On topological filtration for Severi-Brauer varieties II

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Abstract

The topological filtration on K'_0 of a Severi-Brauer variety is computed if the quotient of its index and exponent is a squarefree number and for each prime p dividing this quotient the p-primary component of the corresponding division algebra is decomposable. This gives in particular a description of Ch^2 for such varieties.

Let D be a central simple algebra over a field F and $X = SB(D)$ the Severi-Brauer variety of D [1]. In [3] the topological filtration on the Grothendieck group $K(X)$ has been computed provided that $\operatorname{ind} D = \exp D$. The topic of this note is the case when the quotient $\operatorname{ind} D/\exp D$ is any squarefree number but with one more additional restriction on D: for each prime $p \mid \operatorname{ind} D/\exp D$ the p-primary component of the corresponding division algebra should be decomposable (i.e. isomorphic to a tensor product $D_1 \otimes_F D_2$ with $D_j \neq F$ for both j).

In addition to notations introduced above we fix the following: notations relative to the Grothendieck group as introduced in [3], in particular $G^iK(X)$ is the factorgroup of the topological filtration of codimension i; $\overline{G^iK}(X)$ is the image of the homomorphism $G^iK(X) \to G^iK(\bar{X}) = \mathbf{Z}$ where $\bar{X}$ is the variety X over the algebraic closure of F.

For a prime p, v_p is the p-adic valuation on $\mathbf{Q}$; C_n^k is the binomial coefficient; $(\, , \,)$ is the greatest common divisor.

I owe to A.S. Merkurjev the idea that the cycle $SB(D_1) \times SB(D_2)$ on the variety $SB(D_1 \otimes D_2)$ might be an interesting one.

Theorem 1 *Let D be a central simple algebra with* $\operatorname{ind} D = r$, $\exp D = e$ *and let* $X = SB(D)$.

If r/e is a squarefree number and for each prime $p \mid r/e$ the p-primary component of the similar to D division algebra is decomposable then the map $G^iK(X) \to G^iK(\bar{X}) \quad (0 \le i \le \dim X)$

1. *is injective;*

2. has the image $\frac{r}{(i,r)\prod p} \cdot \mathbf{Z}$ *where the product* $\prod p$ *is taken over all prime* $p \mid r/e$ *such that* $0 < v_p(i+p-1) < v_p(r)$.

Example. Let D be a division algebra of index p^2 and exponent p, $X = SB(D)$. If D is decomposable then for all $1 \le i \le p^2 - 1$

$$G^i K(X) = \begin{cases} p\mathbf{Z}, & \text{if } i \vdots p \text{ or } i-1 \vdots p \text{ without } i = p^2 - p + 1; \\ p^2\mathbf{Z} & \text{otherwise.} \end{cases}$$

Proof of Theorem. It suffices to consider only the case when $r = p^n$ for a prime p. Then e equals to p^n or p^{n-1}. The first case was done in [3]. We suppose that $e = p^{n-1}$ below.

The proof consists of several lemmas.

Consider at first the case when D is a division algebra.

Lemma 2 *If* $D = D_1 \otimes_F D_2$ *is a nontrivial decomposition of a division algebra* D *with* $\operatorname{ind} D = p^n$ *and* $\exp D = p^{n-1}$ *then the index and exponent of one of the factors equal* p *while the index and exponent of the other one equal* p^{n-1}.

Proof. Put $\operatorname{ind} D_j = p^{k_j}$, $\exp D_j = p^{l_j}$. Then $k_1 + k_2 = n$ and $n-1 \ge \max\{k_1, k_2\} \ge \max\{l_1, l_2\} \ge n-1$. So, all inequalities are in fact equalities which implies the statement. □

We fix a decomposition $D = D_1 \otimes_F D_2$ with $\operatorname{ind} D_1 = \exp D_1 = p$ and $\operatorname{ind} D_2 = \exp D_2 = p^{n-1}$ for below.

Lemma 3 *For* $X = SB(D)$ *where* $D = D_1 \otimes_F D_2$ *is a division algebra with* $\operatorname{ind} D_1 = \exp D_1 = p$ *and* $\operatorname{ind} D_2 = \exp D_2 = p^{n-1}$*, it holds*

$$\log_p |K(\bar{X})/K(X)| = n \cdot p^n - \alpha$$

where $\alpha = v_p(p^n!) + p^{n-1} - 1$.

Proof. It is known from [5] that

$$|K(\bar{X})/K(X)| = \prod_{i=1}^{p^n} \operatorname{ind} D^{\otimes i} .$$

Put $\log_p \operatorname{ind} D^{\otimes i} = n - \alpha_i$. If $v_p(i) = 0$ then $\alpha_i = 0$.

If $v_p(i) > 0$ then $\operatorname{ind} D^{\otimes i} = \operatorname{ind} D_2^{\otimes i} = \operatorname{ind} D_2/(i, \operatorname{ind} D_2)$ (see [3] for the last equality), so $\alpha_i = \min\{v_p(i) + 1, n\}$. Consequently

$$\alpha = \sum_{i=1}^{p^n} \alpha_i = v_p(p^n!) + p^{n-1} - 1.$$

□

Before to deal with Lemma 5 let's formulate a fact from [4] which will be needed below.

Proposition 4 ([4]) *For $N \geq 1$ denote by X^N the variety $SB(M_N(D))$ where $M_N(D)$ is the F-algebra of $N \times N$-matrices over an arbitrary central simple algebra D of degree d. Then*

$$\mathrm{Ch}_*(X^N) = \mathrm{Ch}_*(X) \oplus \mathrm{Ch}_{*-d}(X) \oplus \ldots \oplus \mathrm{Ch}_{*-(N-1)d}(X)$$

where Ch_* *denotes the Chow group graded by dimensions of cycles (so to say,* $\mathrm{Ch}(X^N) = (\mathrm{Ch}(X))^N)$ *.* □

Now the main spot in proving of Theorem is coming.

Lemma 5 *As agreed before Proposition, D is a decomposed division algebra of index p^n and exponent p^{n-1}, $X = SB(D)$. If $i-1 \vdots p$ where $0 \leq i \leq p^n-1$ and $i \neq p^n-p+1$ then $\overline{G^iK}(X) \ni p^{n-1}$.*

Proof. Replace the factorgroups of topological filtration by the Chow groups which we'll numerate by dimension. What we need to show is that $\overline{\mathrm{Ch}}_i(X) \ni p^{n-1}$ where $\overline{\mathrm{Ch}}_i(X) = \mathrm{Im}\,(\mathrm{Ch}_i(X) \to \mathrm{Ch}_i(\bar{X}) = \mathbf{Z})$ if $i = kp-2$ and $2 \leq k \leq p^{n-1}$.

Fix a large number N (it's enough to take any $N \geq p$) and consider a closed embedding $X_1 \times X_2^N \hookrightarrow X^N$ where $X_j = SB(D_j)$ induced by tensor product of ideals [1]. According to Proposition it would suffice to show that $p^{n-1} \in \overline{\mathrm{Ch}}_i(X^N)$ if $i = kp-2$ (and $2 \leq k \leq p^{n-1}$ as above).

Consider a commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Ch}_{p-1}(\bar{X}_1) \otimes \mathrm{Ch}_{(k-1)p-1}(\bar{X}_2^N) & \to & \mathrm{Ch}_i(\bar{X}_1 \times \bar{X}_2^N) & \to & \mathrm{Ch}_i(\bar{X}^N) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Ch}_{p-1}(X_1) \otimes \mathrm{Ch}_{(k-1)p-1}(X_2^N) & \to & \mathrm{Ch}_i(X_1 \times X_2^N) & \to & \mathrm{Ch}_i(X^N).
\end{array}$$

The morphism $\bar{X}_1 \times \bar{X}_2^N \hookrightarrow \bar{X}^N$ is a Segre embedding [1]. In the light of this we need now a couple of standard statements on the Segre embeddings.

Sublemma 6 ([2]) *The image of the Segre embedding $\mathbf{P}^{a_1} \times \mathbf{P}^{a_2} \hookrightarrow \mathbf{P}^a$ where $a = a_1a_2 + a_1 + a_2$ has degree $C^{a_1}_{a_1+a_2}$. If $\mathbf{P}^{b_j} \subset \mathbf{P}^{a_j}$ $(j = 1, 2)$ are linear subvarieties then the image of $\mathbf{P}^{b_1} \times \mathbf{P}^{b_2}$ is contained in some linear $\mathbf{P}^b \subset \mathbf{P}^a$ with $b = b_1b_2+b_1+b_2$ and the induced morphism $\mathbf{P}^{b_1} \times \mathbf{P}^{b_2} \hookrightarrow \mathbf{P}^b$ is a Segre embedding as well.* □

As a corollary we see looking at the diagram that image of 1 from the upper left corner in the upper right corner equals C_i^{p-1}.

Further, $\overline{\mathrm{Ch}}_{p-1}(X_1) \ni 1$ in a trivial way and $\overline{\mathrm{Ch}}_{(k-1)p-1}(X_2^N) \ni p^{n-m-2}$ with $m = v_p(k-1)$ according to [3] since $X_2^N = SB(M_N(D_2))$ and $\mathrm{ind}\,(M_N(D_2)) = p^{n-1}$.

Thus, $\overline{\mathrm{Ch}}_i(X^N) \ni C_i^{p-1} \cdot p^{n-m-2}$ and the last observation we need is $v_p(C_i^{p-1}) = m+1$. □

End of Theorem's proof. Put $\log_p |G^*K(\bar{X})/\overline{G^*K}(X)| = n \cdot p^n - \beta$. We would like to show that $\beta \geq \alpha$ for α from Lemma 2 (compare with [3]).

Put for each i $(0 \leq i \leq p^n - 1)$

$$\beta_i = \begin{cases} n, & \text{if } i = 0; \\ v_p(i), & \text{if } i \vdots p \text{ and } i \neq 0; \\ 1, & \text{if } i-1 \vdots p \text{ and } i \neq p^n - p + 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\log_p |G^i K(\bar{X})/\overline{G^i K}(X)| \leq n - \beta_i$. Really, $\frac{p^n}{(i,p^n)} \in \overline{G^i K}(X)$ for each i [3] and $p^{n-1} \in \overline{G^i K}(X)$ if $i-1 \vdots p$ and $i \neq p^n - p + 1$ by Lemma 5 what implies the inequality required.

Adding all the inequalities together we obtain that $\beta \geq \sum_{i=0}^{p^n-1} \beta_i$. It is easy to see that the last sum equals α. Consequently, by [3, proposition], $G^* K(X) \to G^* K(\bar{X})$ is an injection and $p^{n-\beta_i}$ generates $\overline{G^i K}(X)$ for each i.

To finish the proof we should consider the case when D is arbitrary, not necessary with division. Write $D \simeq M_s(D')$ with a skewfield D' and put $X' = SB(D')$. Then

$$|G^* K(\bar{X}')/\overline{G^* K}(X')| = |K(\bar{X}')/K(X')|$$

as shown above. Since $|G^* K(\bar{X})/\overline{G^* K}(X)| = s|G^* K(\bar{X}')/\overline{G^* K}(X')|$ by Proposition and $|K(\bar{X})/K(X)| = s|K(\bar{X}')/K(X')|$ [5] the same equality holds for X. It implies injectivity of $G^* K(X) \to G^* K(\bar{X})$ and the second assertion of Theorem in the same way as above. □

Corollary 7 *If X is a Severi-Brauer variety satisfying conditions of Theorem then* $\mathrm{Ch}^2(X) = \frac{r}{(2,r)} \cdot \mathbf{Z}$. □

References

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