On topological filtration for Severi-Brauer varieties

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Abstract

The topological filtration on K'_0 of a Severi-Brauer variety is computed if the index of the corresponding central simple algebra coincides with its exponent. It gives, in particular, a description of Ch^2 for such varieties.

Let D be a central simple algebra over a field F with index $\operatorname{ind} D = r$; X the corresponding Severi-Brauer variety [1]; $K(X) = K'_0(X)$ the Grothendieck group of X; $K(X) = K(X)^{(0)} \supset K(X)^{(1)} \supset \ldots \supset K(X)^{(d)} \supset K(X)^{(d+1)} = 0$ (where $d = \dim X$) the topological filtration on $K(X)$ [6]; $G^i K(X) = K(X)^{(i/i+1)}$ the i-codimensional factorgroup of the topological filtration; $G^* K(X) = \coprod_{i=0}^{d} G^i K(X)$ the associated graded group. We denote by $\bar{X}$ the variety X over the algebraic closure $\bar{F}$ of F and identify $G^i K(\bar{X})$ for each $0 \le i \le d$ with $\mathbf{Z}$ using the canonical isomorphism.

The groups $G^i K(X)$ are closely related with the Chow groups and some other K-cohomology groups of Severi-Brauer varieties which have been intensively investigated in the last ten years mainly because of connections with the norm residue homomorphism [2, 3, 4, 5, 7].

The group $G^* K(X)$ is known if $\operatorname{ind} D$ is a prime number [3]. The exponent of such D coincides of course with index. It turns out that having the last condition alone is enough to make computation of $G^* K(X)$ rather easy. The main result of these notes is the next

Theorem 1 *If the index r of algebra D coincides with its exponent then for each i, $0 \le i \le d$, the map $G^i K(X) \to G^i K(\bar{X}) = \mathbf{Z}$ is injective and has image $\frac{r}{(i,r)} \cdot \mathbf{Z}$ where (i, r) is the greatest common divisor.*

Proposition 2 *Let X be any variety such that the map $K(X) \to K(\bar{X})$ is injective and the numerator from the fraction below is finite (a Severi–Brauer variety X satisfies these conditions [6]). Then*

$$|\operatorname{Ker}(G^* K(X) \to G^* K(\bar{X}))| = \frac{|G^* K(\bar{X}) / \operatorname{Im} G^* K(X)|}{|K(\bar{X}) / K(X)|}$$

where $|\,.\,|$ denotes the order of a group.

Proof. Using the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & K(\bar{X})^{(i+1)} & \to & K(\bar{X})^{(i)} & \to & G^iK(\bar{X}) & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow f_i & & \\ 0 & \to & K(X)^{(i+1)} & \to & K(X)^{(i)} & \to & G^iK(X) & \to & 0 \end{array}$$

we obtain a long exact sequence

$$0 \to \operatorname{Ker} f_i \to K(\bar{X})^{(i+1)}/K(X)^{(i+1)} \to K(\bar{X})^{(i)}/K(X)^{(i)} \to \operatorname{Coker} f_i \to 0 \,.$$

Hence

$$\frac{|\operatorname{Coker} f_i|}{|\operatorname{Ker} f_i|} = \frac{|K(\bar{X})^{(i)}/K(X)^{(i)}|}{|K(\bar{X})^{(i+1)}/K(X)^{(i+1)}|} \,.$$

If we multiply all such equalities for i from 0 to d we will get what we need. □

Lemma 3 *Again, let X be the Severi-Brauer variety corresponding to an algebra D of index r (without a restriction on the exponent). Then*

$$\operatorname{Im}(G^iK(X) \to G^iK(\bar{X})) \ni \frac{r}{(i,r)} \quad \textit{for any} \quad 0 \le i \le d.$$

Proof. It suffices to consider only the case when the ground field F has no extensions of degree prime to p for some prime number p. In this case r is a power of p, say $r = p^\alpha$, and $(i,r) = p^\beta$ for some $\beta \le \alpha$. Consider an extension E/F of degree $p^{\alpha-\beta}$ such that $\operatorname{ind}(D_E) = p^\beta$. Since i is divisible by p^β the variety X_E contains a closed i-codimensional subvariety Z such that $\bar{Z}$ is a linear subspace in the projective space $\bar{X}$ [1]. We know that $[\bar{Z}] = 1 \in G^iK(\bar{X})$, so

$$1 \in \operatorname{Im}(G^iK(X_E) \to G^iK(\bar{X}))\,.$$

Using the norm map $N_{E/F}$ we obtain that

$$\frac{r}{(i,r)} = p^{\alpha-\beta} \in \operatorname{Im}(G^iK(X) \to G^iK(\bar{X}))\,.$$

□

Proof of Theorem. We consider the fraction from Proposition. According to [6] the denominator is equal to the product $\prod_{i=0}^{d} \operatorname{ind} D^{\otimes i}$. We have

$$\operatorname{ind} D^{\otimes i} \ge \exp D^{\otimes i} = \frac{\exp D}{(i, \exp D)} = \frac{r}{(i,r)}$$

(the last equality holds since $r = \exp D$). So

$$|K(\bar{X})/K(X)| \ \ge \ \prod_{i=0}^{d} \frac{r}{(i,r)} \,.$$

From the other hand, using Lemma we get an inequality for the numerator:

$$|G^*K(\bar{X})/\operatorname{Im} G^*K(X)| = \prod_{i=0}^{d} |G^iK(\bar{X})/\operatorname{Im} G^iK(X)| \le \prod_{i=0}^{d} \frac{r}{(i,r)} \,.$$

Concequently, the denominator and numerator coincide, $|G^iK(\bar{X})/\mathrm{Im}\, G^iK(X)| = \frac{r}{(i,r)}$ for each i and $\mathrm{Ker}\,(G^*K(X) \to G^*K(\bar{X})) = 0$. □

Since the group $G^2K(X)$ is canonically isomorphic to the second Chow group $\mathrm{Ch}^2(X)$ we get

Corollary 4 *If the index r of the algebra D coincides with its exponent then*

$$\mathrm{Ch}^2(X) = \frac{r}{(2,r)} \cdot \mathbf{Z}$$

and

$$\mathrm{Im}\,(\mathrm{Ch}^i(X) \to \mathrm{Ch}^i(\bar{X})) = \frac{r}{(i,r)} \cdot \mathbf{Z}$$

for each i. □

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