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Canonical Dimension of (semi-)spinor Groups of Small Ranks

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Abstract: We show that the canonical dimension $\operatorname{cd}\operatorname{Spin}_{2n+1}$ of the spinor group $\operatorname{Spin}_{2n+1}$ has an inductive upper bound given by $n + \operatorname{cd}\operatorname{Spin}_{2n-1}$. Using this bound, we determine the precise value of $\operatorname{cd}\operatorname{Spin}_n$ for all $n \leq 16$ (previously known for $n \leq 10$). We also obtain an upper bound for the canonical dimension of the semi-spinor group $\operatorname{cd}\operatorname{Spin}_n^\sim$ in terms of $\operatorname{cd}\operatorname{Spin}_{n-2}$. This bound determines $\operatorname{cd}\operatorname{Spin}_n^\sim$ for $n \leq 16$; for any n, assuming a conjecture on the precise value of $\operatorname{cd}\operatorname{Spin}_{n-2}$, this bound determines $\operatorname{cd}\operatorname{Spin}_n^\sim$.

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1. Introduction

Let X be a smooth algebraic variety over a field F. A field extension L/F is called a *splitting field* of X, if $X(L) \neq \emptyset$. A splitting field E of X is called *generic*, if it has an F-place $E \dashrightarrow L$ to any splitting field L of X. Given a prime number p, a splitting field E of X is called *p-generic*, if for any splitting field L of X there exists an F-place $E \dashrightarrow L'$ to some finite extension L'/L of degree prime to p. Note that since X is smooth, the function field $F(X)$ is a generic splitting field of X; besides, any generic splitting field of X is p-generic for any p.

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The canonical dimension $\mathrm{cd}(X)$ of the variety X is defined as the minimum of $\mathrm{tr.deg}_F E$, where E runs over the generic splitting fields of X; the canonical p-dimension $\mathrm{cd}_p(X)$ of X is defined as the minimum of $\mathrm{tr.deg}_F E$, where E runs over the p-generic splitting fields of X. For any p, one evidently has $\mathrm{cd}_p(X) \leq \mathrm{cd}(X)$.

Let G be an algebraic group over F. The notion of canonical dimension $\mathfrak{cd}(G)$ of G is introduced in [1]: $\mathfrak{cd}(G)$ is the maximum of $\mathrm{cd}(T)$, where T runs over the G-torsors over all field extensions K/F. The notion of canonical p-dimension $\mathfrak{cd}_p(G)$ of G is introduced in [3]: $\mathfrak{cd}_p(G)$ is the maximum of $\mathrm{cd}_p(T)$, where T runs over the G-torsors over all field extensions K/F. For any p, one evidently has $\mathfrak{cd}_p(G) \leq \mathfrak{cd}(G)$.

A recipe of computation of $\mathfrak{cd}_p(G)$ for an arbitrary p and an arbitrary split simple algebraic group G is given in [3]; the value of $\mathfrak{cd}_p(G)$ is determined there for all G of classical type (the remaining types are treated in [4]).

Let G be a split simple algebraic group over F and let p be a prime. As follows from the definition of the canonical p-dimension, $\mathfrak{cd}_p(G) \neq 0$ if and only if p is a torsion prime of G. It is shown in [2], that $\mathfrak{cd}(G) = \mathfrak{cd}_p(G)$ for any G possessing a unique torsion prime p with the exception of the case where G is a spinor or a semi-spinor group.

According to [3], for any $n \geq 1$ one has

$$\mathfrak{cd}_2(\mathrm{Spin}_{2n+1}) = \mathfrak{cd}_2(\mathrm{Spin}_{2n+2}) = n(n+1)/2 - 2^l + 1 \ ,$$

where l is the smallest integer such that $2^l \geq n+1$ (the prime 2 is the unique torsion prime of the spinor group). As shown in [1], $\mathfrak{cd}(\mathrm{Spin}_{2n+1}) = \mathfrak{cd}(\mathrm{Spin}_{2n+2})$ for any n and $\mathfrak{cd}(\mathrm{Spin}_n) = \mathfrak{cd}_2(\mathrm{Spin}_n)$ for all $n \leq 10$.

We note that the Spin_{10}-torsors are related to the 10-dimensional quadratic forms of trivial discriminant and trivial Clifford invariant, and that the value of $\mathfrak{cd}(\mathrm{Spin}_{10})$ is obtained due to a theorem of Pfister on those quadratic forms.

In [2], an upper bound on $\mathfrak{cd}(\mathrm{Spin}_{2n+1})$ given by $n(n-1)/2$ is established. If $n+1$ is a power of 2, this upper bound coincides with the lower bound given by the known value of $\mathfrak{cd}_2(\mathrm{Spin}_{2n+1})$. Therefore $\mathfrak{cd}(\mathrm{Spin}_n) = \mathfrak{cd}_2(\mathrm{Spin}_n)$, if n or $n+1$ is a 2 power.

In the current note, we establish for an arbitrary n the following inductive upper bound on $\mathfrak{cd}(\mathrm{Spin}_{2n+1})$ (see Theorem 2.2):

$$\mathfrak{cd}(\mathrm{Spin}_{2n+1}) \leq n + \mathfrak{cd}(\mathrm{Spin}_{2n-1}) \ .$$

This bound together with the computation of $\mathfrak{cd}(\mathrm{Spin}_n)$ for $n \leq 10$, cited above, shows (see Corollary 2.4) that $\mathfrak{cd}(\mathrm{Spin}_n) = \mathfrak{cd}_2(\mathrm{Spin}_n)$ for any $n \leq 16$ (the really new cases are $n \in \{11, 12, 13, 14\}$). More generally, if $\mathfrak{cd}(\mathrm{Spin}_{2^m+1}) = \mathfrak{cd}_2(\mathrm{Spin}_{2^m+1})$ for some positive integer m, then our inductive bound shows that

$\mathfrak{cd}(\mathrm{Spin}_n) = \mathfrak{cd}_2(\mathrm{Spin}_n)$ for any n lying in the interval $[2^m+1,\ 2^{m+1}]$ (see Corollary 2.3).

Note that $\mathfrak{cd}_2(\mathrm{Spin}_{2^m+1}) = \mathfrak{cd}_2(\mathrm{Spin}_{2^m})$. Therefore the crucial statement needed for a further progress on $\mathfrak{cd}(\mathrm{Spin}_n)$ is the statement that $\mathfrak{cd}(\mathrm{Spin}_{17}) = \mathfrak{cd}(\mathrm{Spin}_{16})$. As mentioned above, the similar equality $\mathfrak{cd}(\mathrm{Spin}_9) = \mathfrak{cd}(\mathrm{Spin}_8)$, concerning the previous 2 power, is a consequence of the Pfister theorem.

We finish the introduction by discussing the semi-spinor group $\mathrm{Spin}^{\sim}_n$. Here n is a positive integer divisible by 4. To see the parallels with the spinor case, it is more convenient to speak on $\mathrm{Spin}^{\sim}_{2n+2}$ with n odd. The lower bound on $\mathfrak{cd}(\mathrm{Spin}^{\sim}_{2n+2})$ given by the canonical 2-dimension (the prime 2 is the unique torsion prime of the semi-spinor group) is calculated in [3] as

$$\mathfrak{cd}_2(\mathrm{Spin}^{\sim}_{2n+2}) = n(n+1)/2 + 2^k - 2^l\ ,$$

where k is the largest integer such that 2^k divides $n+1$ (and l is still the smallest integer with $2^l \geq n+1$). The upper bound $\mathfrak{cd}(\mathrm{Spin}^{\sim}_{2n+2}) \leq n(n-1)/2 + 2^k - 1$, established in [2], shows that the canonical 2-dimension is the value of the canonical dimension if $n+1$ is a power of 2. In particular, $\mathfrak{cd}(\mathrm{Spin}^{\sim}_n) = \mathfrak{cd}_2(\mathrm{Spin}^{\sim}_n)$ for $n \in \{4, 8, 16\}$.

In the current note we establish the following general upper bound on the canonical dimension of the semi-spinor group in terms of the canonical dimension of the spinor group (see Theorem 3.1):

$$\mathfrak{cd}(\mathrm{Spin}^{\sim}_{2n+2}) \leq n - 1 + 2^k + \mathfrak{cd}(\mathrm{Spin}_{2n})$$

(with k as above). This bound together with the computation of $\mathfrak{cd}(\mathrm{Spin}_{10})$ shows (see Corollary 3.3) that $\mathfrak{cd}(\mathrm{Spin}^{\sim}_{12}) = \mathfrak{cd}_2(\mathrm{Spin}^{\sim}_{12}) = 11$; therefore the formula $\mathfrak{cd}(\mathrm{Spin}^{\sim}_n) = \mathfrak{cd}_2(\mathrm{Spin}^{\sim}_n)$ holds for all $n \leq 16$ (where the only new case is $n = 12$).

In general, if $\mathfrak{cd}(\mathrm{Spin}_{2n}) = \mathfrak{cd}_2(\mathrm{Spin}_{2n})$ for some (odd) n, then our upper bound on $\mathfrak{cd}(\mathrm{Spin}^{\sim}_{2n+2})$ shows that $\mathfrak{cd}(\mathrm{Spin}^{\sim}_{2n+2}) = \mathfrak{cd}_2(\mathrm{Spin}^{\sim}_{2n+2})$ for this n (see Corollary 3.2).

2. The spinor group

Our main tool is the following general observation made in [2]. Let G be a split semisimple algebraic group over a field F, P a parabolic subgroup of G, P' a special parabolic subgroup of G sitting inside of P. Saying *special*, we mean that any P'-torsor over any field extension K/F is trivial.

For any G-torsor T over F, let us write $\mathrm{cd}'(T/P)$ for $\min\{\dim X\}$, where X runs over all closed subvarieties of the variety T/P admitting a rational morphism $F(T/P') \dashrightarrow X$.

Lemma 2.1 ([2, lemma 5.3]). *In the above notation, one has*

$$\operatorname{cd}(T) \leq \operatorname{cd}'(T/P) + \max_Y \operatorname{cd}(Y) ,$$

where Y runs over all fibers of the projection $T/P' \to T/P$.

In this section, we apply Lemma 2.1 in the following situation: $G = \mathrm{Spin}_{2n+1} = \mathrm{Spin}(\varphi)$, where $\varphi\colon F^{2n+1} \to F$ is a split quadratic form; P is the stabilizer of a rational point x under the standard action of G on the variety of 1-dimensional totally isotropic subspaces of φ; $P' \subset P$ is the stabilizer of a rational point x', lying over x, under the standard action of G on the variety of flags consisting of a 1-dimensional totally isotropic subspace sitting inside of an n-dimensional (maximal) totally isotropic subspace of φ.

The parabolic subgroup P' of G is clearly special.

Let T be a G-torsor over F and let $\psi\colon F^{2n+1} \to F$ be a quadratic form such that the similarity class of ψ is the class corresponding to T in the sense of [3, §8.2]. Note that the even Clifford algebra of ψ is trivial.

The algebraic variety T/P is identified with the projective quadric of ψ; in particular, $\dim(T/P) = 2n - 1$. The variety T/P' is identified with the variety of flags consisting of a 1-dimensional subspace sitting inside of an n-dimensional (maximal) totally isotropic subspace of ψ. The morphism $T/P' \to T/P$ is identified with the natural projection of the flag variety onto the quadric.

Let $X \subset T/P$ be an arbitrary subquadric of dimension n (X is the quadric of the restriction of ψ onto an $(n + 2)$-dimensional subspace of F^{2n+1}). Since over the function field $F(T/P')$ the quadratic form ψ becomes split, the variety $X_{F(T/P')}$ has a rational point, or, in other words, there exists a rational morphism $T/P' \dashrightarrow X$. Therefore $\operatorname{cd}'(T/P) \leq \dim X = n$.

Any fiber Y of the projection $T/P' \to T/P$ is the variety of n-dimensional (maximal) totally isotropic subspaces of ψ, containing a fixed 1-dimensional subspace U. The latter variety is identified with the variety of $(n - 1)$-dimensional (maximal) totally isotropic subspaces of the quotient $U^\perp/U$. Note that we have $\dim U^\perp/U = 2n-1$; besides, the quadratic form on $U^\perp/U$, induced by the restriction of ψ, is Witt-equivalent to ψ and, in particular, its even Clifford algebra is trivial. Since $\operatorname{cd}(\mathrm{Spin}_{2n-1})$ is the maximum of the canonical dimension of the variety of maximal totally isotropic subspaces of a $(2n - 1)$-dimensional quadratic forms with trivial even Clifford algebra, it follows that $\operatorname{cd}(Y) \leq \mathfrak{cd}(\mathrm{Spin}_{2n-1})$. Applying Lemma 2.1, we get our main inequality for the spinor group:

Theorem 2.2. *For any n, one has $\mathfrak{cd}(\mathrm{Spin}_{2n+1}) \leq n + \mathfrak{cd}(\mathrm{Spin}_{2n-1})$.* □

Corollary 2.3. *Assume that* $\mathfrak{cd}(\mathrm{Spin}_{2^m+1}) = \mathfrak{cd}_2(\mathrm{Spin}_{2^m+1})$ *for some positive integer* m*. Then* $\mathfrak{cd}(\mathrm{Spin}_n) = \mathfrak{cd}_2(\mathrm{Spin}_n)$ *for any* n *lying in the interval* $[2^m + 1,\ 2^{m+1}]$.

Proof. Let n be such that $2n \pm 1 \in [2^m,\ 2^{m+1}]$ and $\mathfrak{cd}(\mathrm{Spin}_{2n-1}) = \mathfrak{cd}_2(\mathrm{Spin}_{2n-1})$. Then

$$\mathfrak{cd}(\mathrm{Spin}_{2n+1}) \le n + \mathfrak{cd}(\mathrm{Spin}_{2n-1}) = n + n(n-1)/2 - 2^m + 1 = \\ n(n+1)/2 - 2^m + 1 = \mathfrak{cd}_2(\mathrm{Spin}_{2n+1}) \le \mathfrak{cd}(\mathrm{Spin}_{2n+1})\ .$$

Consequently, $\mathfrak{cd}(\mathrm{Spin}_{2n+1}) = \mathfrak{cd}_2(\mathrm{Spin}_{2n+1})$. □

Since $\mathfrak{cd}(\mathrm{Spin}_n) = \mathfrak{cd}_2(\mathrm{Spin}_n)$ for $n \le 10$ (see [1, example 12.2]), the assumption of Corollary 2.3 holds for $m = 3$, and we get

Corollary 2.4. *The equality* $\mathfrak{cd}(\mathrm{Spin}_n) = \mathfrak{cd}_2(\mathrm{Spin}_n)$ *holds for any* $n \le 16$. □

3. The semi-spinor group

In this section, we apply Lemma 2.1 in the following situation: $G = \mathrm{Spin}^{\sim}_{2n+2} = \mathrm{Spin}^{\sim}(\varphi)$, where $\varphi\colon F^{2n+2} \to F$ is a hyperbolic quadratic form; P is the stabilizer of a rational point x under the standard action of G on the variety of 1-dimensional totally isotropic subspaces of φ; $P' \subset P$ is the stabilizer of a rational point x', lying over x, under the standard action of G on the scheme of flags consisting of a 1-dimensional totally isotropic subspace sitting inside of an $(n+1)$-dimensional (maximal) totally isotropic subspace of φ.

The parabolic subgroup P' of G is clearly special.

Let T be a G-torsor over F and let π be a quadratic pair on a degree $2n+2$ central simple F-algebra A such that the isomorphism class of π corresponds to T in the sense of [3, §8.4]. Note that the discriminant and a component of the Clifford algebra of π are trivial.

The quotient T/P is identified with the variety of rank 1 isotropic ideals of π; in particular, $\dim(T/P) = 2n$. The quotient T/P' is identified with a component of the scheme of flags consisting of a rank 1 ideal sitting inside of a rank $(n+1)$ (maximal) isotropic ideal of π. The morphism $T/P' \to T/P$ is identified with the natural projection.

The index of the degree $2n+2$ central simple algebra A is a 2 power dividing $2n+2$. Therefore A is Brauer-equivalent to a central simple algebra A' of degree $n+1+2^k$, where k is the largest integer such that 2^k divides $n+1$. Let π' be the adjoint quadratic pair on A' and let X be the variety of rank 1 isotropic ideals of π'. The variety X is a closed subvariety of the quotient T/P. Over the function field $F(T/P')$ the variety T/P becomes a hyperbolic quadric and the closed subvariety X becomes its subquadric; since $\dim X > \dim(T/P)$, the variety $X_{F(T/P')}$ has a rational point, or, in other words, there exists a rational morphism $T/P' \dashrightarrow X$. Therefore $\mathrm{cd}'(T/P) \leq \dim X = n - 1 + 2^k$.

Let y be a point of T/P. The algebra $A_{F(y)}$ is isomorphic to the algebra of $(2n+2) \times (2n+2)$ matrices over $F(y)$. Let $\psi\colon F(y)^{2n+2} \to F(y)$ be the adjoint quadratic form. Note that the discriminant and the Clifford algebra of ψ are trivial.

The fiber Y of the projection $T/P' \to T/P$ over the point y is a component of the scheme of rank $n+1$ (maximal) isotropic ideals of π, containing a fixed rank 1 isotropic ideal. Therefore Y is identified with a component of the scheme of $(n+1)$-dimensional (maximal) totally isotropic subspaces of ψ, containing a fixed 1-dimensional subspace U. The latter variety is identified with a component of the scheme of n-dimensional (maximal) totally isotropic subspaces of the quotient $U^\perp/U$. Note that $\dim U^\perp/U = 2n$; besides, the quadratic form on $U^\perp/U$, induced by the restriction of ψ, is Witt-equivalent to ψ and, in particular, its discriminant and Clifford algebra are trivial.

Since $\mathrm{cd}(\mathrm{Spin}_{2n})$ is the maximum of the canonical dimension of a component of the scheme of maximal totally isotropic subspaces of a $2n$-dimensional quadratic form with trivial discriminant and Clifford algebra, it follows that $\mathrm{cd}(Y) \leq \mathfrak{cd}(\mathrm{Spin}_{2n})$. Applying Lemma 2.1, we get our main inequality for the semi-spinor group:

Theorem 3.1. *For any odd n, one has* $\mathfrak{cd}(\mathrm{Spin}^{\sim}_{2n+2}) \leq n-1+2^k+\mathfrak{cd}(\mathrm{Spin}_{2n})$. □

Corollary 3.2. *Assume that $\mathfrak{cd}(\mathrm{Spin}_{2n}) = \mathfrak{cd}_2(\mathrm{Spin}_{2n})$ for some odd n. Then*

$$\mathfrak{cd}(\mathrm{Spin}^{\sim}_{2n+2}) = \mathfrak{cd}_2(\mathrm{Spin}^{\sim}_{2n+2})$$

for this n.

Proof. Let l be the smallest integer such that $2^l \geq n+1$. Since n is odd, l is also the smallest integer such that $2^l \geq n$, therefore $\mathfrak{cd}(\mathrm{Spin}_{2n}) = \mathfrak{cd}_2(\mathrm{Spin}_{2n}) = n(n-1)/2 - 2^l + 1$. By Theorem 3.1 we have

$$\mathfrak{cd}(\mathrm{Spin}^{\sim}_{2n+2}) \leq \big(n-1+2^k\big) + \big(n(n-1)/2 - 2^l + 1\big) = \\ n(n+1)/2 + 2^k - 2^l = \mathfrak{cd}_2(\mathrm{Spin}^{\sim}_{2n+2}) \leq \mathfrak{cd}(\mathrm{Spin}^{\sim}_{2n+2}) \, .$$

Consequently, $\mathfrak{cd}(\mathrm{Spin}^{\sim}_{2n+2}) = \mathfrak{cd}_2(\mathrm{Spin}^{\sim}_{2n+2})$. □

Since the assumption of Corollary 3.2 holds for $n \leq 8$ (see Corollary 2.4), we get

Corollary 3.3. *The equality* $\mathfrak{cd}(\mathrm{Spin}_n^\sim) = \mathfrak{cd}_2(\mathrm{Spin}_n^\sim)$ *holds for any* $n \leq 16$. □

References

[1] G. Berhuy, Z. Reichstein. *On the notion of canonical dimension for algebraic groups.* Adv. Math. **198** (2005), no. 1, 128–171.

[2] N. A. Karpenko. *A bound for canonical dimension of the (semi-)spinor groups.* Duke Math. J. **133** (2006), no. 2, 391–404.

[3] N. A. Karpenko, A. S. Merkurjev. *Canonical p-dimension of algebraic groups.* Adv. Math. **205** (2006), no. 2, 410–433.

[4] K. Zainoulline. *Canonical p-dimensions of algebraic groups and degrees of basic polynomial invariants.* London Math. Bull. **39** (2007), 301-304.

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