ON TOPOLOGICAL FILTRATION FOR TRIQUATERNION ALGEBRA

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Abstract. The topological filtration on the Grothendieck group of the Severi-Brauer variety corresponding to a triquaternion algebra is computed. In particular, it is shown that the second Chow group of the variety is torsionfree.

Let F be a field, A a central simple F-algebra of degree 8 and exponent 2, $X = \mathrm{SB}(A)$ the Severi-Brauer variety of A [2]. Consider the second Chow group $\mathrm{CH}^2(X)$, i.e. the group of 2-codimensional algebraic cycles on X modulo rational equivalence [3, 10]. In [6] it is shown that the group $\mathrm{CH}^2(X)$ can contain a non-trivial torsion.

In this note we study the case when A decomposes (in a tensor product of two smaller algebras) or (what is equivalent [1]) when A is a product of three quaternion algebras $Q_1 \otimes_F Q_2 \otimes_F Q_3$. We compute (almost completely) the topological filtration on the Grothendieck group $K(X) = K'_0(X)$ [3, 10] and as a consequence show that the second Chow group $\mathrm{CH}^2(X)$ is torsionfree.

It deserves to be mentioned that an analogues situation occurs in the case of odd prime exponent too. The group $\mathrm{CH}^2(X')$ where $X' = \mathrm{SB}(A')$ for an algebra A' of an odd prime exponent p and degree p^2 can have a non-trivial torsion [7]. But it is known to be torsionfree in the case when A' decomposes [5].

1. Segre embeddings

Consider the 3-fold Segre embedding

$$\mathbb{P}^1_F \times \mathbb{P}^1_F \times \mathbb{P}^1_F \hookrightarrow \mathbb{P}^3_F \ .$$

In this section we compute the class of the cycle $\mathbb{P}^1_F \times \mathbb{P}^1_F \times \mathbb{P}^1_F$ in $K(\mathbb{P}^3)$.

First we fix a n-dimensional projective space $\mathbb{P}^n$ and consider Hilbert polynomials of its closed subvarieties.

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Lemma 1.1 ([3]). *Hilbert polynomial of an l-dimensional linear subspace equals*

$$\binom{t+l}{l} = \frac{1}{l!}(t+1)\dots(t+l) \, .$$

Lemma 1.2. *Hilbert polynomial defines a group monomorphism*

$$K(\mathbb{P}^n) \hookrightarrow \mathbb{Q}[t] \, .$$

Proof. Consider the graded ring $S = F[x_0, \dots, x_n]$ which is the homogeneous coordinate ring of $\mathbb{P}^n$. If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence of graded S-modules then

$$\dim_F M_i = \dim_F M'_i + \dim_F M''_i$$

for any i. Hence the Hilbert polynomial for M is equal to the sum of the Hilbert polynomials for M' and M''. Thus the homomorphism $K(\mathbb{P}^n) \hookrightarrow \mathbb{Q}[t]$ is well-defined.

The abelean group $K(\mathbb{P}^n)$ is generated by classes of linear subspaces. There is no relations between their images — polynomials $\binom{t+l}{l}$ with $0 \le l \le n$ (by the degree reason) whence injectivity of the homomorphism. □

Lemma 1.3. *Let*

$$\mathbb{P}^{l_1} \times \dots \times \mathbb{P}^{l_k} \hookrightarrow \mathbb{P}^n$$

be a k-fold Segre embedding (where $n = \prod_{i=1}^k (l_i + 1) - 1$). Hilbert polynomial of the subvariety $\mathbb{P}^{l_1} \times \dots \times \mathbb{P}^{l_k}$ is equal to

$$\prod_{i=1}^k \binom{t+l_i}{l_i} \, ,$$

i.e. to the product of Hilbert polynomials of the factors.

Proof. Taking homogeneous coordinate rings $S^1, \dots, S^k$ of $\mathbb{P}^{l_1}, \dots, \mathbb{P}^{l_k}$ one obtaines the homogeneous coordinate ring S of the subvariety $\mathbb{P}^{l_1} \times \dots \times \mathbb{P}^{l_k} \hookrightarrow \mathbb{P}^n$ as the Cartesian product of $S^1, \dots, S^k$ [3]:

$$S_d = \bigotimes_{i=1}^k S^i_d \, .$$

In particular $\dim_F S_d = \prod_{i=1}^k \dim_F S^i_d$ whence the relation for the Hilbert polynomials. □

Corollary 1.4. *Consider the 3-fold Segre embedding*

$$\mathbb{P}^1_F \times \mathbb{P}^1_F \times \mathbb{P}^1_F \hookrightarrow \mathbb{P}^3_F$$

and denote by h^i the class in $K(\mathbb{P}^7)$ of an i-codimensional linear subspace. The following equality holds in $K(\mathbb{P}^7)$:

$$[\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1] = 6h^4 - 6h^5 + h^6 \ .$$

Proof. According to (1.2) it suffices to check the corresponding equality for the Hilbert polynomials. According to (1.1) and (1.3) the corresponding equality is:

$$(t+1)^3 = 6\binom{t+3}{3} - 6\binom{t+2}{2} + \binom{t+1}{1} \ .$$

□

2. Topological filtration

We use the notations introduced in the introduction. The triquaternion algebra A is now supposed to be a skewfield. We denote by E/F a maximal subfield of A (so, $[E : F] = 8$) and identify $K(X)$ with a subgroup of the group $K(X_E)$ which (the latter) is abelean, freely generated by $1, h, \dots, h^7$.

Definition 2.1. Let Y_1, Y_2, Y_3 are conics corresponding to the quaternion algebras Q_1, Q_2, Q_3. We have a closed imbedding $Y_1 \times Y_2 \times Y_3 \hookrightarrow X$ (induced by tensor product of ideals [2]) which is a twisted form of the 3-fold Segre embedding $\mathbb{P}^1_F \times \mathbb{P}^1_F \times \mathbb{P}^1_F \hookrightarrow \mathbb{P}^3_F$.

We define a cycle ζ on X as

$$\zeta = [Y_1 \times Y_2 \times Y_3] - 3[2\mathbb{P}^3] + [8\mathbb{P}^2]$$

where $8\mathbb{P}^2$ is the norm from the extension E/F of a linear subspace $\mathbb{P}^2_E \subset X_E$ and $2\mathbb{P}^3$ is a norm from a quadratic extension L/F for which A_L is no more a division algebra of a linear subspace $\mathbb{P}^3_L \subset X_L$ (more precisely, of a twisted form of $\mathbb{P}^3_L$) [2].

We will consider ζ as an element of $K(X)$.

Lemma 2.2. *It holds: $\zeta = 2h^5 + h^6$ and $\zeta \in K(X)^{(4)}$.*

Proof. By (1.4) we have

$$[Y_1 \times Y_2 \times Y_3] = 6h^4 - 6h^5 + h^6 \ .$$

Since $[2\mathbb{P}^3] = 2h^4$ and $[8\mathbb{P}^2] = 8h^5$ we get $\zeta = 2h^5 + h^6$.

Two first cycles in definition of ζ are 3-dimensional (i.e. 4-codimensional), the last one even 2-dimensional. Hence the whole sum — the cycle ζ itself lies at least in codimension 4. □

Now consider the topological filtration on $K(X)$ [10] and denote by $G^*K(X)$ the adjoint graded group. For each i with $0 \leq i \leq \dim X$ we identify $G^iK(X_E)$ with $\mathbb{Z}$ using the generator $\bar{h}^i$. If for some i the group $G^iK(X)$ is torsionfree then the map

$$G^iK(X) \to G^iK(X_E)$$

is injective and we may identify $G^iK(X)$ with a subgroup of $G^iK(X_E)$.

Lemma 2.3. *The groups*

$$\operatorname{Im}(G^iK(X) \to G^iK(X_E))$$

contain following elements:

- *2 for* $i = 1, 4$*;*
- *4 for* $i = 2, 3, 5, 6$*;*
- *8 for* $i = 7$*.*

Proof. According to [4],

$$\operatorname{Im}(G^iK(X) \to G^iK(X_E)) \ni \frac{r}{(i, r)}$$

where $r = \operatorname{ind} A$ ($r = 8$ in our case) and (i, r) stays for the greatset common divisor. So, we get our statement for $i = 2, 4, 6, 7$. A general statement for $i = 1$ is [2]:

$$G^1K(X) \ni \exp A$$

and $\exp A = 2$ in our case.

To manage the case $i = 5$ take such a quadratic exteension L/F as in (2.1). The group $G^5K(X_L)$ coincides with the group $G^1K(X_L)$ [9] which contains $2 = \exp A_L$. Applying the transfer argument we see that $4 \in \operatorname{Im}(G^5K(X) \to G^5K(X_E))$.

Finally, consider the codimension $i = 3$. Denote by Y the Severi-Brauer variety of the product $Q_2 \otimes Q_3$ and consider the cycle $Y_1 \times Y$ on X. The embedding $Y_1 \times Y \hookrightarrow X$ is a twisted form of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^7$. The cycle $\mathbb{P}^1 \times \mathbb{P}^3$ has degree $\binom{1+3}{1} = 4$ [3]. Whence $4 \in \operatorname{Im}(G^3K(X) \to G^3K(X_E))$. □

We will apply the following

Proposition 2.4 ([4])**.** *Let E/F be a field extension and X any variety over F such that the map $K(X) \to K(X_E)$ is injective and the numerator of the fraction below is finite (a Severi-Brauer variety X satisfies these conditions for any extension of the base field* [10]*). Then*

$$|\operatorname{Ker}(G^*K(X) \to G^*K(X_E))| = \frac{|G^*K(X_E)/\operatorname{Im} G^*K(X)|}{|K(X_E)/K(X)|}$$

where $|.|$ denotes the order of groups.

Theorem 2.5. *Let A be a triquaternion division algebra, $X = \mathrm{SB}(A)$ and ζ the defined in (2.1) cycle on X. The groups*

$$\mathrm{Im}(G^i K(X) \to G^i K(X_E))$$

are generated by

- *2 for $i = 1, 4$;*
- *4 for $i = 2, 3, 6$;*
- *8 for $i = 7$.*

If $\zeta \in K(X)^{(5)}$ then the graded group $G^* K(X)$ is trosionfree and the component $G^5 K(X)$ is generated by 2.

If $\zeta \notin K(X)^{(5)}$ then the torsion in $G^* K(X)$ has order 2 and is generated by $\bar{\zeta} \in G^4 K(X)$; in this case $G^5 K(X)$ is generated by 4.

Proof. The index $[K(X_E) : K(X)]$ equals $\prod_{i=0}^{\deg A - 1} A^{\otimes i}$ [10] what is 2^{12}. Suppose that $\zeta \in K(X)^{(5)}$. Then $2 \in \mathrm{Im}(G^5 K(X) \to G^5 K(X_E))$ by (2.2). From this fact and (2.3) it follows that

$$[G^* K(X_E) : \mathrm{Im}\, G^* K(X)] \le 2^{12} \, .$$

Applying the formula (2.4) we obtain that

$$\mathrm{Ker}(G^* K(X) \to G^* K(X_E)) = 0$$

(i.e. $G^* K(X)$ is torsionfree) and

$$[G^* K(X_E) / G^* K(X)] = 2^{12} \, .$$

Thus the given elements in $\mathrm{Im}(G^i K(X) \to G^i K(X_E))$ are generators.

Now assume that $\zeta \notin K(X)^{(5)}$. Then $\bar{\zeta} \in K(X)^{(4/5)}$ is a non-trivial torsion (at least an inclusion

$$8\zeta = 16h^5 + 8h^6 \in K(X)^{(5)}$$

is clear at once). From the other hand the formula (2.4) tells that

$$|\, \mathrm{Ker}(G^* K(X) \to G^* K(X_E))| \le 2 \, .$$

Thus $\bar{\zeta} \in K(X)^{(4/5)}$ has the order 2 and generates the torsion subgroup of the whole $G^* K(X)$. □

3. Chow groups

Since $G^2 K(X)$ coincides with $\mathrm{CH}^2(X)$ for any Severi-Brauer variety X we obtain

Corollary 3.1. *The second Chow group $\mathrm{CH}^2(X)$ of a Severi-Brauer variety X corresponding to a triquaternion algebra is equal to $4\mathbb{Z}$ (in particular, is torsionfree).*

Proof. In the case when the triquaternion algebra is a division one the statement follows from (2.5). In the other case the statement is trivial. ☐

In the conclusion, we construct a triquaternion algebra for which the torsion in G^*K really appears.

Example 3.2. *Take a field F, a quaternion algebra Q and an algebra A of degree and exponent 4 such that $A \otimes Q$ is a skewfield. Put $X = \mathrm{SB}(A \otimes Q)$ and $Y = \mathrm{SB}(A^{\otimes 2})$. Then $(A \otimes Q)_{F(Y)}$ (where $F(Y)$ denotes the function field) is a triquaternion algebra for which*

$$\zeta \notin K(X_{F(Y)})^{(5)} .$$

Proof. Since $A_{F(Y)}$ has degree 4 and exponent 2 it is biquaternion [1] and so, the algebra $(A \otimes Q)_{F(Y)}$ is triquaternion. It is also easy to see that the latter algebra is not split (in fact, by the index reduction formula [11], $(A \otimes Q)_{F(Y)}$ is a skewfield).

Consider the pull-back

$$f^* : \mathrm{CH}^*(X \times Y) \longrightarrow \mathrm{CH}^*(X_{F(Y)})$$

with respect to the morphism of varieties $f : X_{F(Y)} \to X \times Y$ obtained from $\operatorname{Spec} F(Y) \to Y$ by the base change. It is easy to show that f^* is an epimorphism (see e.g. [8]). Since $X \times Y$ is a projective space bundle over X (via the first projection $pr_X : X \times Y \to X$) the ring $\mathrm{CH}^*(X \times Y)$ is generated by $\mathrm{CH}^1(X \times Y)$ and $pr_X^* \mathrm{CH}^*(X)$ [3]. Thus the ring $\mathrm{CH}^*(X_{F(Y)})$ is generated by $\mathrm{CH}^1(X_{F(Y)})$ and $\mathrm{res}_{F(Y)/F} \mathrm{CH}^*(X)$.

Passing to G^*K by using the canonical epimorphism

$$\mathrm{CH}^* \twoheadrightarrow G^*K$$

we see that the ring $G^*K(X_{F(Y)})$ is generated by $G^1K(X_{F(Y)})$ and $\mathrm{res}_{F(Y)/F} G^*K(X)$.

Now take a maximal subfield E/F of the skewfield $A \otimes Q$. Then $E(Y)/F(Y)$ is a maximal subfield of $(A \otimes Q)_{F(Y)}$ and we see that the graded ring

$$\mathrm{res}_{E(Y)/F(Y)} G^*K(X_{F(Y)})$$

is generated by its first graded component and $\mathrm{res}_{E(Y)/F} G^*K(X)$. Since the algebra $(A \otimes Q)_{F(Y)}$ has exponent 2 we obtain ([2], see also (2.5)) that the group $\mathrm{res}_{E(Y)/F(Y)} G^1K(X_{F(Y)})$ is generated by 2. The groups $\mathrm{res}_{E/F} G^iK(X)$ are computed in [5]. The generators are:

- 2 for $i = 4$;
- 4 for $i = 1, 2, 3, 5, 6$;
- 8 for $i = 7$.

It follows that the group $\mathrm{res}_{E(Y)/F(Y)}\, G^5 K(X_{F(Y)})$ is generated by 4. Hence $\zeta \notin K(X_{F(Y)})^{(5)}$ by (2.5). □

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