic decomposition of $Q_{i\,F(Q_1)}$ obtained in [1] (see also [6]). We assume that $n \geq 3$ and $i \geq 1$. Let q' be an $(n-2)$-dimensional quadratic form over the field $F(Q_1)$ Witt-equivalent to $q_{F(Q_1)}$. Write Q'_j for the orthogonal grassmannians of q'. Then

$$(2.6)\quad M(Q_{i\,F(Q_1)}) \simeq M(Q'_{i-1}) \oplus M(Q'_i)\big((\dim Q_i - \dim Q'_i)/2\big) \oplus M(Q'_{i-1})(\dim Q_i - \dim Q'_{i-1}).$$

3. Odd-dimensional quadratic forms

Let F be a field, m an integer ≥ 0, $n := 2m+1$, q a non-degenerate n-dimensional quadratic form over F. Let us recall the definition and meaning of the J-invariant $J(q)$ given in [18].

Writing the bar $\bar{}$ over an F-variety we mean that we are considering it over an algebraic closure of F. Let $f : \mathrm{Ch}^*(\bar{Q}_1) \to \mathrm{Ch}^{*-m+1}(\bar{Q}_m)$ be the composition of the pull-back with respect to the projection $Q_{1,m} \to Q_1$ followed by the push-forward with respect to the projection $Q_{1,m} \to Q_m$. For $i = 1, \ldots, m$, let us define $z_i \in \mathrm{Ch}^i(\bar{Q}_m)$ as the image under f of the class in $\mathrm{Ch}_{m-i}(\bar{Q}_1)$ of an $(m-i)$-dimensional projective subspace on $\bar{Q}_1$. The J-invariant $J(q)$ is defined as the subset of $\{1, \ldots, m\}$ consisting of those i for which the element z_i *rational* by which we mean that it is in the image of the homomorphism $\mathrm{Ch}^i(Q_m) \to \mathrm{Ch}^i(\bar{Q}_m)$. The ring $\mathrm{Ch}(\bar{Q}_m)$ is known to be generated by z_i, $i \in \{1, \ldots, m\}$. The main result of [18] affirms that the ring of rational elements in $\mathrm{Ch}(\bar{Q}_m)$ is generated by z_i, $i \in J(q)$. Note that $J(q) = \emptyset$ for the generic quadratic form $q := \langle t_0 \rangle \perp [t_1, t_2] \perp \ldots \perp [t_{2m-1}, t_{2m}]$ over the field $F := k(t_0, \ldots, t_{2m})$, where k is any field and $t_0, \ldots, t_{2m}$ are variables, [19, Statement 3.6] (for a treatment including characteristic 2 the reader might look at [4, §88]).

Theorem 3.1. *Let q be a non-degenerate $(2m+1)$-dimensional quadratic form over a field F such that $J(q) = \emptyset$. Then for any i with $0 \le i \le m$, the motive of the variety Q_i is indecomposable. In particular, all Q_i are 2-incompressible.*

Proof. We induct on m. The induction base is the trivial case of $m = 0$. Now we assume that $m \ge 1$.

We do a descending induction on i. The induction base is the case of $i = m$ which follows directly from [18, Main Theorem 5.8] (the characteristic $\ne 2$ assumption made in [18] is omitted in [4, Theorem 87.7]). Indeed, the result cited tells us that the image of $\mathrm{Ch}^i(Q_m) \to \mathrm{Ch}^i(\bar{Q}_m)$ is trivial for any $i > 0$ provided that $J(q) = \emptyset$. As the motive of $Q_m \times Q_m$ is a direct sum of shifted copies of $M(Q_m)$, it follows that the diagonal class is the only non-zero rational element in $\mathrm{Ch}^{\dim Q_m}(\bar{Q}_m \times \bar{Q}_m)$.

Now we assume that $i < m$. Since the case of $i = 0$ is trivial, we may assume that $i > 0$.

We are using notation of Example 2.5. According to [19, Corollary 3.5] (see [4, §88] for a proof including a positive characteristic), the assumption on the J-invariant still holds for the quadratic form q'. By the induction hypothesis, each of the three summands of the decomposition (2.6) is indecomposable.

It follows by Proposition 2.4 that if the motive of Q_i (over F) is decomposable, then it has an indecomposable summand M such that M_L, where $L := F(Q_1)$, is a shift of $M(Q'_i) = U(Q'_i)$. Indeed, if one of the two extreme summands of the decomposition 2.6 is "defined over F", then the other extreme summand is "defined over F" by the last statement of Proposition 2.4, so that the remaining (interior) summand has also to be "defined over F".

Note that the varieties Q'_i and $Q_{i+1\,L}$ are equivalent. By Proposition 2.4, $M \simeq U(Q_{i+1})$, that is, $U(Q_{i+1})_L \simeq M(Q'_i)$. By the induction hypothesis, the motive of Q_{i+1} is indecomposable, i.e., $U(Q_{i+1}) = M(Q_{i+1})$. Therefore we have an isomorphism $M(Q_{i+1})_L \simeq M(Q'_i)$ and, in particular, $\dim Q_{i+1} = \dim Q'_i$. However

$$\dim Q_{i+1} - \dim Q'_i = n - i - 2 = 2m - i - 1 > m - 1 \ge 0. \qquad \square$$

4. Even-dimensional quadratic forms of trivial discriminant

Let F be a field, m an integer ≥ 1, $n := 2m$, q a non-degenerate n-dimensional quadratic form over F of trivial discriminant. In this case the variety Q_m (of totally isotropic m-dimensional subspaces in q) has two (isomorphic) connected components, and we write Q_m^+ for a component of the variety Q_m. The J-invariant $J(q)$ is defined in [18] as the subset $J(q) := \{0\} \cup J(q^1)$ of $\{0, 1, \ldots, m-1\}$, where q^1 is an arbitrary non-degenerate subform in q of codimension 1.

Note that the variety Q_m^+ is isomorphic to the variety Q_{m-1}^1 of totally isotropic $(m-1)$-dimensional subspaces in q^1. Therefore we have $J(q) = \{0\}$ for the generic n-dimensional quadratic form q with trivial discriminant constructed as follows: the base field F is the discriminant quadratic extension over $k(t_1, \ldots, t_n)$ of the quadratic form

$$[t_1, t_2]\perp \ldots \perp[t_{n-1}, t_n],$$

where k is a field and $t_1, \ldots, t_n$ are variables, and $q = ([t_1, t_2]\perp \ldots \perp[t_{n-1}, t_n])_F$. The motive of Q_m^+ is indecomposable and the variety is 2-incompressible for general q provided that $J(q) = \{0\}$.

Theorem 4.1. *Let q be a non-degenerate $(2m)$-dimensional quadratic form over a field F such that the discriminant of q is trivial and $J(q) = \{0\}$. Then for any i with $0 \leq i \leq m-2$, the motive of the variety Q_i is indecomposable. In particular, Q_i is 2-incompressible for such i.*

Proof. We induct on m. The induction base is the vacuous case of $m = 1$. Now we assume that $m \geq 2$.

We do a descending induction on $i \leq m-2$. Since the case of $i = 0$ is trivial, we may assume that $i > 0$ (and, in particular, $m \geq 3$).

We are using notation of Example 2.5. The discriminant of the quadratic form q' is also trivial. According to [19, Corollary 3.5] (see [4, §88] for a proof including a positive characteristic), the assumption on the J-invariant holds for q'. By the induction hypothesis, the motive $M(Q'_{i-1})$ appearing in the decomposition (2.6) is indecomposable. However the motive $M(Q'_i)$ – the middle summand of the decomposition (up to a shift) – is indecomposable if $i \neq m-2$. Let us treat the case $i = m-2$ first.

The variety Q'_{m-2} is a rank $m-2$ projective bundle over Q'^+_{m-1} (a component of Q'_{m-1}). Indeed, any totally isotropic $(m-2)$-dimensional subspace in q' is contained in a unique totally isotropic $(m-1)$-dimensional subspace in q' lying on Q'^+_{m-1}. This provides us with a morphism $Q'_{m-2} \to Q'^+_{m-1}$ which is a projective bundle: the fiber over a point of Q'^+_{m-1} given by a space W is given by all 1-codimensional subspaces in W and is the dual projective space of W. Therefore the complete decomposition of $M(Q'_{m-2})$ looks as follows:

$$M(Q'_{m-2}) \simeq M(Q'^+_{m-1}) \oplus M(Q'^+_{m-1})(1) \oplus \cdots \oplus M(Q'^+_{m-1})(m-2).$$

It follows that $U(Q_{m-1}) = U(Q_m^+) = M(Q_m^+)$ is not a shift of a summand of $M(Q_{m-2})$. Indeed, otherwise

$$M(Q_m^+)_L \simeq M(Q'^+_{m-1}) \oplus M(Q'^+_{m-1})(m-1),$$

where $L := F(Q_1)$, would be a shift of a summand of $M(Q_{m-2})_L$ but it is not because the motives $M(Q'_{m-3}) = U(Q'_{m-3})$ and $U({Q'}^{+}_{m-1}) = M({Q'}^{+}_{m-1})$ are not isomorphic (e.g., because $\dim Q'_{m-3} \neq \dim {Q'}^{+}_{m-1}$).

It follows that the motive of Q_{m-2} is indecomposable. This is the base case of our descending induction on i. Below we assume that $i < m-2$.

Now each of the three summands of the decomposition of $M(Q_i)_L$, given in (2.6), is indecomposable. It follows by Proposition 2.4 that if the motive of Q_i (over F) is decomposable, then it has an indecomposable summand M with M_L isomorphic to a shift of $M(Q'_i)$. By Proposition 2.4, M (over F) is isomorphic to a shift of $U(Q_{i+1})$, so that $U(Q_{i+1})_L \simeq M(Q'_i)$. By the induction hypothesis, the motive of Q_{i+1} is indecomposable, i.e., $U(Q_{i+1}) = M(Q_{i+1})$. Therefore we have an isomorphism $M(Q_{i+1})_L \simeq M(Q'_i)$ and, in particular, $\dim Q_{i+1} = \dim Q'_i$. However

$$\dim Q_{i+1} - \dim Q'_i = n - i - 2 = 2m - i - 2 > m \geq 3. \qquad \square$$

5. Even-dimensional quadratic forms of non-trivial discriminant

Let F be a field, m an integer ≥ 1, $n := 2m$, q a non-degenerate n-dimensional quadratic form over F of non-trivial discriminant. In this case, the J-invariant $J(q)$ is defined in [18] as the subset $J(q) := J(q^1_E)$ of $\{0, 1, \ldots, m-1\}$, where q^1 is an arbitrary non-degenerate subform in q of codimension 1 and E is the quadratic extension field of F given by the discriminant of q. We have $J(q) = \emptyset$ for the generic n-dimensional quadratic form q.

Theorem 5.1. *Let q be a non-degenerate $(2m)$-dimensional quadratic form over a field F such that $J(q) = \emptyset$ (in particular, the discriminant of q is non-trivial and the variety Q_m is connected). For any i with $0 \leq i \leq m$, the motive of the variety Q_i is indecomposable. In particular, all Q_i are 2-incompressible.*

Proof. We induct on m. The induction base is the trivial case of $m = 1$. Now we assume that $m \geq 2$.

We do a descending induction on $i \leq m$. The induction base $i = m$ holds by our assumption on q. Below we are assuming that $i < m$. Since the case of $i = 0$ is trivial, we may assume that $i > 0$.

We are using notation of Example 2.5 and set $L := F(Q_1)$. Since F is algebraically closed in L, the discriminant of the quadratic form q' is non-trivial. Moreover, according to [19, Corollary 3.5] (see also [4, §88]), the assumption on the J-invariant holds for q'. By the induction hypothesis, each of the three summands of the decomposition (2.6) is indecomposable.

(We apologize for repeating almost word by word the end of the proof of Theorem 3.1 below.) It follows by Proposition 2.4 that if the motive of Q_i (over F) is decomposable, then it has an indecomposable summand M with M_L isomorphic to a shift of $M(Q'_i)$. By Proposition 2.4, M (over F) is isomorphic to a shift of $U(Q_{i+1})$, so that $U(Q_{i+1})_L \simeq M(Q'_i)$. By the induction hypothesis, the motive of Q_{i+1} is indecomposable, i.e., $U(Q_{i+1}) = M(Q_{i+1})$. Therefore we have an isomorphism $M(Q_{i+1})_L \simeq M(Q'_i)$ and, in particular, $\dim Q_{i+1} = \dim Q'_i$. However $\dim Q_{i+1} - \dim Q'_i = 2m - i - 2 > m - 2 \geq 0$. $\square$

6. Canonical dimension

In this section, we make some development of the theory of canonical dimension of general projective homogeneous varieties which might be of independent interest and which will be used in the next section. We fix a prime p. Let G be a semisimple affine algebraic group over a field F such that G_E is of inner type for some finite galois field extension E/F of degree a power of p ($E = F$ is allowed). Let X be a projective G-homogeneous F-variety. We refer to [7] for a definition and discussion of the notion of *canonical p-dimension* $\operatorname{cdim}_p X$ of X. Actually, canonical p-dimension is defined in the context of more general algebraic varieties. For any irreducible smooth projective variety X, $\operatorname{cdim}_p X$ is the minimal dimension of a closed subvariety $Y \subset X$ with a 0-cycle of p-coprime degree on $Y_{F(X)}$. Recall that a smooth projective X is *p-incompressible*, if it is irreducible and $\operatorname{cdim}_p X = \dim X$. We write CH for the Chow groups with integer coefficients.

Proposition 6.1. *Let X be a projective G-homogeneous variety with G as above. For $d := \operatorname{cdim}_p X$, there exist a cycle class $\alpha \in \mathrm{CH}^d X_{F(X)}$ (over $F(X)$) of codimension d and a cycle class $\beta \in \mathrm{CH}_d X$ (over F) of dimension d such that the degree of the product $\beta_{F(X)} \cdot \alpha$ is not divisible by p.*

Proof. We use Chow motives with coefficients in $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. By [7, Theorem 5.1 and Proposition 5.2], the upper motive $U(X)$ of X (being by definition a direct summand of $M(X)$) is also a direct summand of $M(X)(d-m)$, where $m := \dim X$. The composition

$$M(X) \to U(X) \to M(X)(d-m)$$

is given by a correspondence $f \in \mathrm{Ch}^d(X \times X)$; the composition

$$M(X)(d-m) \to U(X) \to M(X)$$

is given by a correspondence $g \in \mathrm{Ch}_d(X \times X)$. The composition of correspondences $g \circ f \in \mathrm{Ch}_m(X \times X)$ is a projector on X such that $U(X) = (X, g \circ f)$. In particular, the multiplicity $\operatorname{mult}(g \circ f)$ of the correspondence $g \circ f$ is $1 \in \mathbb{F}_p$. Taking for α an integral representative of the pull-back of f with respect to the morphism

$$\operatorname{Spec} F(X) \times X \to X \times X$$

induced by the generic point of the first factor, and taking for β an integral representative of the push-forward of g with respect to the projection of $X \times X$ onto the first factor, we get that

$$\deg(\beta_{F(X)} \cdot \alpha) \pmod p = \operatorname{mult}(g \circ f) = 1 \in \mathbb{F}_p. \qquad \square$$

Corollary 6.2. *The canonical p-dimension $\operatorname{cdim}_p X$ of X is the minimal integer d such that there exist a cycle class $\alpha \in \mathrm{Ch}^d X_{F(X)}$ and a cycle class $\beta \in \mathrm{Ch}_d X$ with $\deg(\beta_{F(X)} \cdot \alpha) = 1 \in \mathbb{F}_p$.*

Proof. We only need to show that $\operatorname{cdim}_p X \leq d$. The proof is similar to [12, Proof of $\leq$ in Theorem 5.8]. Since $\deg(\beta_{F(X)} \cdot \alpha) = 1 \in \mathbb{F}_p$ for some $\beta \in \mathrm{Ch}_d X$ (and some α), there exists a closed irreducible d-dimensional subvariety $Y \subset X$ such that $\deg([Y]_{F(X)} \cdot \alpha) \neq 0 \in \mathbb{F}_p$ (with the same α). Since the product $[Y]_{F(X)} \cdot \alpha$ can be represented by a cycle on $Y_{F(X)}$, the variety $Y_{F(X)}$ has a 0-cycle of p-coprime degree. Therefore $\operatorname{cdim}_p X \leq \dim Y = d$. $\square$

Corollary 6.3. *In the situation of Proposition 6.1, for any field extension L/F, the change of field homomorphism $\mathrm{Ch}_d X \to \mathrm{Ch}_d X_L$ is non-zero.*

Proof. The image of $\beta \mod p \in \mathrm{Ch}_d X$ in $\mathrm{Ch}_d X_L$ is non-zero because $\deg(\beta_{L(X)} \cdot \alpha_{L(X)}) \not\equiv 0 \pmod p$. □

Remark 6.4. If the variety X is *generically split* (meaning that the motive of $X_{F(X)}$ is a sum of Tate motives (this implies that the adjoint algebraic group acting on X is of inner type)), then [12, Theorem 5.8] says that $\mathrm{cdim}_p X$ is the minimal d with non-zero $\mathrm{Ch}_d X \to \mathrm{Ch}_d X_L$ for any L. Corollary 6.3 can be considered as a generalization of a part of [12, Theorem 5.8] to the case of a projective G-homogeneous variety X which is not necessarily generically split with G not necessarily of inner type. Note that the statement of [12, Theorem 5.8] in whole fails in such generality. Corollary 6.2 is its correct replacement (giving the original statement in the case of generically split X).

Lemma 6.5. *In the situation of Proposition 6.1, let $\alpha, \alpha' \in \mathrm{Ch}^d X_{F(X)}$ and $\beta, \beta' \in \mathrm{Ch}_d X$ be cycle classes with $\deg(\beta_{F(X)} \cdot \alpha) = 1 = \deg(\beta'_{F(X)} \cdot \alpha')$. Then*

$$\deg(\beta_{F(X)} \cdot \alpha') \neq 0 \neq \deg(\beta'_{F(X)} \cdot \alpha).$$

Proof. We fix an algebraically closed field containing $F(X)$ and write $\bar{\cdot}$ when considering a variety or a cycle class over that field. The surjectivity of the pull-back with respect to the flat morphism $\mathrm{Spec}\, F(X) \times X \to X \times X$ induced by the generic point of the first factor of the product $X \times X$, tells us that the group $\mathrm{Ch}^d(\bar{X} \times \bar{X})$ contains a *rational* (i.e., coming from $\mathrm{Ch}^d(X \times X)$) cycle class of the form $[\bar{X}] \times \bar{\alpha} + \cdots + \bar{\gamma} \times [\bar{X}]$ with some $\gamma \in \mathrm{Ch}^d X_{F(X)}$, where $\dots$ is in the sum of products $\mathrm{Ch}^i \bar{X} \otimes \mathrm{Ch}^j \bar{X}$ with $0 < i, j < d$ and $i + j = d$. Multiplying by $[\bar{X}] \times \bar{\beta}$, we get a rational cycle class of the form $[\bar{X}] \times [\mathbf{pt}] + \cdots + \bar{\gamma} \times \bar{\beta}$, where $\mathbf{pt}$ is a rational point on $\bar{X}$ and $\dots$ is now in the sum of $\mathrm{Ch}^i \bar{X} \otimes \mathrm{Ch}_i \bar{X}$ with $0 < i < d$. The composition of the obtained correspondence with itself equals $[\bar{X}] \times [\mathbf{pt}] + \cdots + \deg(\gamma \cdot \beta)(\bar{\gamma} \times \bar{\beta})$. Since an appropriate power of this correspondence is a multiplicity 1 *projector* (cf. [8, Corollary 3.2] or [11]) and $d = \mathrm{cdim}_p X$, it follows by [7, Theorem 5.1] that $\deg(\gamma \cdot \beta) \neq 0$. Now multiplying $[\bar{X}] \times \bar{\alpha} + \cdots + \bar{\gamma} \times [\bar{X}]$ by $\bar{\beta} \times [\bar{X}]$, transposing, and raising to $(p-1)$th power (by means of composition of correspondences), we get a rational cycle of the form $[\bar{X}] \times [\mathbf{pt}] + \cdots + \bar{\alpha} \times \bar{\beta}$.

Similarly, there is a rational cycle of the form $[\bar{X}] \times [\mathbf{pt}] + \cdots + \bar{\alpha'} \times \bar{\beta}'$. One of its compositions with the previous one produces $[\bar{X}] \times [\mathbf{pt}] + \cdots + \deg(\beta' \cdot \alpha)(\bar{\alpha'} \times \bar{\beta})$, therefore $\deg(\beta' \cdot \alpha) \neq 0 \in \mathbb{F}_p$. The other composition produces $[\bar{X}] \times [\mathbf{pt}] + \cdots + \deg(\beta \cdot \alpha')(\bar{\alpha} \times \bar{\beta}')$, so that $\deg(\beta \cdot \alpha') \neq 0$. □

Below we are using notation of Example 2.5:

Corollary 6.6. *If $\mathrm{cdim}_2 Q_i = \mathrm{cdim}_2 Q'_{i-1} = \dim Q'_{i-1}$ for some i with $0 < i < n/2$, then the variety Q_i has a 0-cycle of degree 2^{i-1}.*

Proof. The statement is trivial for $i = 1$. Indeed, in this case $n \geq 3$ so that the variety Q_1 is geometrically integral. The condition of Corollary 6.6 says that $\mathrm{cdim}_2 Q_1 = 0$, and it follows that Q_1 has a rational point. Below we are assuming that $i \geq 2$.

For $d := \mathrm{cdim}_2 Q_i$, using Proposition 6.1, we find some $\alpha \in \mathrm{CH}^d Q_{i\,F(Q_i)}$ and $\beta \in \mathrm{CH}_d Q_i$ with odd $\deg(\beta_{F(Q_i)} \cdot \alpha)$. Note that $\mathrm{cdim}_2 Q_{i\,F(Q_1)} = \mathrm{cdim}_2 Q'_{i-1} = d$ (because the varieties

$Q_{i\,F(Q_1)}$ and Q'_{i-1} are equivalent). We construct some special $\alpha' \in \mathrm{CH}^d\, Q_{i\,F(Q_1)(Q_i)}$ and $\beta' \in \mathrm{CH}_d\, Q_{i\,F(Q_1)}$ with $\deg(\beta'_{F(Q_1)(Q_i)} \cdot \alpha') = 1$ as follows. Let us consider the variety $Q_{1,i}$ of $(1,i)$-flags of totally isotropic subspaces in q together with the projections $Q_{1,i} \to Q_1, Q_i$. We define β' as the pull-back via $Q_{1,i\,F(Q_1)} \to Q_{1\,F(Q_1)}$ followed by the push-forward via $Q_{1,i\,F(Q_1)} \to Q_{i\,F(Q_1)}$ of the rational point class l_0 on $Q_{1\,F(Q_1)}$ (notice that the variety $Q_{1\,F(Q_1)}$ has a rational point). Thus $\bar\beta'$ is the d-dimensional *standard elementary class* on $\bar Q_i$ as defined in [20, §2] (notation Q_i used here corresponds to $F(Q, i-1)$ of [20]). Since $d = \dim Q'_{i-1}$, the codimension of this standard elementary class is equal to $\dim Q_i - \dim Q'_{i-1} = n - i - 1$.

We define α' as the product of the elements $z_j \in \mathrm{Ch}^j\, Q_{i\,F(Q_1)(Q_i)}$, $j = n-i-2, \dots, n-2i$ defined as the pull-back via the projection $Q_{1,i\,F(Q_1)(Q_i)} \to Q_{1\,F(Q_1)(Q_i)}$ followed by the push-forward via the projection $Q_{1,i\,F(Q_1)(Q_i)} \to Q_{i\,F(Q_1)(Q_i)}$ of the class $l_{n-i-1-j}$ of an $(n-i-1-j)$-dimensional projective subspace on $Q_{1\,F(Q_1)(Q_i)}$ (note that the quadric $Q_{1\,F(Q_1)(Q_i)}$ contains an $(i-1)$-dimensional projective subspace). (Thus $\bar z_j$ is the j-codimensional standard elementary class on $\bar Q_i$ as defined in [20, §2].) Note that the codimension of α' is indeed d.

The degree condition on α' and β' is satisfied by [20, Statement 2.15]. Fixing an algebraically closed field containing $F(Q_1)(Q_i)$, we see by Lemma 6.5 that the product $\bar\beta \cdot \bar\alpha'$ is an odd degree 0-cycle class on $\bar Q_i$. Moreover, the class $\bar\beta$ is rational. Since $2\bar z_j$ is rational for every j (by the reason that $2l_j$ is rational), the class $2^{i-1}\bar\beta\bar\alpha$ is also rational and it follows that Q_i has a 0-cycle of degree 2^{i-1}. □

7. Mathews' conjecture

Theorem 7.1, proved below, has been conjectured in [15]. It is known for $i = 1$ by [16]. The case of maximal i, i.e., of $i = [n/2]$, is also known by [18, Proposition 6.5]. For $i = 2$ and odd-dimensional q, Theorem 7.1 has been proved in [15] (the proof for $i = 2$ given here is different; in particular, it does not make use of the motivic decompositions of [3] for products of projective homogeneous varieties).

Theorem 7.1. *Let q be a non-degenerate quadratic form over a field F. Let i be an integer satisfying $1 \le i \le m$. If the degree of every closed point on Q_i is divisible by 2^i and the Witt index of the quadratic form $q_{F(Q_i)}$ equals i, then the variety Q_i is* 2-incompressible *(i.e., $\mathrm{cdim}_2\, Q_i = \dim Q_i$).*

Proof. We set $n := \dim q$. Note that for $i = n/2$ (and even n) the condition on closed points on $Q_{n/2}$ ensures that $\mathrm{disc}\, q$ is non-trivial. In particular, $Q_{n/2}$ is irreducible. Moreover, the variety $Q_{n/2}$ is isomorphic to the orthogonal grassmannian of totally isotropic $(n/2-1)$-planes of q^1 considered as a variety over F, where q^1 is any 1-codimensional non-degenerate subform in q_E, and E/F is the quadratic field extension given by the discriminant of q. Therefore the statement of Theorem 7.1 for $i = n/2$ follows from the statement for $i = (n-1)/2$. By this reason, we do not consider the case of $i = n/2$ below. In particular, Q_i below is a projective G-homogeneous variety.

We induct on n. There is nothing to prove for $n < 3$. Below we are assuming that $n \ge 3$.

Over the field $F(Q_1)$, the motive of $Q_{i\,F(Q_1)}$ decomposes as in (2.6). Since $n' := \dim q' = n-2 < n$, the variety Q'_{i-1} is 2-incompressible by the induction hypothesis (more precisely, the induction hypothesis is applied if $i \geq 2$, for $i = 1$ the statement if trivial). Indeed, since the extension $F(Q_1)/F$ is a tower of a purely transcendental extension followed by a quadratic one, the degree of any closed point on Q'_{i-1} is divisible by 2^{i-1}; the Witt index of $q'_{F(Q_1)(Q'_{i-1})}$ is $i-1$, that is, the Witt index of $q_{F(Q_1)(Q'_{i-1})}$ is i because the field extension $F(Q_1)(Q'_{i-1})(Q_i)/F(Q_i)$ is purely transcendental.

By Theorem 2.2 and Example 2.5, the motive of Q'_{i-1} decomposes in a direct sum of one copy of $U(Q'_{i-1})$ (we recall that the variety Q'_{i-1} is 2-incompressible), shifts of $U(Q'_j)$ with various $j \geq i$, and (in the case of even n and non-trivial $\operatorname{disc} q$) shifts of $U(Q'_{j\,E})$ with $j \geq i-1$ (where E/F is the quadratic field extension corresponding to $\operatorname{disc} q$). The motive of Q'_i decomposes in a direct sum of shifts of $U(Q'_j)$ and (in the case of even n and non-trivial $\operatorname{disc} q$) shifts of $U(Q'_{j\,E})$ with various $j \geq i$. Note that for any $j \geq i$ the motive $U(Q'_{i-1})$ is not isomorphic to $U(Q'_j)$ since the varieties Q'_{i-1} and Q'_j are not equivalent due to our condition on the Witt index of the form $q_{F(Q_i)}$. Besides $U(Q'_{i-1}) \not\simeq U(Q'_{j\,E})$ because every closed point on the $F(Q'_{i-1})$-variety $Q'_{j\,E(Q'_{i-1})}$ is of even degree. Therefore the complete motivic decomposition of $Q_{i\,F(Q_1)}$ contains one copy of $U(Q'_{i-1})$, one copy of $U(Q'_{i-1})(\dim Q_i - \dim Q'_{i-1})$ and no other shifts of $U(Q'_{i-1})$.

The complete decomposition of $U(Q_i)_{F(Q_1)}$ contains the summand $U(Q'_{i-1})$. If it also contains the second (shifted) copy of $U(Q'_{i-1})$, then $\operatorname{cdim}_2 Q_i = \dim Q_i$ by the 2-incompressibility of Q'_{i-1}, and we are done. Otherwise, by Proposition 2.4, $\operatorname{cdim}_2 Q_i = \operatorname{cdim}_2 Q'_{i-1} = \dim Q'_{i-1}$, and we get by Corollary 6.6 that Q_i has a closed point of degree not divisible by 2^i. $\square$

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