CHARACTERIZATION OF MINIMAL PFISTER NEIGHBORS VIA ROST PROJECTORS

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Abstract. Let ϕ be an anisotropic 9-dimensional quadratic form over a field (of characteristic $\neq 2$). We show that the projective quadric given by ϕ possesses a Rost projector if and only if ϕ is a Pfister neighbor. The following consequence of this result gives the initial step in the construction [10] of the filed with the u-invariant 9: if ϕ is not a Pfister neighbor and the Schur index of its even Clifford algebra is at least 4, then ϕ stays anisotropic over the function field of any 9-dimensional form non-similar to ϕ.

Contents

1. Main results and their proofs

The main results of this article are Theorem 1.7 and its consequence — Theorem 1.13, the latter being used in the recent proof [10] of the existence of a field with the u-invariant 9 (see section 2).

We work with non-degenerate quadratic forms over fields of characteristic not 2 and use the standard terminology concerned (cf. [22] and [29]). In particular, a *Pfister neighbor* is a quadratic form of dimension $\geq 2^r+1$ similar to a subform of an $(r+1)$-fold Pfister form for some $r \geq 1$.

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Definition 1.1. A *minimal Pfister neighbor* is a Pfister neighbor of dimension $2^r + 1$ with some $r \geq 1$.

Let X be a smooth n-dimensional (with $n \geq 1$) projective quadric over a field F (where $\operatorname{char} F \neq 2$). We write $\bar{X}$ for X over an algebraic closure $\bar{F}$ of F and denote by l_0 the class of a rational point in the Chow group of 0-dimensional cycles $\mathrm{CH}^n(\bar{X})$; $1 \in \mathrm{CH}^0(X)$ stays for the class of X.

Definition 1.2. A cycle $\rho \in \mathrm{CH}^n(X \times X)$ is called a *Rost correspondence on* X, if over the algebraic closure $\bar{F}$ of F it is equal to the cycle

$$l_0 \times 1 + 1 \times l_0 \in \mathrm{CH}^n(\bar{X} \times \bar{X}) .$$

Definition 1.3. A Rost correspondence which is a projector (i.e., an idempotent in the ring of correspondences on X) is called a *Rost projector*. The motive (X, ρ) determined by a Rost projector ρ in the category of the Grothendieck Chow-motives is called a *Rost motive*.

Remark 1.4. Note that a Rost correspondence becomes a projector over $\bar{F}$ (because $l_0 \times 1$ and $1 \times l_0$ are orthogonal projectors). Therefore, some power of a Rost correspondence will be a projector as well (Corollary 4.2). In particular, a Rost projector does exist on a given quadric if and only if a Rost correspondence does.

Remark 1.5. M. Rost himself introduced and studied Rost projectors only on the norm quadrics (see [26], [27]). In contrast to this, the quadric X is arbitrary in our context. In particular, the case, where X is given by a minimal Pfister neighbor (which is conjecturally the only possible case — see Conjecture 1.6) will be studied in details here; it will be shown that any such X *possesses* a *unique* Rost projector (see section 6).

Let ϕ be a quadratic form over a field. We write Q_ϕ for the projective quadric $\phi = 0$.

Conjecture 1.6. *If an anisotropic quadric $X = Q_\phi$ possesses a Rost correspondence, then the quadratic form ϕ is a minimal Pfister neighbor.*

Our main result is a proof of the above conjecture in the case $\dim \phi = 9$ (see also Propositions 10.8 and 10.10):

Theorem 1.7. *Conjecture 1.6 is true for 9-dimensional quadratic forms.*

Conjecture 1.6 is a direct consequence of three other conjectures:

Conjecture 1.8. *If an anisotropic quadric $X = Q_\phi$ possesses a Rost correspondence, then* $\dim X = 2^r - 1$ *for some* $r \geq 1$ *(i.e.,* $\dim \phi = 2^r + 1$*).*

Conjecture 1.9. *If an anisotropic $(2^r - 1)$-dimensional quadric X possesses a Rost correspondence, then the homomorphism $H^{r+1}(F) \to H^{r+1}(F(X))$ of the Galois cohomology groups with $\mathbb{Z}/2$-coefficients is not injective.*

Conjecture 1.10 (cf. [14, introduction: discussion of cor. 2 (2)])**.** *If the homomorphism $H^{r+1}(F) \to H^{r+1}(F(X))$ is not injective for some $(2^r - 1)$-dimensional quadric X, then X is determined by a Pfister neighbor.*

Remark 1.11. Voevodsky's announcement of the proof of the Milnor conjecture [35] contains a claim on the existence of certain cohomological operations in the motivic cohomology. Using some expected properties of these operation one may "prove" (at least in characteristic 0) Conjecture 1.8 (cf. [34, proof of statement 6.1]) as well as Conjecture 1.9 (cf. [9]).

Conjecture 1.10 seems to be most difficult. The cases with $r \leq 3$ are settled however. The first non-trivial case — $r = 2$ — is due to J. Arason ([1]). We need this conjecture in the case $r = 3$ settled in [14, thm. 1].

Proof of Theorem 1.7. Let X be an anisotropic projective quadric of dimension 7 possessing a Rost correspondence. According to the above discussion we only need to verify that Conjecture 1.9 holds for X.

As noticed above, a power of the Rost correspondence is a Rost projector. Therefore X possesses a Rost projector ρ. Consider the cycle

$$x := \rho_*(h^4) \in \mathrm{CH}^4(X) ,$$

where ρ_* is the endomorphism of $\mathrm{CH}^4(X)$ given by ρ while $h \in \mathrm{CH}^1(X)$ is the class of a hyperplane section of X. According to Corollary 9.6 and Proposition 7.1, x is a non-zero element of $\mathrm{CH}^4(X)$ vanishing in $K_0(X)^{(4)}/K_0(X)^{(5)}$ under the canonical epimorphism $\mathrm{CH}^4(X) \twoheadrightarrow K_0(X)^{(4)}/K_0(X)^{(5)}$ being the border effect of the BGQ spectral sequence [25, §7.5] of X. Therefore the E_2-term of this spectral sequence has a non-zero differential ending in $\mathrm{CH}^4(X) = E_2^{4,-4}$ or (more generally) the E_l-term for some $l \geq 2$ has a non-zero differential d ending in $E_l^{4,-4}$.

Let us look at where the differential d can start in. Since

$$E_2^{0,-1} = H^0(X, K_1) = F^* = E_\infty^{0,-1}$$

and $E_2^{1,-2} = H^1(X, K_2) = F^* \cdot h = E_\infty^{1,-2}$ ([15, thm. 4.1]), there are no non-trivial differentials starting in $E_l^{0,-1}$ or in $E_l^{1,-2}$. Thus the only possible point for d to start in is the K-cohomology group $H^2(X, K_3) = E_2^{2,-3}$.

Let L/F be a quadratic extension such that X_L has a rational point. Since $H^2(X_L, K_3) = L^* \cdot h^2$, the group $H^2(X, K_3)$ decomposes as

$$F^* \oplus \mathrm{Ker}(H^2(X, K_3) \to H^2(X_L, K_3)) .$$

Since $d(F^*) = 0$ while $d \neq 0$, we conclude that the second summand of the decomposition is non-zero.

As the last step of the proof, we use the isomorphism of [24, prop. 1]:

$$\mathrm{Ker}(H^2(X, K_3) \to H^2(X_L, K_3)) \simeq \mathrm{Ker}(H^4(F) \to H^4(F(X)))$$

(this isomorphism is established in [24] for any projective quadric X provided that $\dim X \geq 3$ and the quadratic form giving X is not similar to a 3-fold

Pfister neighbor (the latter condition is in fact superfluous, cf. [14, proof of thm. 5.1]); for our purposes only the injectivity is needed). □

We would like to mention an evident consequence:

Corollary 1.12. *If a 9-dimensional anisotropic form ϕ is not a Pfister neighbor, then for any odd field extension E/F the form ϕ_E is not a Pfister neighbor as well.*

Proof. If ϕ_E is a Pfister neighbor, the quadric X_E possesses a Rost correspondence ρ. Then $N_{E/F}(\rho)$ is a correspondence on X which is equal to $[E : F] \cdot (1 \times l_0 + l_0 \times 1)$ over $\bar{F}$. Since the correspondence $1 \times h^7 + h^7 \times 1$ on X is equal to $2 \cdot (l_0 \times 1 + 1 \times l_0)$ over $\bar{F}$, it follows that X possesses a Rost correspondence. Therefore, by Theorem 1.7, ϕ is a Pfister neighbor. □

A more interesting consequence is the following

Theorem 1.13. *Let ϕ be an anisotropic 9-dimensional quadratic form which is not a Pfister neighbor and whose even Clifford algebra $C_0(\phi)$ is of index ≥ 4. Let ψ be any quadratic form of dimension 9. If the form ϕ becomes isotropic over the function field of ψ (i.e., over the function field of Q_ψ), then ψ is similar to ϕ.*

Ideas of the proof. This is only a pretaste of a proof. The real proof (which is quite elementary and even does not use the notion of motive) will be given in section 11.

Since ϕ becomes isotropic over $F(\psi)$, ψ is also isotropic over $F(\phi)$ ([8]). Therefore, by [34, §3.3], the motive of Q_ϕ has a direct summand isomorphic to a direct summand of the motive of Q_ψ.

It is not hard to show, that the only way to decompose for the motive of Q_ϕ is to contain a Rost motive as a direct summand. Thus, by Theorem 1.7, the motive of Q_ϕ is indecomposable (cf. Corollary 10.14). Consequently, the whole motive of Q_ϕ is isomorphic to a direct summand in the motive of Q_ψ.

Now we conclude by the dimension reason that the motives of Q_ϕ and of Q_ψ are isomorphic. By [7, cor. 2.9], it can happen only if the forms ϕ and ψ are similar. □

2. A comment on the u-invariant 9

In this section we briefly (and not trying to be precise) explain how Theorem 1.13 leads to a field with u-invariant 9. For details see [10].

We start with an essential form (Definition 10.12) ϕ over a field F. We are looking for an extension E/F such that all 10-dimensional forms over E are isotropic while the form ϕ_E is still anisotropic.

Let ψ be a 10-dimensional quadratic form over F or a 9-dimensional form non-similar to ϕ. We like to know that the form ϕ_L, where $L = F(\psi)$, is still essential. Since we have Theorem 1.13 (see also Remark 11.4) and the index reduction formula [23], the only thing to check is that ϕ does not become a Pfister neighbor over L. Let us assume the contrary: assume that ϕ_L *is* a neighbor of a 4-fold Pfister L-form π. An important observation due to O. Izhboldin is as follows: the element of $H^4(F)$ determined by π is non-ramified, and if by some reason any non-ramified element of $H^4(L)$ is defined over F, then π is defined over F in a sense what can be led to a contradiction with the assumption that ϕ is not a Pfister neighbor over F.

Thus we may split the form ψ (i.e., we may pass to L), if the cokernel of the restriction map $H^4(F) \to H^4_{\mathrm{nr}}(L)$ to the non-ramified cohomology is 0. This cokernel was related in [14] to the Chow group $\mathrm{CH}^3(Q_\psi)$. In particular, the cokernel turns out to be 0 if the group $\mathrm{CH}^3(Q_\psi)$ is torsion-free.[1]

Below there is a list of some 9- and 10-dimensional quadratic forms ψ over F having the following properties:

- the groups $\mathrm{CH}^3(Q_\psi)$ are torsion-free;
- for any 9-dimensional ψ from the list, the index of the even Clifford algebra $C_0(\psi)$ is less than 4 what guaranties that ψ is not similar to the form ϕ;
- if this list is empty (for a given field F), then any 10-dimensional quadratic form over F is isotropic.

Consequently, the required extension E/F (with $u(E) = 9$) can be obtained by a like [23, §3] infinite procedure involving the function fields of the forms from the list.

Here is the list (just one possible choice of it):

1. Anisotropic 10-dimensional quadratic forms ψ with $\operatorname{ind} C_0(\psi) \geq 4$.
2. 9-dimensional quadratic forms ψ containing a 7-dimensional Pfister neighbor as a subform and such that the 10-dimensional form $\psi \perp \langle -\det \psi \rangle$ is anisotropic.
3. Anisotropic 9-dimensional forms ψ containing an 8-dimensional subform ψ' with $\det \psi' = 1$ and $\operatorname{ind} C(\psi') = 2$.
4. Anisotropic 10-dimensional forms ψ containing an 8-dimensional subform ψ' with $\det \psi' = 1$ and $\operatorname{ind} C(\psi') = 4$.

The claim that the groups $\mathrm{CH}^3(Q_\psi)$ are torsion-free for any ψ from the list can be verified by using the techniques of [16]. Let us check that any 10-dimensional quadratic form is really isotropic, if the list is empty.

Assume that the list is empty and let q be a 10-dimensional quadratic form over F. If $\operatorname{ind} C_0(q) \geq 4$, then q is isotropic (see Item 1 of the list). If $\operatorname{ind} C_0(q) \leq 2$ and $\operatorname{disc} q = 1$, then q can be represented (up to similarity) as $\psi \perp \langle -\det \psi \rangle$, where ψ is a 9-dimensional form containing a 7-dimensional Pfister neighbor; therefore q is also isotropic in this case (see Item 2 of the list). In particular, we have already shown that any 10-dimensional F-form of trivial discriminant is isotropic. Therefore, q contains an 8-dimensional subform ψ' with $\det \psi' = 1$. If $\operatorname{ind} C_0(q) \leq 2$, then $\operatorname{ind} C_0(\psi') \leq 4$. If $\operatorname{ind} C_0(\psi') = 4$, the form q is isotropic in view of Item 4. If $\operatorname{ind} C_0(\psi') = 2$, the form q is isotropic

[1] Here ψ should be not a Pfister neighbor; the case of a Pfister neighbor should be treated in a different way, see [10].

in view of Item 3. Finally, if $\operatorname{ind} C_0(\psi') = 0$ (i.e., if ψ' is similar to a 3-fold Pfister form), the form q is isotropic in view of Item 2.

3. Cycles, correspondences, and motives

In this section we introduce some notation and recall some notions concerning the Chow groups, Chow-correspondences, and Grothendieck Chow-motives.

Let F be a field (of any characteristic) and let X be a smooth F-variety. We write $\mathrm{CH}(X)$ for the Chow group of X, i.e., for the group of algebraic cycles on X modulo rational equivalence (although the elements of $\mathrm{CH}(X)$ are *classes of cycles*, we will mostly refer to them simply as to *cycles*). This is a graded ring, where the gradation $\mathrm{CH}^*(X)$ is given by the codimension of cycles, while the multiplication is served by the intersection theory. Sometimes we use the "lower" indices $\mathrm{CH}_*(X)$ meaning the gradation by dimension of cycles; in particular, for an equidimensional X, one has $\mathrm{CH}_i(X) = \mathrm{CH}^{\dim X - i}(X)$.

Let E/F be a field extension and X an F-variety. We say that a cycle $\alpha \in \mathrm{CH}(X_E)$ is *defined over* F, if it lies in the image of the restriction homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(X_E)$.

For smooth projective F-varieties X and Y, the Chow group $\mathrm{CH}(X \times Y)$ is also called the *group of correspondences* from X to Y and denoted by $\mathrm{Corr}(X, Y)$. The elements of $\mathrm{Corr}(X, Y)$ are called correspondences; in what follows, we use the classical notion of the composition for correspondences, see [3, def. 16.1.1]. We write $\beta \circ \alpha \in \mathrm{Corr}(X, Z)$ for the composite of correspondences $\alpha \in \mathrm{Corr}(X, Y)$ and $\beta \in \mathrm{Corr}(Y, Z)$. In order to distinguish with the product of cycles, we write $\alpha^{\circ N}$ for the N-th power of a correspondence $\alpha \in \mathrm{Corr}(X, X)$ on a variety X (although sometimes we write simply α^N, if it does not lead to the misunderstanding).

Recall that for any $r \in \mathbb{Z}$, any smooth projective X, and any equidimensional smooth projective Y, the correspondences in $\mathrm{Corr}(X, Y)$ given by the elements of the group $\mathrm{CH}^{\dim Y + r}(X \times Y)$ are called the correspondences of degree r. The group of all correspondences of degree r is denoted by $\mathrm{Corr}^r(X, Y)$. In the case of an arbitrary smooth projective Y, one sets $\mathrm{Corr}^r(X, Y) := \bigoplus_i \mathrm{Corr}^r(X, Y_i)$, where Y_i are the components of Y. Note that the degrees of correspondences are added when the correspondences are composed ([3, example 16.1.1]).

Sometimes we work in the category of the Grothendieck Chow-motives $\mathcal{M}(F)$ (these are the only motives we work with). The following very simple definition of $\mathcal{M}(F)$ is due to Jannsen, [13] (see also [30, §1.4]): the objects of $\mathcal{M}(F)$ are the triples (X, p, l), where X is a smooth projective F-variety, p is a projector on X, and $l \in \mathbb{Z}$; the group $\mathrm{Hom}((X, p, l), (Y, q, m))$ is defined as

$$q \circ \mathrm{Corr}^{m-l}(X, Y) \circ p \subset \mathrm{Corr}^{m-l}(X, Y) \ .$$

Let us accept the usual agreement that in the notation (X, p, l), one may omit p, if $p = id_X$; also one may omit l, if $l = 0$. We write simply X for the motive

(X) of a variety X. We write $M(l)$ for the twist $(X, p, l+m)$ of a motive $M = (X, p, m)$.

We write **pt** for the variety $\operatorname{Spec} F$ (and its motive).

4. Some consequences of the nilpotence theorem

The following statement, due to M. Rost, is called the *nilpotence theorem* (for projective quadrics):

Theorem 4.1 ([27, prop. 9]). *Let X be a projective quadric and let $\alpha \in \operatorname{Corr}^0(X, X)$ be a degree 0 correspondence on X. If $\alpha_E = 0 \in \operatorname{Corr}^0(X_E, X_E)$ for some field extension E/F, then the correspondence α is nilpotent.*

Corollary 4.2. *Let X be a projective quadric and $\alpha \in \operatorname{Corr}^0(X, X)$. If α_E is a projector for some field extension E/F, then $\alpha^{\circ N}$ is a projector for some N.*

Proof. Since α_E is a projector, the difference $\varepsilon := \alpha^{\circ 2} - \alpha$ vanishes over E. Therefore, $\varepsilon^{\circ 2^N} = 0$ and $2^N \varepsilon = 0$ for certain positive integer N (the latter is achived by the transfer argument). We claim that $\alpha^{\circ 4^N}$ is a projector.

To see it, take the 4^N-th power of the equality $\alpha^{\circ 2} = \varepsilon + \alpha$. Since ε commutes with α, we get the sum of $\binom{4^N}{i}\varepsilon^{\circ i}\alpha^{\circ(4^N - i)}$ on the right. Each summand with $i > 0$ is however zero, because $\varepsilon^{\circ i} = 0$ if i is divisible by 2^N, and $\binom{4^N}{i}$ is divisible by 2^N otherwise. Thus $(\alpha^{\circ 4^N})^{\circ 2} = \alpha^{\circ 4^N}$. □

Corollary 4.3. *Let X_1 and X_2 be projective quadrics; ρ_1 and ρ_2 some projectors on X_1 and X_2; $n_1, n_2 \in \mathbb{Z}$. If some morphisms of motives*

$$f \in \operatorname{Hom}\big((X_1, \rho_1, n_1), (X_2, \rho_2, n_2)\big) \quad \textit{and} \quad g \in \operatorname{Hom}\big((X_2, \rho_2, n_2), (X_1, \rho_1, n_1)\big)$$

are such that for some field extension E/F the morphism g_E is a left inverse to f_E, then there exists a left inverse to f (though it is not g in general). In particular, if f_E and g_E are mutually inverse isomorphisms for some E/F, then f and g themselves are isomorphisms (though not mutually inverse ones in general).

Proof. Put $\alpha := g \circ f - \rho_1$. Then $\alpha_E = 0$ and so $\alpha^{\circ 2^n} = 0 = 2^n \cdot \alpha$ for some n.

Since $f = f \circ \rho_1$ and $g = \rho_1 \circ g$, the correspondence ρ_1 commutes with $g \circ f$. Therefore α commutes with ρ_1 and so $(\alpha + \rho_1)^{\circ 4^n} = \rho_1^{\circ 4^n} = \rho_1$.

Thus, we have $(g \circ f)^{\circ N} = \rho_1$ for some N. Then $g \circ (f \circ g)^{\circ(N-1)}$ is a left inverse to f. □

5. Isotropic quadrics

Let $X = Q_\phi$ be an isotropic projective quadric. Write ϕ as $\phi = \phi' \perp \mathbb{H}$, where $\mathbb{H}$ is the hyperbolic plane, and put $X' = Q_{\phi'}$.

Lemma 5.1. *The correspondence $l_0 \times 1 + 1 \times l_0$ is the only Rost projector on X.*

Proof. The motivic decomposition [27, prop. 2] of X produces the decomposition

$$\operatorname{End}(X) \simeq \mathbb{Z} \times \operatorname{End}(X') \times \mathbb{Z}$$

of the ring $\operatorname{End}(X) := \operatorname{Corr}^0(X, X)$ in the direct product of rings. Therefore an arbitrary Rost correspondence on X looks out as $\rho = (l_0 \times 1 + 1 \times l_0) + \alpha$, where $\alpha \in \operatorname{End}(X')$ and $\alpha_{\bar{F}} = 0$. By the nilpotence theorem, $\alpha^{\circ N} = 0$ for some big N. Since

$$\alpha \circ (l_0 \times 1 + 1 \times l_0) = 0 = (l_0 \times 1 + 1 \times l_0) \circ \alpha \ ,$$

we have $\rho^{\circ N} = l_0 \times 1 + 1 \times l_0 + \alpha^{\circ N} = l_0 \times 1 + 1 \times l_0$ for this N. Consequently, if ρ is a projector, then $\rho = l_0 \times 1 + 1 \times l_0$. □

Corollary 5.2. *If ρ is a Rost projector on a n-dimensional isotropic quadric X, then for any i with $1 \leq i \leq n-1$ the homomorphism $\rho_* \colon \operatorname{CH}_i(X) \to \operatorname{CH}_i(X)$ is zero.*

Proof. Let $x \in \operatorname{CH}_i(X)$. If $i < n$, then $(l_0 \times 1) \cdot (x \times 1) = 0 \in \operatorname{CH}(X \times X)$ because $l_0 \cdot x = 0$ by the dimension reason. If $i > 0$, then $(pr_2)_*\big((1 \times l_0) \cdot (x \times 1)\big) = (pr_2)_*(x \times l_0) = 0$, where $pr_2 \colon X \times X \to X$ is the projection onto the second factor. Thus, for $0 < i < n$ one has $\rho_*(x) = 0$. □

6. Rost projectors on minimal neighbors

In this section we prove for any minimal neighbor the existence (Proposition 6.2) and the uniqueness (Proposition 6.4) of the Rost projector, and compute the summand complement to the Rost motive (Proposition 6.3). All these was done previously by M. Rost ([27]) in the specific case of a norm quadric.

For this section we fix a minimal neighbor ϕ of a Pfister form π, put $X = Q_\phi$, $P = Q_\pi$, and $Z = Q_\psi$, where $\phi \perp \psi = \pi$, and write n for the dimension of X (of course $n = 2^r - 1$ for some r).

The following statement first appeared in [17]. We publish it here because we are not going to publish the preprint [17] in a journal.

Lemma 6.1. *The cycle $l_n \times 1 + 1 \times l_n \in \operatorname{CH}^n(\bar{P} \times \bar{P})$, where $l_n \in \operatorname{CH}^n(\bar{P})$ is a class of a (maximal) n-dimensional linear subspace of $\bar{P}$, is defined over F.*

Proof. The Chow groups of completely split quadrics and their products are easy to calculate (cf. [6], [32, thm. 13.3], [15, §2.1], or [27, §2.3]). We need a description of $\operatorname{CH}^*(\bar{P} \times \bar{P})$. One has

$$\operatorname{CH}^*(\bar{P} \times \bar{P}) = \operatorname{CH}^*(\bar{P}) \otimes \operatorname{CH}^*(\bar{P})$$

where the inverse isomorphism is given by the outer product of cycles. Furthermore, the group $\operatorname{CH}^*(\bar{P})$ is torsion-free. The component $\operatorname{CH}^i(\bar{Y})$ is generated by h^i if $i < n$ (notice that $\dim P = 2n$) and by $h^i/2$ if $i > n$ (the generator $h^i/2$ coincides with the class l_{2n-i} of a totally isotopic subspace of the appropriate dimension). The "middle" component $\operatorname{CH}^n(\bar{Y})$ has (unlikely to the others) two free generators: h^n and l_n.

It follows that $\mathrm{CH}^n(\bar{P}\times\bar{P})$ is a free abelian group on $h^i\times h^{n-i}$ $(i=0,\dots,n)$, $l_n\times 1$, and $1\times l_n$. Note that all but two last generators are defined over F.

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^n(\bar{P}\times\bar{P}) & \xrightarrow{(id_{\bar{P}}\times\bar{f})^*} & \mathrm{CH}^n(\bar{P}_{\bar{F}(\bar{P})}) \\ {\scriptstyle \mathrm{res}_{\bar{F}/F}}\uparrow & & \uparrow{\scriptstyle \mathrm{res}_{\bar{F}(\bar{P})/F(P)}} \\ \mathrm{CH}^n(P\times P) & \xrightarrow{(id_P\times f)^*} & \mathrm{CH}^n(P_{F(P)}) \end{array}$$

where the horizontal arrows are the pull-backs with respect to the flat morphisms $id_P\times f: P_{F(P)}\to P\times P$ and $id_{\bar{P}}\times\bar{f}: \bar{P}_{\bar{F}(\bar{P})}\to\bar{P}\times\bar{P}$ and where f (resp. $\bar{f}$) is the generic point morphism of P (resp. $\bar{P}$).

Since the Pfister form π is isotropic over $F(P)$, it is hyperbolic over this function field ([22, cor. 10.1.6]). Thereby $l_n\in\mathrm{CH}^n(\bar{P}_{\bar{F}(\bar{P})})$ is defined over $F(P)$. Since $(id_P\times f)^*$ is evidently surjective (see, for example, [12, prop. 5.1]), it follows that there exists a defined over F cycle $\alpha\in\mathrm{CH}^n(\bar{P}\times\bar{P})$ such that $(id_{\bar{P}}\times\bar{f})^*(\alpha)=l_n$.

It is easy to see how the homomorphism $(id_{\bar{P}}\times\bar{f})^*$ acts on the generators of the group $\mathrm{CH}^n(\bar{P}\times\bar{P})$:

$$\begin{aligned} &(id_{\bar{P}}\times\bar{f})^*(h^i\times h^{n-i})=0 && \text{for } i=0,\dots,n-1,\\ &(id_{\bar{P}}\times\bar{f})^*(h^n\times 1)=h^n\ , &&\\ &(id_{\bar{P}}\times\bar{f})^*(1\times l_n)=0\ , && \text{and}\\ &(id_{\bar{P}}\times\bar{f})^*(l_n\times 1)=l_n\ . && \end{aligned}$$

Since $(id_{\bar{P}}\times\bar{f})^*(\alpha)=l_n$, it follows that

$$\alpha=l_n\times 1+a\cdot(1\times l_n)+\sum_{i=0}^{n}a_i\cdot(h^i\times h^{n-i})$$

with some integers a_i and a (one has additionally $a_n=0$ but we don't care about this). Since the generators $h^i\times h^{n-i}$ are defined over F and since the cycle $2(1\times l_n)$ is defined over F (because $2l_n\in\mathrm{CH}^n(\bar{P})$ is defined over F by the transfer argument), it follows that either $l_n\times 1+1\times l_n$ or $l_n\times 1$ is defined over F. If we are in the first case, we are done. Since the sum of $l_n\times 1$ and its transpose gives the cycle $l_n\times 1+1\times l_n$, this sum is defined over F also in the second case (in fact the second case is not possible, if the quadratic form π is anisotropic). □

The construction of the Rost projector given in the proof of Proposition 6.2 is probably the simplest and shortest possible one. It is shorter as the one in [17] (and in contrast to [17], can be applied to an arbitrary minimal neighbor).

Proposition 6.2. *The quadric X possesses a Rost projector.*

Proof. By Lemma 6.1, the cycle $l_n\times 1+1\times l_n\in\mathrm{CH}^n(\bar{P}\times\bar{P})$ is defined over F. Taking its pull-back to $\mathrm{CH}^n(\bar{X}\times\bar{X})$ with respect to the (defined over

F) imbedding $\bar{X} \times \bar{X} \hookrightarrow \bar{P} \times \bar{P}$, we get the cycle $l_0 \times 1 + 1 \times l_0$. We have shown that X possesses a Rost correspondence. Since certain power of a Rost correspondence always will be a Rost projector (see Corollary 4.2), we are done. □

Let us write Δ_X for the diagonal class in $\mathrm{CH}^n(X \times X)$. This is the identity correspondence on X.

Proposition 6.3. *The motive $(X, \Delta_X - \rho)$, where ρ is a Rost projector on X, is isomorphic to the motive $\mathbb{Z}(1)$.*

Proof. Over $\bar{F}$ the mutually inverse isomorphisms required are given by the cycle

$$\alpha := (h \times h^{n-2} + h^2 \times h^{n-3} + \cdots + h^{n-2} \times h + h^{n-1} \times 1)/2 \in \mathrm{CH}^{n-1}(\bar{X} \times \bar{Z})$$

and its transpose (cf. [18, §4]). According to Corollary 4.3 it suffices to show that the cycle α is defined over F.

Suppose that X has a rational point. Then the Pfister form π is split and so $\phi \simeq \psi \perp \mathbb{H}$. Using the motivic decomposition $X \simeq \mathbf{pt} \oplus Z(1) \oplus \mathbf{pt}(n)$, we get an isomorphism $\mathrm{CH}^{n-1}(X \times Z) \simeq \mathrm{CH}^{n-2}(Z \times Z)$. The required cycle on $X \times Z$ is the cycle corresponding to the diagonal on $Z \times Z$ under this isomorphism.

In the general case, we can therefore get the required cycle if we pass over the function field $F(X)$. Let us take a preimage of this cycle with respect to the epimorphism

$$\mathrm{CH}^{n-1}\big(X \times (X \times Z)\big) \twoheadrightarrow \mathrm{CH}^{n-1}\big((X \times Z)_{F(X)}\big) .$$

The result is a cycle on $X \times X \times Z$, looking over $\bar{F}$ as

$$(*) \qquad 1 \times \alpha + \sum_i \beta_i \times \gamma_i \ ,$$

where β_i are some homogeneous cycles on $\bar{X}$ while γ_i are some cycles on $\bar{X} \times \bar{Z}$ and moreover $\operatorname{codim} \beta_i > 0$ and $\dim \beta_i > 0$ (the last inequality is evidently caused by the fact that $\beta_i \times \gamma_i \in \mathrm{CH}^{n-1}(\bar{X} \times \bar{X} \times \bar{Z})$ while $\dim \bar{X} = n$). Consider the cycle $(*)$ as a correspondence in $\mathrm{Corr}(X, X \times Z)$ and compose it with $\rho \in \mathrm{Corr}(X, X)$. We get a defined over F cycle which is equal to $1 \times \alpha$ over $\bar{F}$ (since over $\bar{F}$ one has $(\beta_i \times \gamma_i) \circ (l_0 \times 1) = 0$ because $\operatorname{codim} \beta_i > 0$, while $(\beta_i \times \gamma_i) \circ (1 \times l_0) = 0$ because $\dim \beta_i > 0$). Taking the pull-back with respect to the diagonal of X

$$\mathrm{CH}^{n-1}(X \times X \times Z) \to \mathrm{CH}^{n-1}(X \times Z)$$

we get the required cycle because $1 \times \alpha \mapsto \alpha$ over $\bar{F}$. □

Proposition 6.4. *If ρ and ρ' are two Rost projectors on X, then $\rho = \rho'$ (to be understood as an equality in the Chow group $\mathrm{CH}^n(X \times X)$).*

Proof. It is enough to show that $\rho = \rho' \circ \rho$ and that $\rho = \rho \circ \rho'$.

To prove the first equality, consider the difference $\rho - \rho' \circ \rho = (\Delta_X - \rho') \circ \rho$. This correspondence is a morphism of motives $(X, \rho) \to (X, \Delta_X - \rho')$ and to show that it is zero we show that the whole group $\mathrm{Hom}((X, \rho), (X, \Delta_X - \rho'))$

is so. To compute this group, we replace (using Proposition 6.3) the motive $(X, \Delta_X - \rho')$ by $Z(1)$. Applying the definition (see section 3), we get

$$\operatorname{Hom}((X,\rho), Z(1)) = \rho \circ \operatorname{Corr}^1(X,Z)\ .$$

Note that trying to prove the second equality (i.e., the equality $\rho = \rho \circ \rho'$) in the similar way, we come to the group

$$\operatorname{Hom}(Z(1),(X,\rho)) = \operatorname{Corr}^{-1}(Z,X) \circ \rho \simeq \rho^t \circ \operatorname{Corr}^1(X,Z)\ .$$

Since ρ^t is also a Rost projector, the both cases are covered by the following

Lemma 6.5 (cf. [27, prop. 1])**.** *The action of ρ on the group $\operatorname{Corr}^1(X,Z)$ is zero.*

Proof. Let us consider the following filtration on the group $\operatorname{Corr}^1(X,Z) = \operatorname{CH}^{n-1}(X \times Z)$: for any $p \geq 0$ the term $\mathcal{F}^p \operatorname{CH}^{n-1}(X \times Z)$ is the subgroup of $\operatorname{CH}^{n-1}(X \times Z)$ generated by the classes of the simple cycles α satisfying the condition $\operatorname{codim}_Z pr_Z(\alpha) \geq p$, where $pr_Z \colon X \times Z \to Z$ is the projection. This is a finite filtration (one evidently has $\mathcal{F}^{n-1} = 0$) agreed with the action of ρ (cf. [27, proof of prop. 1]). For any $0 \leq p \leq n-2$ there is an epimorphism onto the successive quotient $\mathcal{F}^p/\mathcal{F}^{p+1}$ of this filtration, which starts in the group $\coprod_{z \in Z^p} \operatorname{CH}^{n-1-p}(X_{F(z)})$. This epimorphism is agreed with the action of ρ if we let ρ act on each of the direct summands the way as the correspondence $\rho_{F(z)}$ acts on $\operatorname{CH}^*(X_{F(z)})$ (cf. [27, proof of prop. 1]). Since for every point $z \in Z$ the quadric $X_{F(z)}$ is isotropic, a Rost projector acts by zero on $\operatorname{CH}^i(X_{F(z)})$ for all $i \neq 0, n$ (Corollary 5.2). Since $n-1-p \neq 0, n$, we see that ρ acts by zero on the successive quotients of the filtration $\mathcal{F}^p$. Therefore the action of ρ on $\operatorname{Corr}^1(X,Z)$ is nilpotent. Since ρ is a projector, it follows that the action is in fact zero. □

Proposition 6.4 is proven. □

7. "Vanishing in K_0"

In this section, X is an arbitrary smooth projective quadric of any dimension $n \geq 3$ and ρ is a fixed Rost projector on X.

We write simply $K(X)$ for the Grothendieck group $K'_0(X) = K_0(X)$. We write $K(X)^{(i)}$ for the topological filtration on $K(X)$ and put $G^iK(X) := K(X)^{(i)}/K(X)^{(i+1)}$.

Proposition 7.1. *For any i with $1 \leq i \leq n-2$, the group $\rho_*(\operatorname{CH}^i(X))$ "vanishes in K_0", that is, it lies in the kernel of the canonical epimorphism $\operatorname{CH}^i(X) \twoheadrightarrow G^iK(X)$, $[Z] \mapsto [\mathcal{O}_Z] \mod K(X)^{(i+1)}$.*

This Proposition is applied (in the proof of Theorem 1.7) in the only case with $n = 7$ and to the only element $h^4 \in \operatorname{CH}^4(X)$. The proof in this specific case is however not simpler as in the general one. It is given in the end of this section, after several preparative lemmas. To begin, let us take the image of ρ in $G^nK(X \times X)$ and choose its representative (just any) $\mathcal{P} \in K(X \times X)^{(n)}$.

Since the restriction homomorphisms $K(X\times X)\to K(\bar X\times\bar X)$ and $K(X)\to K(\bar X)$ are injective, we may identify $K(X\times X)$ with a subgroup of $K(\bar X\times\bar X)$ and we may identify $K(X)$ with a subgroup of $K(\bar X)$. As an element of $K(\bar X\times\bar X)$, the representative $\mathcal{P}$ can be written down as $\mathcal{P}=l_0\times 1+1\times l_0+\delta$, where l_0 is now the class of a rational point in $K(\bar X)$, and δ is an element of $K(\bar X\times\bar X)^{(n+1)}$.

One may consider elements of $K(X\times X)$ as K-correspondences on X. In particular, the elements $\mathcal{P}$ and δ produce endomorphisms $\mathcal{P}_*$ and δ_* of $K(X)$ and of $K(\bar X)$.

Lemma 7.2. *For any i, one has $\delta_*(K(\bar X)^{(i)})\subset K(\bar X)^{(i+1)}$ and $\mathcal{P}_*(K(X)^{(i)})\subset K(X)^{(i)}$.*

Proof. The first claim is a direct consequence of the definition of δ_*, of the inclusion $\delta\in K(\bar X\times\bar X)^{(n+1)}$, and of the fact the flat pull-back and the multiplication on K are agreed with the filtration by the codimension $K^{(i)}$, while the proper push-forward is agreed with the filtration by the dimension.

The second claim relies on the same arguments and on the inclusion $\mathcal{P}\in K(X\times X)^{(n)}$. □

We write $h\in K(X)$ for the class of a hyperplane section.

Lemma 7.3. $\mathcal{P}_*(h^{n-1})=a\cdot l_0$ *for some odd $a\in\mathbb{Z}$.*

Proof. Since $h^{n-1}=2l_1-l_0$ ([15]), where $l_1\in K(\bar X)$ is the class of a line, it suffices to show that $\mathcal{P}_*(l_0)=l_0$.

Let us substitute $\mathcal{P}=l_0\times 1+1\times l_0+\delta$. Since $l_0\in K(\bar X)^{(n)}$, $\delta_*(l_0)\in K(\bar X)^{(n+1)}=0$.

It remains to compute $(l_0\times 1+1\times l_0)_*(l_0)$. By definition this is

$$(pr_2)_*\big((l_0\times 1)\cdot(l_0\times 1+1\times l_0)\big)\ ,$$

where $pr_2:\bar X\times\bar X\to\bar X$ is the projection onto the second factor. Since $l_0\cdot l_0\in K(\bar X)^{(2n)}=0$, we finally get $(pr_2)_*(l_0\times l_0)=l_0$. □

Corollary 7.4. $l_0\in K(X)^{(n-1)}$.

Proof. Since $h^{n-1}\in K(X)^{(n-1)}$, one has $\mathcal{P}_*(h^{n-1})\in K(X)^{(n-1)}$ by Lemma 7.2. Thus, taking Lemma 7.3 into account, we see that $a\cdot l_0\in K(X)^{(n-1)}$ for some odd integer a. This finishes the proof because $2\cdot l_0=h^n\in K(X)^{(n)}\subset K(X)^{(n-1)}$. □

Lemma 7.5. *There is an inclusion $(l_0\times 1+1\times l_0)_*(K(\bar X)^{(1)})\subset\mathbb{Z}\cdot l_0=K(\bar X)^{(n)}$.*

Proof. Let $\alpha\in K(\bar X)^{(1)}$. Since $\alpha\cdot l_0=0$, the direct computation gives

$$(l_0\times 1+1\times l_0)_*(\alpha)=(1\times l_0)_*(\alpha)=pr_*(\alpha)\cdot l_0\ ,$$

where pr_* is the push-forward $K(\bar X)\to K(\bar F)=\mathbb{Z}$ with respect to the structure morphism $pr\colon\bar X\to\bar F$ of $\bar X$. □

Corollary 7.6. *For any i with $1 \le i \le n-1$, there is an inclusion*

$$\mathcal{P}_*(K(\bar{X})^{(i)}) \subset K(\bar{X})^{(i+1)} .$$

Proof. This is a direct consequence of Lemma 7.5 and the first claim of Lemma 7.2. ☐

Applying Corollary 7.6 $n-1$ times and taking into account that $K(\bar{X})^{(n)} = \mathbb{Z} \cdot l_0$, we get

Corollary 7.7. $(\mathcal{P}_*)^{\circ(n-1)}(K(\bar{X})^{(1)}) \subset \mathbb{Z} \cdot l_0$. ☐

Since $K(X)^{(1)} \subset K(\bar{X})^{(1)}$ and $l_0 \in K(X)^{(n-1)}$ (Corolary 7.4), we achieve

Corollary 7.8. $(\mathcal{P}_*)^{\circ(n-1)}(K(X)^{(1)}) \subset K(X)^{(n-1)}$. ☐

Proof of Proposition 7.1. Let the integer i be as required and let us take a cycle $\alpha \in \mathrm{CH}^i(X)$. Since the homomorphism $\mathrm{CH}^* \to G^*K$ agrees with the flat pull-backs, proper push-forwards, and with the multiplication of cycles, the image of $(\rho_*)^{\circ(n-1)}(\alpha)$ in $G^iK(X)$ coincides with the co-set of $(\mathcal{P}_*)^{\circ(n-1)}(A) \in K(X)^{(i)}$ in the factorgroup $G^iK(X) = K(X)^{(i)}/K(X)^{(i+1)}$, where $A \in K(X)^{(i)}$ is a representative of the image of α. Since $i < n-1$, this co-set is zero by Corollary 7.8. On the other hand, since ρ is a projector, we have $(\rho_*)^{\circ(n-1)}(\alpha) = \rho_*(\alpha)$. ☐

8. Norm quadrics

Definition 8.1. A *norm form* is a quadratic form of the kind $\pi \perp \langle -a \rangle$, where $a \in F^*$ and π is a Pfister form (note that it is a minimal neighbor of the Pfister form $\pi \langle\langle a \rangle\rangle$). A *norm quadric* is the projective quadric given by a norm form.

Let $a_1, a_2, \dots, a_r, a_{r+1} \in F^*$ and let ϕ_r be the norm form

$$\langle\langle a_1, \dots, a_r \rangle\rangle \perp \langle -a_{r+1} \rangle \ .$$

For every i from 0 up to r, consider also the norm subform $\phi_i \subset \phi_r$, $\phi_i = \langle\langle a_1, \dots, a_i \rangle\rangle \perp \langle -a_{i+1} \rangle$. Note that Q_{ϕ_i} is a closed subvariety of Q_{ϕ_r}.

Theorem 8.2 (M. Rost, [26]). *Assume that the norm form ϕ_r is anisotropic. Let ρ be a Rost projector on $X := Q_{\phi_r}$. For any $i = 0, \dots, r$, the $(2^i - 1)$-dimensional Chow group $\rho_* \mathrm{CH}_{2^i-1}(X)$ of the Rost motive (X, ρ) is generated by the element $\rho_*([Q_{\phi_i}])$ having the infinite order for $i = 0, r$ and the order 2 otherwise. For $j \ne 2^i - 1$, the group $\rho_* \mathrm{CH}_j(X)$ is zero.*

Since $[Q_{\phi_i}] = h^{2^r - 2^i}$ in the Chow group of X, we in particular get

Corollary 8.3. *For any $i = 0, \dots, r$, the element $\rho_*(h^{2^r-2^i}) \in \mathrm{CH}^{2^r-2^i}(X)$ is non-zero.* ☐

Remark 8.4. The preprint [26] does not contain any proofs, so that the reference we use for Theorem 8.2 is not a honest one. The proofs of some statements of [26] appeared in [27], but a proof of Theorem 8.2 was not included there. Below we give a proof of Corollary 8.3 for the case of $r = 3$ and $i = 2$, because this is the only case needed in the proof of Theorem 1.7.

In the rest of this section, we prove Corollary 8.3 in the case $r = 3$ and $i = 2$. We fix the following notation: a_1, a_2, a_3, a_4 are elements of F^* such that the form $\phi := \phi_3 = \langle\langle a_1, a_2, a_3\rangle\rangle \perp \langle -a_4\rangle$ is anisotropic; π is the Pfister form $\langle\langle a_1, a_2, a_3\rangle\rangle \subset \phi$; $\psi := \phi_2 = \langle\langle a_1, a_2\rangle\rangle \perp \langle -a_3\rangle$; $X := Q_\phi$; $P := Q_\pi$; $Y := Q_\psi$; U is the affine quadric $X \setminus P$, which is determined by the equation $\pi = a_4$.

We will work with the K-groups $K_0(X)$ and $K_1(X)$ (by that reason, in contrast to section 7, we do not omit the index 0 in the notation $K_0(X)$ here) and write $G^*K_0(X)$, $G^*K_1(X)$ for the graded groups associated with the topological filtration on $K_0(X)$ and $K_1(X)$.

Lemma 8.5. $\operatorname{Tors} G^i K_0(X) \simeq \begin{cases} 0 & \text{for } i = 0,1,2,7; \\ \mathbb{Z}/2 & \text{for } i = 3,4,5,6. \end{cases}$

Proof. According to [15, thm. 3.8], for every i, the group $\operatorname{Tors} G^i K_0(X)$ is either zero or of the order 2. Since $C_0(\phi)$ is a split central simple F-algebra of degree 2^4, the order of the whole group $\operatorname{Tors} G^* K_0(X)$ equals 2^4 by [15, thm. 3.8]. The group $G^i K_0(X)$ is torsion-free for $i = 0$, $i = 1$, $i = 2$ ([15, thm. 6.1]), and $i = \dim X = 7$ ([33]). Consequently, for each of the other four values of i, namely for $i = 3,4,5,6$, the group $\operatorname{Tors} G^i K_0(X)$ is of the order 2. □

Lemma 8.6. *For the Rost projector ϱ on Y, one has*

$$\operatorname{Tors} \varrho_* \operatorname{CH}^i(Y) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 2; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Write C for the conic $\langle 1, -a_1, -a_2\rangle$. The motivic decomposition $Y \simeq (Y, \varrho) \oplus C(1)$ of Proposition 6.3 gives the decomposition $\operatorname{CH}^i(Y) \simeq \varrho_* \operatorname{CH}^i(Y) \oplus \operatorname{CH}^{i-1}(C)$. Since the Chow group $\operatorname{CH}^*(C)$ of C is torsion-free, while

$$\operatorname{Tors} \operatorname{CH}^*(Y) = \operatorname{Tors} \operatorname{CH}^2(Y) \simeq \mathbb{Z}/2$$

([15, thm. 5.3]), we are done. □

Corollary 8.7. $\operatorname{Tors} \operatorname{CH}^i(P) = \begin{cases} 0 & \text{for } i = 0,1,6; \\ \mathbb{Z}/2 & \text{for } i = 2,3,4,5. \end{cases}$

Proof. The motivic decomposition $P \simeq \oplus_{l=0}^3 (Y, \varrho, l)$ of [27, prop. 19] gives the decomposition $\operatorname{CH}^i(P) \simeq \oplus_{l=0}^3 \operatorname{CH}^i(Y, \varrho, l) = \oplus_{l=0}^3 \varrho_* \operatorname{CH}^{i-l}(Y)$. Applying Lemma 8.6, we finish the proof. □

Corollary 8.8. *For every $i \neq 4$, the canonical epimorphism $\operatorname{CH}^i(X) \twoheadrightarrow G^i K_0(X)$ is bijective.*

Proof. Since the kernel of the epimorphism $\operatorname{CH}^i(X) \twoheadrightarrow G^i K_0(X)$ is contained in $\operatorname{Tors} \operatorname{CH}^i(X)$, the group $\operatorname{Tors} \operatorname{CH}^i(X)$ is mapped surjectively onto $\operatorname{Tors} G^i K_0(X)$ and

$$\operatorname{Ker}\Big(\operatorname{CH}^i(X) \twoheadrightarrow G^i K_0(X)\Big) = \operatorname{Ker}\Big(\operatorname{Tors} \operatorname{CH}^i(X) \twoheadrightarrow \operatorname{Tors} G^i K_0(X)\Big).$$

The groups $\operatorname{Tors} G^i K_0(X)$ are computed in Lemma 8.5. To show that

$$\operatorname{Ker}\Big(\operatorname{Tors}\operatorname{CH}^i(X) \twoheadrightarrow \operatorname{Tors} G^i K_0(X)\Big) = 0$$

for $i \neq 4$, it suffices to show that $\operatorname{Tors}\operatorname{CH}^i(X) = 0$ for $i = 1, 2, 7$ and that the order of $\operatorname{Tors}\operatorname{CH}^i(X)$ is at most 2 for $i = 3, 5, 6$.

For any positive i, the right-hand side term of the exact sequence

$$\operatorname{CH}^{i-1}(P) \to \operatorname{CH}^i(X) \to \operatorname{CH}^i(U)$$

is zero by Theorem A.4. Therefore, the homomorphism $\operatorname{CH}^{i-1}(P) \twoheadrightarrow \operatorname{CH}^i(X)$ is surjective. Since for any $i \neq 4$, the group $\operatorname{CH}^{i-1}(P)/\operatorname{Tors}\operatorname{CH}^{i-1}(P)$ is generated by h^{i-1} ([15, §2.7]), we get a surjection $\operatorname{Tors}\operatorname{CH}^{i-1}(P) \twoheadrightarrow \operatorname{Tors}\operatorname{CH}^i(X)$ for any $i \neq 4$. Corollary 8.7 finishes the proof. □

Lemma 8.9. *$G^2 K_1(X) = F^* \cdot h^2$.*

Proof. Since the even Clifford algebra of the quadratic form ϕ is split, it follows from [31] that $K_1(X) = F^* \cdot K_0(X)$. Furthermore, for every i, there is an inclusion $F^* \cdot K_0(X)^{(i)} \subset K_1(X)^{(i)}$, which produces a homomorphism on the successive quotients $f_i : F^* \cdot G^i K_0(X) \to G^i K_1(X)$. Hereby the subgroup $F^* \cdot h^i \subset F^* \cdot G^i K_0(X)$ maps injectively into $G^i K_1(X)$. Now we take in account that for every $i \leq 2$ the group $G^i K_0(X)$ is generated by h^i (because it is torsion-free), whereby f_i is an injection for $i \leq 2$. We finish the proof as follows: since the epimorphism f_0 is injective, it is an isomorphism, i.e., $K_1(X)^{(1)} = F^* \cdot K_0(X)^{(1)}$; therefore, the injection f_1 is surjective and so $K_1(X)^{(2)} = F^* \cdot K_0(X)^{(2)}$; finally, the injection f_2 turns out to be surjective and gives the required isomorphism. □

Corollary 8.10. $\operatorname{Ker}\Big(\operatorname{Tors}\operatorname{CH}^4(X) \twoheadrightarrow \operatorname{Tors} G^4 K_0(X)\Big) \neq 0.$

Proof. The kernel of the restriction map $H^4(F) \to H^4(F(X))$ of the Galois cohomology groups (with $\mathbb{Z}/2$-coefficients) contains a non-zero element given by the class (a_1, a_2, a_3, a_4) of the quadratic form $\langle\langle a_1, a_2, a_3, a_4\rangle\rangle$ in $H^4(F)$. In particular, $\operatorname{Ker}\Big(H^4(F) \to H^4(F(X))\Big) \neq 0$. Applying the isomorphism [24, prop. 1], we get that $\operatorname{Ker}\Big(H^2(X, K_3) \to H^2(\bar{X}, K_3)\Big) \neq 0$. The group $H^2(X, K_3)$ coincides with the term $E_2^{2,-3}$ of the BGQ spectral sequence [25, §7.5] for X and decomposes in the direct sum of the subgroups $F^* \cdot h^2$ and $\operatorname{Ker}\Big(H^2(X, K_3) \to H^2(\bar{X}, K_3)\Big)$. The term $E_\infty^{2,-3}$ is hereby $G^2 K_1(X)$ and equals $F^* \cdot h^2$ by Lemma 8.9. Thus, $E_2^{2,-3} \supsetneq E_\infty^{2,-3}$. Consequently, in the BGQ spectral sequence, there exists a non-zero differential starting in $E_2^{2,-3}$ or (more generally) for some $l \geq 2$, the E_l term of the BGQ spectral sequence contains a non-zero differential d starting in $E_l^{2,-3}$.

Let us look at where the differential d can end in. Since for any $i \neq 4$, the epimorphism $E_2^{i,-i} = \operatorname{CH}^i(X) \twoheadrightarrow E_\infty^{2,-3} = G^i K_0(X)$ is bijective (Corollary 8.8), there are no non-zero differentials ending in $E_l^{i,-i}$ for any $l \geq 2$ and $i \neq 4$.

Thus, the only possible point for d to end in is the group $E_2^{4,-4} = \mathrm{CH}^4(X)$. Consequently, the epimorphism $\mathrm{CH}^4(X) \twoheadrightarrow G^4K_0(X)$ has a non-zero kernel, which is equivalent to the statement required. $\square$

Corollary 8.11. *The order of torsion in* $\mathrm{CH}^4(X)$ *is greater that* 2.

Proof. According to Corollary 8.10, this order is greater that the order of torsion in $G^4K_0(X)$. The latter equals 2 by Lemma 8.5. $\square$

Corollary 8.12. *For the Rost projector* ρ *on* X, *one has* $\rho_* \mathrm{CH}^4(X) \neq 0$.

Proof. The motivic decomposition

$$X \simeq (X, \rho) \oplus (Y, \varrho, 1) \oplus (Y, \varrho, 2) \oplus (Y, \varrho, 3)$$

of [27, thm. 17] implies that

$$\mathrm{CH}^4(X) \simeq \rho_* \mathrm{CH}^4(X) \oplus \varrho_* \mathrm{CH}^3(Y) \oplus \varrho_* \mathrm{CH}^2(Y) \oplus \varrho_* \mathrm{CH}^1(Y) .$$

Since $\mathrm{Tors}\, \varrho_* \mathrm{CH}^3(Y) = 0$, $\mathrm{Tors}\, \varrho_* \mathrm{CH}^2(Y) \simeq \mathbb{Z}/2$, and $\mathrm{Tors}\, \varrho_* \mathrm{CH}^1(Y) = 0$ (Lemma 8.6), while $|\,\mathrm{Tors}\, \mathrm{CH}^4(X)| > 2$, it follows that $\rho_* \mathrm{CH}^4(X) \neq 0$. $\square$

Lemma 8.13. *The homomorphism* $\varrho_* \mathrm{CH}^{i-4}(Y) \to \rho_* \mathrm{CH}^i(X)$, *induced by the push-forward* $\mathrm{CH}^{i-4}(Y) \to \mathrm{CH}^i(X)$ *with respect to the imbedding* $Y \hookrightarrow X$ *is for any* $i > 0$ *an epimorphism.*

Proof. According to Theorem A.4, for any $i > 0$, the push-forward homomorphism $\mathrm{CH}^{i-1}(P) \to \mathrm{CH}^i(X)$ is surjective. The motivic decomposition

$$P \simeq \bigoplus_{l=0}^{3} (Y, \varrho, l)$$

produces the decomposition

$$\mathrm{CH}^{i-1}(P) = \bigoplus_{l=0}^{3} \mathrm{CH}^{i-1}(Y, \varrho, l) = \bigoplus_{l=0}^{3} \varrho_* \mathrm{CH}^{i-1-l}(Y) .$$

Thus, we complete the proof, when we show that the morphism of motives $(Y, \varrho, l+1) \to (X, \rho)$ induced by the imbedding $P \hookrightarrow X$ is zero for $l < 3$ and is induced by the imbedding $Y \hookrightarrow X$ for $l = 3$.

Since the motivic morphism $(Y, \varrho, 3) \to P$ is induced by the imbedding $Y \hookrightarrow P$, the second statement (i.e., the statement on $l = 3$) is clear. For the statement on $l < 3$, we identify the direct sum of motives $\oplus_{l=0}^{2}(Y, \varrho, l)$ with the motive of the projective quadric Z, determined by the pure subform $\langle -a_1, -a_2, -a_3, a_1a_2, a_1a_3, a_2a_3, -a_1a_2a_3 \rangle$ of the Pfister form $\pi = \langle\langle a_1, a_2, a_3 \rangle\rangle$, via the motivic isomorphism [27, thm. 17 (9)]. Since $\mathrm{Hom}(Z(1), (X, \rho)) = 0$ (Lemma 6.5), the statement on $l < 3$ follows. $\square$

Corollary 8.14. *The group* $\rho_* \mathrm{CH}^4(X)$ *is generated by the element* $\rho_*(h^4)$.

Proof. Applying Lemma 8.13 for $i = 4$, we see that the composition

$$\varrho_* \mathrm{CH}^0(Y) = \mathrm{CH}^0(Y) \xrightarrow{\text{push-forward}} \mathrm{CH}^4(X) \xrightarrow{\rho_*} \rho_* \mathrm{CH}^4(X)$$

is an epimorphism. Since the group $\mathrm{CH}^0(Y)$ is generated by the class of Y, which is equal to h^4 in $\mathrm{CH}^4(X)$, we get what we want. $\square$

Proof of Corollary 8.3 for $r = 3$ and $i = 2$. The group $\rho_* \operatorname{CH}^4(X)$ is non-zero (Corollary 8.12) and is generated by the element $\rho_*(h^4)$ (Corollary 8.14). Therefore $\rho_*(h^4) \neq 0$. □

9. Minimal neighbors

Let ϕ be a minimal neighbor of an anisotropic Pfister form π. We put $X = Q_\phi$, $P = Q_\pi$, and $n = \dim X$. We denote by in the closed imbedding $X \hookrightarrow P$. Let $\mathcal{P} \in \operatorname{CH}^n(P \times P)$ be a cycle such that $\mathcal{P}_{\bar F} = l_n \times 1 + 1 \times l_n$ (which exists by Lemma 6.1).

Lemma 9.1. *The push-forward homomorphism $in_* : \operatorname{CH}_*(X) \to \operatorname{CH}_*(P)$ is injective on the subgroup $\rho_*(\operatorname{CH}_*(X)) \subset \operatorname{CH}_*(X)$, where ρ is the Rost projector on X.*

Proof. It suffices to show that the morphism of motives $f\colon (X, \rho) \to P$, induced by the imbedding in, possesses a left inverse. It is so indeed, because the motive (X, ρ) is a direct summand of P. To give a more direct proof, consider the morphism $g\colon P \to (X, \rho)$, given by the correspondence

$$\rho \circ (id_P \times in)^*(\mathcal{P}) \ .$$

Since $f_{\bar F} = l_0 \times h^n + 1 \times l_0 \in \operatorname{CH}^{2n}(\bar X \times \bar P)$ and $g_{\bar F} = l_n \times 1 + 1 \times l_0$, we have $g_{\bar F} \circ f_{\bar F} = l_0 \times 1 + 1 \times l_0 = \rho_{\bar F}$. Therefore, $g_{\bar F}$ is a left inverse to $f_{\bar F}$. It follows that f possesses a left inverse (Corollary 4.3). □

Now denote by δ_X the Rost correspondence $(in \times in)^*(\mathcal{P})$ on X. Also put $\delta_P := \mathcal{P} \cdot (1 \times h^n) \in \operatorname{CH}^{2n}(P \times P) = \operatorname{Corr}^0(P, P)$.

Lemma 9.2. *There is the following commutation formula:*

$$in_* \circ (\delta_X)_* = (\delta_P)_* \circ in_* \in \operatorname{Hom}\big(\operatorname{CH}_*(X), \operatorname{CH}_*(P)\big) \ .$$

Proof. For any $x \in \operatorname{CH}_*(X)$ one has:

$$\begin{aligned} in_*\big((\delta_X)_*(x)\big) &\overset{1)}{=} \big(in_* \circ (pr_2)_*\big)\big(\delta_X \cdot (x \times 1)\big) = \\ &\overset{2)}{=} \big((pr_2)_* \circ (in \times in)_*\big)\big((in \times in)^*(\mathcal{P}) \cdot (x \times 1)\big) = \\ &\overset{3)}{=} (pr_2)_*\big(\mathcal{P} \cdot (in \times in)_*(x \times 1)\big) = \\ &\overset{4)}{=} (pr_2)_*\big(\mathcal{P} \cdot (in_*(x) \times h^n)\big) = \\ &\overset{5)}{=} (pr_2)_*\big((\mathcal{P} \cdot (1 \times h^n) \cdot (in_*(x) \times 1))\big) \overset{6)}{=} (\delta_P)_*\big(in_*(x)\big) \ , \end{aligned}$$

where

1) is the definition of the push-forward with respect to δ_X;
2) holds since the diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{in \times in} & P \times P \\ \Big\downarrow{\scriptstyle pr_2} & & \Big\downarrow{\scriptstyle pr_2} \\ X & \xrightarrow{in} & P \end{array}$$

commutes (the definition of δ_X is also applied);

3) is the projection formula for the morphism $in \times in$;
4) $(in \times in)_*(x \times 1) = in_*(x) \times in_*(1)$ and $in_*(1) = [X] = h^n \in \mathrm{CH}^n(P)$;
5) $in_*(x) \times h^n = (in_*(x) \times 1) \cdot (1 \times h^n)$;
6) is the definition of the correspondence δ_P and the definition of the push-forward homomorphism.

□

Corollary 9.3. *For any $N \geq 1$, one has $in_* \circ (\delta_X^{\circ N})_* = (\delta_P^{\circ N})_* \circ in_*$.* □

Corollary 9.4. *Let ϕ_1 and ϕ_2 be minimal neighbors of the same Pfister form π, $X_1 = Q_{\phi_1}$, $X_2 = Q_{\phi_2}$, and $P = Q_\pi$. Put $x_1 := (\rho_1)_*(h^i)$ and $x_2 := (\rho_2)_*(h^i)$, where i is a natural number, while ρ_1 and ρ_2 are the Rost projectors on X_1 and X_2. Then $(in_1)_*(x_1) = (in_2)_*(x_2)$ for the imbeddings $in_1 \colon X_1 \hookrightarrow P$ and $in_2 \colon X_2 \hookrightarrow P$.*

Proof. Put $\delta_1 := (in_1 \times in_1)^*(\mathcal{P})$ and $\delta_2 := (in_2 \times in_2)^*(\mathcal{P})$. Since δ_1 and δ_2 are Rost correspondences, one has $\delta_1^{\circ N_1} = \rho_1$ and $\delta_2^{\circ N_2} = \rho_2$ for some N_1 and N_2. Put $N := N_1 \cdot N_2$. Then $\delta_1^{\circ N} = \rho_1$ and $\delta_2^{\circ N} = \rho_2$. According to Corollary 9.3, we have

$$(in_1)_*(x_1) = \big((in_1)_* \circ (\rho_1)_*\big)(h^i) = \big((in_1)_* \circ (\delta_1^{\circ N})_*\big)(h^i) = \\ = \big((\delta_P^{\circ N})_* \circ (in_1)_*\big)(h^i) = (\delta_P^{\circ N})_*(h^{n+i}) ,$$

because $(in_1)_*(h^i) = h^{n+i}$. Similarly, we have $(in_2)_*(x_2) = (\delta_P^{\circ N})_*(h^{n+i})$. □

Corollary 9.5. *Let ϕ be an anisotropic minimal neighbor of dimension $2^r + 1$, and let ρ the Rost projector on $X = Q_\phi$. Then $\rho_*(h^{2^r - 2^i}) \neq 0$ for all $i = 0, \dots, r$.*

Proof. Let π be the Pfister form where ϕ is a neighbor. Let ϕ' be a norm form which is a neighbor of π and $X' := Q_{\phi'}$. According to Corollary 8.3, $x' := \rho'_*(h^{2^r - 2^i}) \neq 0$, where ρ' is the Rost projector on X'. Therefore, by Lemma 9.1, $in'_*(x') \neq 0$. Since $in'_*(x') = in_*(x)$, where $x := \rho_*(h^{2^r - 2^i})$ (Corollary 9.4), it follows that $in_*(x) \neq 0$, whereby $x \neq 0$. □

Corollary 9.6. *Let X be an arbitrary $(2^r - 1)$-dimensional anisotropic projective quadric possessing a Rost projector ρ. For any $i = 0, 1, \dots, r$, the element $\rho_*(h^{2^r - 2^i}) \in \mathrm{CH}_{2^i - 1}(X)$ is non-zero.*

Proof. Let ϕ be a quadratic form determining X. Since $\dim \phi = 2^r + 1$, there exists a field extension E/F such that ϕ_E is an anisotropic minimal Pfister neighbor ([4, thm. 2]). By Corollary 9.5, the restriction $\rho_*(h^{2^r - 2^i})_E$ of the element $\rho_*(h^{2^r - 2^i})$ is non-zero. Consequently, the element $\rho_*(h^{2^r - 2^i})$ itself is non-zero as well. □

10. Types of correspondences on odd-dimensional quadrics

Let X and Y be some (smooth) projective quadrics of an odd dimension n. We need a description of $\mathrm{CH}^*(\bar{X} \times \bar{Y})$ (cf. the beginning of the proof of Lemma 6.1). One has

$$\mathrm{CH}^*(\bar{X} \times \bar{Y}) = \mathrm{CH}^*(\bar{X}) \otimes \mathrm{CH}^*(\bar{Y}) ,$$

where the inverse isomorphism is given by the outer product of cycles. Furthermore, the groups $\mathrm{CH}^*(\bar{X})$ and $\mathrm{CH}^*(\bar{Y})$ are torsion-free. We give a description of $\mathrm{CH}^*(\bar{X})$ (of course, the same description is valid for $\mathrm{CH}^*(\bar{Y})$ as well). Write, as usually, h for the class in $\mathrm{CH}^1(\bar{X})$ of a hyperplane section of $\bar{X}$ (more precisely, h is defined as the pull-back of the hyperplane class with respect to the embedding of the hypersurface $\bar{X}$ into the projective space). The group $\mathrm{CH}^i(\bar{X})$ is generated by h^i if $i < n/2$ and by $h^i/2$ if $i > n/2$ (the generator $h^i/2$ coincides with the class of a totally isotropic subspace of the appropriate (co)dimension; in particular, $l_0 = h^n/2 \in \mathrm{CH}^n(\bar{X})$.

It follows that $\mathrm{CH}^n(\bar{X} \times \bar{Y})$ is a free abelian group on the generators

$$(h^i \times h^{n-i})/2 \ , \quad i = 0, \dots, n \ .$$

Now take a correspondence $\alpha \in \mathrm{Corr}^0(X, Y)$ and write down it over $\bar{F}$ as

$$\alpha_{\bar{F}} = \sum_{i=0}^{n} a_i (h^i \times h^{n-i})/2$$

with some integers a_i. We refer to the sequence of integers $(a_0, a_1, \dots, a_n)$ as to the *pretype* of α and we refer to the sequence

$$(a_0 \mod 2, \ a_1 \mod 2, \ \dots, \ a_n \mod 2)$$

as to the *type* of α.

Example 10.1. Let $(a_0, \dots, a_n)$ be the pretype of a correspondence $\alpha \in \mathrm{Corr}^0(X, Y)$. The integer a_0 coincides with the *index* (or *degree*) of α over X in the sense of [3, example 16.1.4], while the integer a_n is the index of α over Y.

Example 10.2 ([19, example 1.2])**.** If α is the closure of a graph of a rational morphism $f\colon X \to Y$, then the pretype $(a_0, \dots, a_n)$ of α satisfies the condition $a_0 = 1$.

Lemma 10.3. *For any correspondence $\alpha \in \mathrm{Corr}^0(X, Y)$ there can be found a correspondence $\alpha' \in \mathrm{Corr}^0(X, Y)$ of the same type as α and of the pretype containing only* 0*-s and* 1*-s.*

Proof. For any i, the cycle $h^i \times h^{n-i}$ is defined over F. $\square$

Lemma 10.4. *The pretype (as well as the type) of the diagonal class $\Delta_X \in \mathrm{Corr}^0(X, X)$ is $(1, 1, \dots, 1)$.*

Proof. To show that $\Delta_{\bar{X}} = \sum_{i=0}^{n}(h^i \times h^{n-i})/2$, it suffices to show that the correspondence $\sum_{i=0}^{n}(h^i \times h^{n-i})/2$ is an identity. For this, it suffices to check that it acts (by composition) identically on every $(h^i \times h^{n-i})/2$. The last assertion is a consequence of the fact that the correspondences $(h^i \times h^{n-i})/2$ are orthogonal projectors (cf. [18, lemma 4.3 (1)]). □

Lemma 10.5 (cf. [18, lemma 4.5]). *If a correspondence $\alpha \in \mathrm{Corr}^0(X,Y)$ is of the (pre)type $(a_0,\dots,a_n)$ while a correspondence $\beta \in \mathrm{Corr}^0(Y,X)$ is of the (pre)type $(b_0,\dots,b_n)$, then the composite $\beta\circ\alpha \in \mathrm{Corr}^0(X,X)$ is of the (pre)type $(a_0b_0,\dots,a_nb_n)$.* □

Lemma 10.6. *Assume that $X = Q_\phi$, $\phi = \mathbb{H}\perp\phi'$, and $X' = \phi'$. If X possesses a correspondence of the (pre)type $(a_0,a_1,\dots,a_{n-1},a_n)$, then X' possesses a correspondence of the (pre)type $(a_1,\dots,a_{n-1})$.*

Proof. The motivic decomposition $X \simeq \mathbf{pt} \oplus X'(1) \oplus \mathbf{pt}(n)$ produces the decomposition

$$\mathrm{Corr}^0(X,X) \simeq \mathbb{Z} \times \mathrm{Corr}^0(X',X') \times \mathbb{Z}\ .$$

If $\alpha \in \mathrm{Corr}^0(X,X)$ has the (pre)type $(a_0,\dots,a_n)$, then the projection of α on $\mathrm{Corr}^0(X',X')$ has the required (pre)type $(a_1,\dots,a_{n-1})$. □

Lemma 10.7. *If X is an anisotropic quadric, then the type $(a_0,\dots,a_n)$ of any correspondence on X satisfies the relation $\sum_{i=0}^{n} a_i = 0 \in \mathbb{Z}/2$.*

Proof. Consider the pull-back $\Delta_X^*\colon \mathrm{Corr}^0(X,X) = \mathrm{CH}^n(X\times X) \to \mathrm{CH}^n(X) = \mathrm{CH}_0(X)$ with respect to the diagonal morphism $\Delta_X : X \hookrightarrow X\times X$. If $\alpha \in \mathrm{Corr}^0(X,X)$ and $\alpha_{\bar{F}} = \sum_{i=0}^{n} a_i(h^i \times h^{n-i})/2$, then $\Delta_X^*(\alpha)$ is a 0-cycle on X of the degree $\sum_{i=0}^{n} a_i$. According to the Springer theorem ([22, thm. 7.2.3]), degree of any 0-cycle on an anisotropic quadric is even. □

Proposition 10.8. *Conjecture 1.6 holds for the quadratic forms of dimension 5.*

Proof. Let ϕ be a 5-dimensional quadratic form and $X = Q_\phi$. If X possesses a Rost projector ρ, then $l_0 \in K_0(X)$ by Corollary 7.4. It follows that the even Clifford algebra $C_0(\phi)$ of the form ϕ is not a skewfield ([31], see also [15, §3]) and so ϕ is a Pfister neighbor. □

Corollary 10.9. *If ϕ is a 5-dimensional quadratic form with* $\operatorname{ind} C_0(\phi) = 4$, *then any correspondence on $X = Q_\phi$ is of the type $(0,0,0,0)$ or $(1,1,1,1)$.*

Proof. Let (a_0,a_1,a_3,a_4) be the type of some correspondence α on X. Since the Witt index of the form ϕ over the function field $F(X)$ equals 1, it follows (see Lemmas 10.6 and 10.7) that $a_1 = a_2$. Replacing, if necessary, α by $\Delta_X - \alpha$ (Lemma 10.4), we come to the situation where $a_1 = a_2 = 0$. By Lemma 10.7, it follows that $a_0 = a_3$. If $a_0 = a_3 = 1$, then, by Lemma 10.3, X possesses a Rost correspondence what contradicts to Proposition 10.8. Therefore $a_0 = a_3 = 0$ and the corollary is proved. □

Proposition 10.10. *Conjecture 1.6 holds for the quadratic forms ϕ of dimension 7 such that the form $\phi\perp\langle\det\phi\rangle$ is anisotropic.*

Proof. Let E/F be the function field of the Severi-Brauer variety of the algebra $C_0(\phi)$. The form $\pi := (\phi\perp\langle\det\phi\rangle)_E$ is anisotropic (for the case where $\operatorname{ind} C_0(\phi) = 8$ see [2], [11, cor. 0.3], or [19]; for the case where $\operatorname{ind} C_0(\phi) \leq 4$ see [21, thm. 4]), whereby ϕ_E is anisotropic as well. Since the form π is similar to a 3-fold Pfister form, the quadric $P := Q_\pi$ possesses a correspondence $\mathcal{P} \in \operatorname{CH}^3(P\times P)$ as in the beginning of section 9. Taking the pull-back of $\mathcal{P}$ to $X_E\times X_E$ and multiplying the result by $1\times h^3$, we get a degree 0 correspondence on X_E having the type $(1,0,0,1,0,0)$.

Now assume that X possesses a Rost correspondence. Passing to E and composing it with the correspondence constructed right above, we get a correspondence on X_E of the type

$$(1,0,0,1,0,0)\cdot(1,0,0,0,0,1) = (1,0,0,0,0,0)\ .$$

In view of Lemma 10.7, the existence of a correspondence of such type on X_E contradicts to the anisotropy of X_E. □

Corollary 10.11. *If ϕ is a quadratic form of dimension 7 such that the index of its even Clifford algebra $C_0(\phi)$ is at least 4 and the form $\phi\perp\langle\det\phi\rangle$ is anisotropic, then any correspondence on $X = Q_\phi$ is either of the type $(1,1,\dots,1)$ or of the type $(0,0,\dots,0)$.*

Proof. Let α be an arbitrary correspondence on X and let $(a_0,\dots,a_5)$ be its type. Note that by [5, thm. 4.1] (since $\operatorname{ind} C_0(\phi) > 1$) the anisotropic part of the form $\phi_{F(X)}$ has the dimension 5. According to the index reduction formula for quadrics [23, thm. 1], this is a 5-dimensional form satisfying the condition of Corollary 10.9 on the Schur index of its even Clifford algebra. Therefore, taking Lemma 10.6 in account, we see that $a_1 = a_2 = a_3 = a_4$. Replacing eventually α by $\Delta_X - \alpha$, we reduce our consideration to the case where $a_1 = a_2 = a_3 = a_4 = 0$. Now, by Lemma 10.7, we should have $a_0 = a_5$. If $a_0 = a_5 = 1$, then, by Lemma 10.3, X possesses a Rost correspondence, what contradicts to Proposition 10.10. Consequently, $a_0 = a_5 = 0$ and the corollary is proved. □

Definition 10.12. An anisotropic 9-dimensional quadratic form ϕ is called *essential*, if it is not a Pfister neighbor and $\operatorname{ind} C_0(\phi) \geq 4$.

Lemma 10.13. *Let ϕ be an essential form. Then the anisotropic part of the form $\phi_{F(\phi)}$ is a 7-dimensional quadratic form satisfying the conditions of Corollary 10.11.*

Proof. The dimension of the anisotropic part is 7 according to [5, thm. 4.1]. The index of the even Clifford algebra is at least 4 according to [23, thm. 1].

It remains to show that $\dim(\psi_{F(\phi)})_{\mathrm{an}} = 8$, where $\psi := \phi\perp\langle-\det\phi\rangle$. The quadratic form ψ is a 10-dimensional form of trivial discriminant and is not a Pfister neighbor (since ϕ is not a Pfister neighbor). Therefore, by [5, thm.

5.1], $\dim(\psi_{F(\psi)})_{\mathrm{an}} = 8$. Since $\dim\phi = 9 > 8$, [4, thm. 1] tells us that the form $(\psi_{F(\psi)})_{\mathrm{an}}$ remains anisotropic over the field $F(\psi)(\phi)$. Since $F(\psi)(\phi) \supset F(\phi)$, there is an inequality $\dim(\psi_{F(\psi)(\phi)})_{\mathrm{an}} \leq \dim(\psi_{F(\phi)})_{\mathrm{an}}$, and we are done. □

Corollary 10.14. *Let ϕ be an essential (9-dimensional) quadratic form. Then the type of any correspondence on $X := Q_\phi$ is $(1, 1, \dots, 1)$ or $(0, 0, \dots, 0)$.*

Proof. Let $(a_0, \dots, a_7)$ be the type of a correspondence on X. Passing to the field $F(X)$ and using Corollary 10.11 (take in account Lemma 10.13) with Lemma 10.6, we see that $a_1 = \dots = a_6$. We may assume that $a_1 = \dots = a_6 = 0$. Then $a_0 = a_7$ by Lemma 10.7. If $a_0 = a_7 = 1$, then, by Lemma 10.3, X possesses a Rost correspondence, what contradicts to Theorem 1.7, since ϕ is not a Pfister neighbor. Consequently, $a_0 = a_7 = 0$, and we are done. □

11. Isotropy of essential forms

The aim of this section is Theorem 1.13. For the beginning we formulate several facts we need to prove this theorem.

Theorem 11.1 ([8]). *Let ϕ be an anisotropic quadratic form of dimension $2^r + 1$ with some $r \geq 1$ and let ψ be a quadratic form with $\dim\psi \geq \dim\phi$. If the form $\phi_{F(\psi)}$ is isotropic, then the form $\psi_{F(\phi)}$ is as well isotropic.*

Theorem 11.2 ([34, statement 1.4.1], [18, criterion 0.1]). *Let ϕ and ψ be quadratic forms of the same dimension. The motives of the quadrics Q_ϕ and Q_ψ are isomorphic if and only if for any field extension E/F the forms ϕ_E and ψ_E have the same Witt index.*

Theorem 11.3 ([7]). *Let ϕ and ψ be quadratic forms of the same odd dimension. If over any field extension E/F the forms ϕ_E and ψ_E have the same Witt index, then ϕ and ψ are similar.*

Proof of Theorem 1.13. Assume that the form ϕ becomes isotropic over the function field $F(\psi)$. Then there exists a rational morphism $Q_\psi \to Q_\phi$; denote by $\alpha \in \mathrm{Corr}^0(Q_\psi, Q_\phi)$ the closure of its graph. The type $(a_0, \dots, a_7)$ of the correspondence α (see section 10) satisfies the condition $a_0 = 1$ (cf. Example 10.2).

By Theorem 11.1, the form ψ also becomes isotropic over $F(\phi)$; let $\beta \in \mathrm{Corr}^0(Q_\phi, Q_\psi)$ the the closure of the graph of a rational morphism $Q_\phi \to Q_\psi$. Its type $(b_0, \dots, b_7)$ satisfies the condition $b_0 = 1$.

The composite $\alpha \circ \beta$ is a correspondence on Q_ϕ of the type $(c_0, \dots, c_7) = (a_0 b_0, \dots, a_7 b_7)$. Since $c_0 = a_0 b_0 = 1$, we have $c_0 = c_1 = \dots = c_7 = 1$ by Corollary 10.14. Therefore, $a_0 = a_1 = \dots = a_7 = 1$ and $b_0 = b_1 = \dots = b_7 = 1$. Together with Lemma 10.3 this implies that there exist $f \in \mathrm{Corr}^0(Q_\psi, Q_\phi)$ and $g \in \mathrm{Corr}^0(Q_\phi, Q_\psi)$ such that

$$f_{\bar{F}} = \sum_{i=0}^{7} (h^i \times h^{n-i})/2 \quad \text{and} \quad g_{\bar{F}} = \sum_{i=0}^{7} (h^i \times h^{n-i})/2\ ,$$

so that $f_{\bar{F}}$ and $g_{\bar{F}}$ are mutually inverse isomorphisms of the motives of the quadrics over $\bar{F}$. It follows by Corollary 4.3 that the motives of Q_ϕ and Q_ψ are isomorphic (over F already). With Theorems 11.2 and 11.3 we conclude that the forms ϕ and ψ are similar. □

Remark 11.4. Using the similar technique one may show that an essential form ϕ remains anisotropic over the function field of any quadratic form ψ with $\dim\psi \geq 10$. However this statement is deduced from Theorem 1.13 in [10]. By that reason we do not prove it here.

Appendix. Chow groups of affine norm quadrics

The aim of this appendix is Theorem A.4, which is needed in the proof of Corollary 8.3. This theorem and some ideas of the proof was communicated by A. A. Suslin at a seminar of LOMI[2] in 1990 with a comment that this theorem is due to M. Rost and is used by M. Rost in his proof of Theorem 8.2. The proof of Theorem A.4 given here is a reconstruction of the proof described by A. A. Suslin, undertaken in 1990 by the author with a participation of A. I. Panin.

Definition A.1. Let π be a Pfister form over F and let $c \in F$. The affine quadric U given by the equation $\pi = c$ is called an *affine norm quadric*. The affine norm quadric U is called *degenerate*, if $c = 0$; otherwise U is called *non-degenerate*.

Definition A.2. Let ϕ be a quadratic form over F (in this definition we allow ϕ to be degenerate!). For every $n \geq 0$, we define a group $D_n(\phi)$ as in [20, §4] (see also [20, lemma 4.2]). In particular,

$$D_0(\phi) = \begin{cases} 2\mathbb{Z}, & \text{if } \phi \text{ is anisotropic;} \\ \mathbb{Z}, & \text{if } \phi \text{ is isotropic,} \end{cases}$$

and $D_1(\phi)$ is the subgroup of F^* generated by the norms $N_{E/F}(E^*)$ for all field extensions E/F such that the form ϕ_E is isotropic.

Remark A.3 (cf. [20, rem. of §4])**.** There are two variants of the definition of $D_n(\phi)$ for higher values of n: $D_n(\phi)$ is defined either as a subgroup of the Quillen K-group $K_n(F)$, or as a subgroup of the Milnor K-group $K_n^{\mathrm{M}}(F)$. The both choices are admissible for us here. If the Quillen K-groups are chosen, then they have to be used in all calculations below, i.e., one should understand everywhere under the K-cohomology group the Quillen ones; if the Quillen K-groups are replaced by the Milnor ones in the definition of $D_n(\phi)$, then this replacement should be made everywhere (cf. [20, examples 2.1 and 2.2]).

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Theorem A.4 (M. Rost). *Let U be a non-degenerate affine norm quadric $\pi = c$. Then*

$$\mathrm{CH}^i(U) = \begin{cases} \mathbb{Z} & \text{for } i = 0; \\ 0 & \text{for } 0 < i < \dim U; \\ \dfrac{D_0(\pi \perp \langle -c\rangle)}{D_0(\pi)} & \text{for } i = \dim U. \end{cases}$$

In particular, if the form $\pi \perp \langle -c\rangle$ is anisotropic, then $\mathrm{CH}^i(U) = 0$ for all $i > 0$.

Proof. For $i = 0$, the statement is trivial; for the statement on $i = \dim U$ see [20, lemma 4.1]. The proof of the theorem for the intermediate values of i will be made by induction on the "foldness" of π. Note that in the case of a 1-fold Pfister form there are no intermediate i-s, so that we have no problem with the base of the induction.

Now assume that the foldness of π is greater that 1 and write down π as $\langle\langle a\rangle\rangle\, \chi = \chi \perp -a\chi$, where χ is a Pfister form and $a \in F^*$. Let V be the vector space of definition of π and let $V = V' \oplus V''$ be the direct decomposition corresponding to the decomposition $\pi = \chi \perp -a\chi$. The composite of the imbedding $U \hookrightarrow \mathbb{A}(V)$ of U in the affine space $\mathbb{A}(V)$ (where U seats as a hypersurface) with the projection $\mathbb{A}(V) \to \mathbb{A} := \mathbb{A}(V'')$ is a flat morphism $f : U \to \mathbb{A}$ such that its fiber U_x over a point $x \in \mathbb{A}$ is the norm quadric (possibly degenerate) $\chi_{F(x)} = a\chi(x) + c$ over the residue field $F(x)$, where χ on the right-hand side of the equation is considered as a rational function on $\mathbb{A}$ (so that the expression $\chi(x)$ makes a sense and determines an element of $F(x)$). Note that $\dim \mathbb{A} = k$, $\dim U = 2k - 1$, and $\dim U_x = k - 1$, where $k := \dim \chi$.

For an integer n, let us consider the spectral sequence

$$E_1^{p,q}(n) = \coprod_{x \in (\mathbb{A})^p} H^q(X, K_{n-p})$$

of [20, thm. 3.1] related with the morphism f and "calculating" the K-cohomology groups $H^{p+q}(U, K_n)$. Note that $E_1^{p,q}(n)$ is concentrated in the area $0 \le p \le k$, $0 \le q \le k-1$, $p+q \le n$. The terms on the diagonal $p+q = n$, which is "responsible" for $\mathrm{CH}^{p+q}(U) = H^{p+q}(U, K_{p+q})$, look as follows:

$$E_1^{p,q}(p+q) = \coprod_{x \in (\mathbb{A})^p} \mathrm{CH}^q(U_x)\ .$$

We **claim** that $E_\infty^{p,q}(p+q) = 0$, if $0 < p+q < 2k-1$ (and therefore $\mathrm{CH}^{p+q}(U) = 0$, q.e.d.). First of all we prove this for $q = 0$, i.e., we prove that $E_\infty^{p,0}(p) = 0$ for any positive p. For this, it suffices to show that the homomorphism $E_1^{p,0}(p) \to \mathrm{CH}^p(U)$ is 0. And indeed, the image of this homomorphism coincides with the image of the pull-back homomorphism $f^* : \mathrm{CH}^p(\mathbb{A}) \to \mathrm{CH}^p(U)$, which is 0 since the group $\mathrm{CH}^p(\mathbb{A})$ is so (for any positive p).

Now consider the case where $0 < q < k-1$. If the the fiber U_x is a *non-degenerate* affine norm quadric, then $\mathrm{CH}^q(U_x) = 0$ by the induction hypotheses. Thus, we only have to "struggle" with the degenerate fibers.

Let U_x be a degenerate fiber over a p-codimensional point $x \in (\mathbb{A})^p$. Any element in the image of the homomorphism $\mathrm{CH}^q(U_x) \to \mathcal{F}^{(p)}/\mathcal{F}^{(p+1)}$, where $\mathcal{F}$ is the filtration on $\mathrm{CH}^{p+q}(U)$ associated with the spectral sequence, can be then represented by a cycle on $U' \times U''$, where U' is the affine quadric $\chi = 0$ while U'' is the affine quadric $-a\chi = c$ (note that $U' \times U''$ is a closed subvariety of U).

Lemma A.5. *The pull-back* $pr_1^* \colon \mathrm{CH}^n(U') \to \mathrm{CH}^n(U' \times U'')$ *with respect to the projection* $pr_1 \colon U' \times U'' \to U'$ *is surjective (for any n).*

Proof. Consider the spectral sequence $E(n)$ associated with the morphism pr_1. The term $E_1^{n-i,i}(n) = \coprod_{x \in (U')^{n-i}} \mathrm{CH}^i(U''_{F(x)})$ is zero for any positive i, because the affine quadric $U''_{F(x)}$ is determined by the equation $\chi_{F(x)} = -c/a$ where the form $\chi_{F(x)}$ is hyperbolic. Therefore, the only (possibly) non-zero term on the responsible for $\mathrm{CH}^n(U' \times U'')$ diagonal is the term $E_1^{n,0}(n)$. Since the image of this term in $\mathrm{CH}^n(U' \times U'')$ always coincides with the image of pr_1^*, we are done. □

Using this lemma, we may represent any cycle on U, coming from $U' \times U''$, as $\alpha \times U''$, where α is a cycle on U'. However $\alpha \times U''$ is the pull-back of α with respect to the composite

$$U \hookrightarrow \mathbb{A}(V) = \mathbb{A}(V') \times \mathbb{A}(V'') \xrightarrow{pr_1} \mathbb{A}(V')$$

(here we consider α as a cycle on $\mathbb{A}(V') \supset U'$; the statement is obtained from the cartesian square

$$\begin{array}{ccc} U' \times U'' & \hookrightarrow & U \\ \downarrow & & \downarrow \\ U' & \hookrightarrow & \mathbb{A}(V') \end{array}$$

by applying [3, prop. 1.7]). Since $\mathrm{CH}^n(\mathbb{A}(V')) = 0$ for positive n-s, we have finished the proof of our claim for $q < k-1$.

It remains to handle the case $q = k-1$, i.e., to show that the term $E_1^{p,k-1} = \coprod_{x \in (\mathbb{A})^p} \mathrm{CH}^{k-1} U_x$ becomes 0 by passing to the limit.

First of all note that for $p = 0$ the group $E_1^{0,k-1} = \dfrac{D_0(\chi_{F(\mathbb{A})} \perp \langle -a\chi(x) - c \rangle)}{D_0(\chi)}$, where $x \in (\mathbb{A})^0$ is the generic point, is zero: if the form χ is anisotropic while the form $\chi_{F(\mathbb{A})} \perp \langle -a\chi(x) - c \rangle$ is isotropic, then the anisotropic form $\chi_{F(\mathbb{A})}$ represents the element $a\chi(x) + c \in F(\mathbb{A})$ and therefore, by the "third representation theorem" [22, thm. 9.2.8], the form χ contains a subform isomorphic to the form $a\chi \perp \langle c \rangle$, what is impossible by the dimension reason.

Thus, we need to consider only the case where $p > 0$. We claim that in fact already $E_2^{p,k-1} = 0$. This $E_2^{p,k-1}$ is the cokernel of the E_1-differential

$$E_1^{p-1,k-1} = \coprod_{x \in (\mathbb{A})^{p-1}} H^{k-1}(U_x, K_k) \longrightarrow E_1^{p,k-1} = \coprod_{x \in (\mathbb{A})^p} \mathrm{CH}^{k-1}(U_x)\,.$$

The commutative diagram

$$\begin{array}{ccc}
\coprod_{x\in(\mathbb{A})^{p-1}} H^{k-1}(U_x, K_k) & \longrightarrow & \coprod_{x\in(\mathbb{A})^{p}} \mathrm{CH}^{k-1}(U_x) \\
\big\downarrow \text{onto} & & \big\downarrow \wr \\
\coprod_{x\in(\mathbb{A})^{p-1}} \dfrac{D_1(\chi_{F(x)}\perp\langle -a\chi(x)-c\rangle)}{D_1(\chi_{F(x)})} & \longrightarrow & \coprod_{x\in(\mathbb{A})^{p}} \dfrac{D_0(\chi_{F(x)}\perp\langle -a\chi(x)-c\rangle)}{D_0(\chi_{F(x)})} \\
\big\uparrow & & \big\uparrow \text{onto} \\
\coprod_{x\in(\mathbb{A})^{p-1}} D_1(\chi_{F(x)}\perp\langle -a\chi(x)-c\rangle) & \longrightarrow & \coprod_{x\in(\mathbb{A})^{p}} D_0(\chi_{F(x)}\perp\langle -a\chi(x)-c\rangle)
\end{array}$$

(see [20, lemma 4.1] for an explanation on the upper vertical arrows) shows that the cokernel we are interested in (i.e., the cokernel of the upper horizontal arrow) coincides with the cokernel of the middle horizontal arrow and is a homomorphic image of the cokernel of the lower horizontal arrow. The cokernel of the lower horizontal arrow is (by definition, see [20]) the D-cohomology group $H^p(\mathbb{A}, D_p(\chi\perp\langle -a\chi-c\rangle))$, and to finish the proof of Theorem A.4 is suffices to show that

$$H^p(\mathbb{A}, D_p(\chi\perp\langle -a\chi-c\rangle)) = 0 \quad \text{for all } 0<p<k.$$

This is a particular case of the following

Proposition A.6. *Let χ be a Pfister form over F; η a subform of χ of a dimension $k\geq 1$; $\mathbb{A}$ the affine space of the vector space, where η is defined; $a, b\in F$ and $a\neq 0$. Then*

$$H^p(\mathbb{A}, D_p(\chi\perp -a\langle \eta+b\rangle)) =$$
$$= \begin{cases} D_0(\chi\perp\langle -a\rangle)\cap D_0(\eta'\perp\langle -b\rangle)+D_0(\chi) & \text{for } p=0; \\ 0 & \text{for } 0<p<k; \\ \dfrac{D_0(\chi\perp -a(\eta\perp\langle b\rangle))}{D_0(\chi\perp\langle -a\rangle)} & \text{for } p=k, \end{cases}$$

where η' is the subform of χ complement to η, i.e., $\chi=\eta\perp\eta'$.

Proof. First we consider the case of $p=0$. We evidently have

$$H^0(\mathbb{A}, D_0(\chi\perp -a\langle\eta+b\rangle)) = D_0(\chi_{F(\mathbb{A})}\perp -a\langle \eta(x)+b\rangle)\ ,$$

where x is the generic point of $\mathbb{A}$. For $b=0$ we get

$$D_0(\chi_{F(\mathbb{A})}\perp -a\langle\eta(x)\rangle) = D_0((\chi\perp\langle -a\rangle)_{F(\mathbb{A})}) = D_0(\chi\perp\langle -a\rangle)$$

(because $\chi_{F(\mathbb{A})}\perp -a\langle\eta(x)\rangle$ and $(\chi\perp\langle -a\rangle)_{F(\mathbb{A})}$ are neighbors of the same Pfister form $\chi\langle\langle a\rangle\rangle_{F(\mathbb{A})}$ and because the extension $F(\mathbb{A})/F$ is purely transcendental) and

$$D_0(\chi\perp\langle -a\rangle) = D_0(\chi\perp\langle -a\rangle)\cap D_0(\eta'\perp\langle -b\rangle)+D_0(\chi)$$

(since $D_0(\chi) \subset D_0(\chi\perp\langle -a\rangle)$) and $D_0(\eta'\perp\langle -b\rangle) = \mathbb{Z}$ for $b = 0$). For $b \neq 0$, the group $D_0(\chi_{F(\mathbb{A})}\perp - a\langle \eta(x) + b\rangle)$ coincides with the answer of the proposition according to the following

Lemma A.7. *The quadratic $F(\mathbb{A})$-form $\chi_{F(\mathbb{A})}\perp - a\langle\eta(x) + b\rangle$ (with $b \neq 0$) is isotropic if and only if the F-form χ is isotropic or the two F-forms $\chi\perp\langle -a\rangle$ and $\eta'\perp\langle -b\rangle$ are simultaneously isotropic.*

Proof. If the form χ is isotropic, then, clearly, the form $\chi_{F(\mathbb{A})}\perp - a\langle\eta(x) + b\rangle$ is isotropic as well.

If the forms $\chi\perp\langle -a\rangle$ and $\eta'\perp\langle -b\rangle$ are isotropic, then χ represents a, while η' represents b. Since $\chi = \eta\perp\eta'$, the latter condition implies that the form $\chi_{F(\mathbb{A})}$ represents the element $\eta(x) + b$. By the multiplicative property [22, cor. 10.1.7] of the Pfister form $\chi_{F(\mathbb{A})}$ we conclude that it represents the element $a(\eta(x) + b)$, i.e., the form $\chi_{F(\mathbb{A})}\perp - a\langle\eta(x) + b\rangle$ is isotropic.

To prove the inverse implication, assume that the form $\chi_{F(\mathbb{A})}\perp - a\langle\eta(x) + b\rangle$ is isotropic, while the form χ is anisotropic. Then, by the third representation theorem [22, thm. 9.2.8], $a(\eta\perp\langle b\rangle)$ is a subform of χ. In particular, $a\eta$ is a subform of χ and therefore $\chi_{F(\mathbb{A})}$ represents the element $a\eta(x)$. Since $\chi_{F(\mathbb{A})}$ also represents $\eta(x)$ (because η is a subform of χ), we conclude by the multiplicative property [22, cor. 10.1.7] of the Pfister form $\chi_{F(\mathbb{A})}$ that it represents a, i.e., the form $\chi\perp\langle -a\rangle$ is isotropic over $F(\mathbb{A})$. Since the field extension $F(\mathbb{A})/F$ is purely transcendental, this form should be already isotropic over F.

It remains to show that the form η' represents b. Since $\chi_{F(\mathbb{A})}$ represents the elements a and $a(\eta(x) + b)$, it also represents $\eta(x) + b$, whereby $\eta\perp\langle b\rangle$ is a subform of $\chi = \eta\perp\eta'$. Cancelling by η, we get that $\langle b\rangle$ is a subform of η', q.e.d. $\square$

We continue the proof of Proposition A.6. By now, we have already finished the case $p = 0$. Let us prove the opposite border case of $p = k$. The D-cohomology group $H^k(\mathbb{A}, D_k(\chi\perp - a\langle\eta + b\rangle))$ we have to calculate is the cokernel of the upper horizontal arrow in the commutative square

$$\begin{array}{ccc}
\coprod_{x\in(\mathbb{A})^{k-1}} D_1(\chi_{F(x)}\perp - a\langle\eta(x) + b\rangle) & \longrightarrow & \coprod_{x\in(\mathbb{A})^{k}} D_0(\chi_{F(x)}\perp - a\langle\eta(x) + b\rangle) \\
\uparrow & & \uparrow \\
\coprod_{x\in(\mathbb{A})^{k-1}} D_1(\chi_{F(x)}) & \longrightarrow & \coprod_{x\in(\mathbb{A})^{k}} D_0(\chi_{F(x)})
\end{array}$$

Since the lower horizontal arrow is surjective ([20, lemma 4.9]), this cokernel coincides with the cokernel of the upper horizontal arrow in the commutative square

$$\begin{array}{ccc}
\coprod_{x\in(\mathbb{A})^{k-1}} \dfrac{D_1(\chi_{F(x)}\perp -a\langle\eta(x)+b\rangle)}{D_1(\chi_{F(x)})} & \longrightarrow & \coprod_{x\in(\mathbb{A})^{k}} \dfrac{D_0(\chi_{F(x)}\perp -a\langle\eta(x)+b\rangle)}{D_0(\chi_{F(x)})} \\
\uparrow & & \uparrow \\
\coprod_{x\in(\mathbb{A})^{k-1}} H_0(U_{(\chi_{F(x)}=a(\eta(x)+b))}, K_1) & \longrightarrow & \coprod_{x\in(\mathbb{A})^{k}} \mathrm{CH}_0(U_{(\chi_{F(x)}=a(\eta(x)+b))})
\end{array}$$

(see section 3 for an explanation on the lower indices in CH_* and see [28] or [20, §2] for an explanation on the lower indices in H_*), where $U_{(*)}$ stays for the affine quadric determined by the equation $(*)$. Since the right-hand side vertical arrow is an isomorphism while the left-hand side vertical arrow is an epimorphism ([20, lemma 4.1]), the cokernel of the upper horizontal arrow in this square coincides with the cokernel of the lower horizontal arrow. The lower horizontal arrow however is an E_1-differential of the spectral sequence associated with the morphism $U_{(\chi\perp -a\eta=ab)} \to \mathbb{A}$. Therefore, as one sees by passing to the limit, its cokernel is equal to

$$\mathrm{CH}_0(U_{(\chi\perp -a\eta=ab)}) = \frac{D_0(\chi\perp -a(\eta\perp\langle b\rangle))}{D_0(\chi\perp -a\eta)} .$$

Since the forms $\chi\perp -a\eta$ and $\chi\perp\langle -a\rangle$ are neighbors of the same Pfister form $\chi\langle\langle a\rangle\rangle$, we may replace the denominator by $D_0(\chi\perp\langle -a\rangle)$.

We have finished the proof of Proposition A.6 for the border values of p. The proof for the intermediate values goes by an induction on k. The induction base $k=1$ has no intermediate values.

Assume that $k>1$. Let us decompose η in a direct sum $\tilde\eta\perp\langle c\rangle$ and consider the corresponding decomposition $\mathbb{A}=\tilde{\mathbb{A}}\times\mathbb{A}^1$, where $\tilde{\mathbb{A}}$ is the affine space of the vector space of definition of the quadratic form $\tilde\eta$. The following exact sequence is produced by the spectral sequence associated with the projection $\mathbb{A}\to\mathbb{A}^1$:

$$\begin{aligned}
&H^{p-1}(\tilde{\mathbb{A}}_{F(\mathbb{A}^1)}, D_p(\chi_{F(\mathbb{A}^1)}\perp -a\langle\tilde\eta_{F(\mathbb{A}^1)}+(cx^2+b)\rangle))\to\\
&\quad\to \coprod_{x\in(\mathbb{A}^1)^1} H^{p-1}(\tilde{\mathbb{A}}_{F(x)}, D_{p-1}(\chi_{F(x)}\perp -a\langle\tilde\eta_{F(x)}+(cx^2+b)\rangle))\to\\
&\qquad\to H^p(\mathbb{A}, D_p(\chi\perp -a\langle\eta+b\rangle))\to\\
&\qquad\to H^p(\tilde{\mathbb{A}}_{F(\mathbb{A}^1)}, D_p(\chi_{F(\mathbb{A}^1)}\perp -a\langle\tilde\eta_{F(\mathbb{A}^1)}+(cx^2+b)\rangle))\to 0 .
\end{aligned}$$

For $p\neq k-1$, the right-hand side term $H^p(\tilde{\mathbb{A}}_{F(\mathbb{A}^1)}, D_p(\dots))$ is zero by the induction hypotheses, because $\dim\tilde{\mathbb{A}}=k-1$. Let us show that for $p=k-1$ this term is zero as well. By the already proved part of Proposition A.6, this term is equal to

$$\frac{D_0(\chi_{F(\mathbb{A}^1)}\perp -a(\tilde\eta_{F(\mathbb{A}^1)}\perp\langle cx^2+b\rangle))}{D_0(\chi\perp\langle -a\rangle)} .$$

To check that the quotient is zero, one has to check that the isotropy of the quadratic form in the numerator implies the isotropy of the form in the denominator.

So, assume that the form $\chi_{F(\mathbb{A}^1)}\perp -a(\tilde{\eta}_{F(\mathbb{A}^1)}\perp \langle cx^2+b\rangle)$ is isotropic, i.e., the form $\chi\perp -a\tilde{\eta}$ represents the element $a(cx^2+b)$ over $F(\mathbb{A}^1)$. If the form $\chi\perp -a\tilde{\eta}$ is already isotropic, then $\chi\perp\langle -a\rangle$ is isotropic, because these two forms are neighbors of the same Pfister form $\chi\,\langle\langle a\rangle\rangle$. Thus, we may assume that the form $\chi\perp -a\tilde{\eta}$ is anisotropic and apply the "third representation theorem". This way we get (even if $b=0$) that ac is a value of $\chi\perp -a\tilde{\eta}$, i.e., the form

$$\chi\perp -a\tilde{\eta}\perp -a\,\langle c\rangle = \chi\perp -a\eta$$

is isotropic. Since $\chi\perp -a\eta\subset\chi\,\langle\langle a\rangle\rangle$, the Pfister form $\chi\,\langle\langle a\rangle\rangle$ turns out to be isotropic, what implies once again the isotropy of $\chi\perp\langle -a\rangle$ (the computation of the D-cohomology group involved here makes use of the computation of the K-cohomology of the affine line, see [25] or [28]).

We have shown that the right-hand side term of the exact sequence is always zero. In order to finish the proof of Proposition A.6, we show that the cokernel of the left-hand side arrow is always zero as well. For any $p\neq 1$, this holds simply by the induction hypotheses. For $p=1$, this arrow is the upper side of the following commutative square

$$\begin{array}{ccc}
F(\mathbb{A}^1)^*\cap D_1(\chi_{F(\mathbb{A})}\perp -a\,\langle\tilde{\eta}(y)+(cx^2+b)\rangle)\to & & \\
\quad\to \coprod_{x\in(\mathbb{A}^1)^1} D_0(\chi_{F(x)}\perp\langle -a\rangle)\cap D_0(\tilde{\eta}'_{F(x)}\perp\langle -cx^2-b\rangle)+D_0(\chi_{F(x)}) & & \\
\uparrow\text{injection} & & \uparrow\text{injection} \\
D_1(\chi_{F(\mathbb{A}^1)}) & \longrightarrow & \coprod_{x\in(\mathbb{A}^1)^1} D_0(\chi_{F(x)})
\end{array}$$

where y is the generic point of $\tilde{\mathbb{A}}$ and $\tilde{\eta}':=\eta'\perp\langle c\rangle$ is the complement of $\tilde{\eta}$ in χ. Since the lower horizontal arrow is surjective ([20, lemma 4.9]), it suffices to show that the homomorphism of cokernels of the vertical arrows is surjective. This homomorphism is the upper side of the following commutative square

$$\begin{array}{ccc}
\dfrac{F(\mathbb{A}^1)^*\cap D_1(\chi_{F(\mathbb{A})}\perp -a\,\langle\tilde{\eta}(y)+(cx^2+b)\rangle)}{D_1(\chi_{F(\mathbb{A}^1)})}\to & & \\
\quad\to\coprod_{x\in(\mathbb{A}^1)^1}\dfrac{D_0(\chi_{F(x)}\perp\langle -a\rangle)\cap D_0(\tilde{\eta}'_{F(x)}\perp\langle -cx^2-b\rangle)}{D_0(\chi_{F(x)})\cap D_0(\tilde{\eta}'_{F(x)}\perp\langle -cx^2-b\rangle)} & & \\
\uparrow\alpha & & \beta\uparrow\text{onto} \\
H_0(U_{(\chi_{F(\mathbb{A}^1)}=a)}\times U_{(\tilde{\eta}'_{F(\mathbb{A}^1)}=cx^2+b)},K_1) & \to & \coprod_{x\in(\mathbb{A}^1)^1}\mathrm{CH}_0(U_{(\chi_{F(x)}=a)}\times U_{(\tilde{\eta}'_{F(x)}=cx^2+b)})
\end{array}$$

The two lemmas below explain the existence of α and the existence (and surjectivity) of β.

Lemma A.8. *Let φ_1 and φ_2 be two quadratic forms over F and $a_1, a_2 \in F$. Assume that φ_1 and φ_2 become isotropic over a common quadratic field extension of F. Then there is an epimorphism*

$$\beta\colon \mathrm{CH}_0(U_{(\varphi_1=a_1)} \times U_{(\varphi_2=a_2)}) \twoheadrightarrow \frac{D_0(\varphi_1 \perp \langle -a_1\rangle) \cap D_0(\varphi_2 \perp \langle -a_2\rangle)}{D_0(\varphi_1) \cap D_0(\varphi_2 \perp \langle -a_2\rangle) + D_0(\varphi_1 \perp \langle -a_1\rangle) \cap D_0(\varphi_2)} .$$

Moreover, if at least one of the forms φ_1 and φ_2 is isotropic, then

$$\mathrm{CH}_0(U_{(\varphi_1=a_1)} \times U_{(\varphi_2=a_2)}) = 0 .$$

Proof. Set $X_1 := Q_{\varphi_1 \perp \langle -a_1\rangle}$, $X_2 := Q_{\varphi_2 \perp \langle -a_2\rangle}$, $Y_1 := Q_{\varphi_1}$, $Y_2 := Q_{\varphi_2}$, $U_1 := U_{(\varphi_1=a_1)}$, and $U_2 := U_{(\varphi_2=a_2)}$. Note that $U_1 \simeq X_1 \setminus Y_1$ and $U_2 \simeq X_2 \setminus Y_2$.

Let us check that the image of the degree homomorphism

$$\deg\colon \mathrm{CH}_0(X_1 \times X_2) \to \mathbb{Z}$$

coincides with the numerator $D_0(\varphi_1 \perp \langle -a_1\rangle) \cap D_0(\varphi_2 \perp \langle -a_2\rangle)$. It is clear that $\operatorname{Im}\deg$ is contained in the numerator (because the residue field of any point on $X_1 \times X_2$ makes the both forms $\varphi_1 \perp \langle -a_1\rangle$ and $\varphi_2 \perp \langle -a_2\rangle$ isotropic). Moreover, since the forms φ_1 and φ_2 have a common quadratic isotropy field extension, $2 \in \operatorname{Im}\deg$. If at least one of the forms $\varphi_1 \perp \langle -a_1\rangle$ and $\varphi_2 \perp \langle -a_2\rangle$ is anisotropic, the numerator equals $2\mathbb{Z}$, and there is nothing more to prove. If the both forms $\varphi_1 \perp \langle -a_1\rangle$ and $\varphi_2 \perp \langle -a_2\rangle$ are isotropic, then the product $X_1 \times X_2$ possesses a closed rational point, therefore $1 \in \operatorname{Im}\deg$ and so the both groups are $\mathbb{Z}$.

Now consider the exact sequence of Chow groups

$$\mathrm{CH}_0(Y_1 \times X_2) \oplus \mathrm{CH}_0(X_1 \times Y_2) \to \mathrm{CH}_0(X_1 \times X_2) \to \mathrm{CH}_0(U_1 \times U_2) \to 0 ,$$

given by the closed subvariety

$$Y_1 \times X_2 \cup X_1 \times Y_2 = (X_1 \times X_2) \setminus (U_1 \times U_2) \subset X_1 \times X_2 .$$

Since

$$\deg(\mathrm{CH}_0(Y_1 \times X_2)) = D_0(\varphi_1) \cap D_0(\varphi_2 \perp \langle -a_2\rangle) ,$$

while $\deg(\mathrm{CH}_0(X_1 \times Y_2)) = D_0(\varphi_1 \perp \langle -a_1\rangle) \cap D_0(\varphi_2)$ (in fact, we need only the inclusions $\subset$ here), we see that the degree homomorphism on $\mathrm{CH}_0(X_1 \times X_2)$ induces the required epimorphism β.

Now, to prove the second part of the lemma, we assume that the form φ_1 is isotropic. Then the group $\mathrm{CH}_0(X_1 \times X_2)$ is isomorphic to $\mathrm{CH}_0(X_2)$ and therefore the degree homomorphism on $\mathrm{CH}_0(X_1 \times X_2)$ is a monomorphism (cf. [33]). Consequently, β is an isomorphism. On the other hand, since now $D_0(\varphi_1) = D_0(\varphi_1 \perp \langle -a_1 \rangle) = \mathbb{Z}$, the quotient, where the isomorphism β takes its values, is zero. □

Lemma A.9. *Let χ be a Pfister form over F with a decomposition $\chi = \eta \perp \eta'$, $\mathbb{A}$ the affine space of the vector space of definition of η, y the generic point of $\mathbb{A}$, and $a, b \in F^*$. There is a canonical homomorphism*

$$\alpha \colon H_0(U_{(\chi=a)} \times U_{(\eta'=b)}, K_1) \simeq \frac{F^* \cap D_1(\chi_{F(\mathbb{A})} \perp -a \langle \eta(y) + b \rangle)}{D_1(\chi)} .$$

Proof. We set $X_1 := Q_{\chi \perp \langle -a \rangle}$, $X_2 := Q_{\eta' \perp \langle -b \rangle}$, $Y_1 := Q_\chi$, $Y_2 := Q_{\eta'}$, $U_1 := U_{(\chi=a)}$, and $U_2 := U_{(\eta'=b)}$.

We consider the norm homomorphism $N : H_0(X_1 \times X_2, K_1) \to F^*$. Since over the residue field $F(x)$ of any point of $x \in X_1 \times X_2$ the forms $\chi \perp \langle -a \rangle$ and $\eta' \perp \langle -b \rangle$ become isotropic, the form $\chi_{F(\mathbb{A})} \perp -a \langle \eta(y) + b \rangle$ also becomes isotropic over the field extension $F(x)(\mathbb{A})/F(\mathbb{A})$ (Lemma A.7); therefore

$$\operatorname{Im} N \subset D_1(\chi_{F(\mathbb{A})} \perp -a \langle \eta(y) + b \rangle) .$$

Now with a help of the exact sequence

$$H_0(Y_1 \times X_2, K_1) \oplus H_0(X_1 \times Y_2, K_1) \to H_0(X_1 \times X_2, K_1) \to H_0(U_1 \times U_2, K_1) \to 0$$

one sees that the required homomorphism α is induced by N, if the groups $N(H_0(Y_1 \times X_2, K_1))$ and $N(H_0(X_1 \times Y_2, K_1))$ are in $D_1(\chi)$. Since $Y_1 = Q_\chi$, the residue field of any point on $Y_1 \times X_2$ makes the form χ isotropic and so, the first inclusion holds. Since $Y_2 = Q_{\eta'}$, the residue field of any point on $X_1 \times Y_2$ makes the form $\eta' \subset \chi$ isotropic and so, the second inclusion holds, too. □

We come back to the proof of Proposition A.6. To finish it we have to show that the cokernel of the homomorphism

$$H_0(U_{(\chi_{F(\mathbb{A}^1)}=a)} \times U_{(\tilde{\eta}'_{F(\mathbb{A}^1)}=cx^2+b)}, K_1) \to \coprod_{x \in (\mathbb{A}^1)^1} \mathrm{CH}_0(U_{(\chi_{F(x)}=a)} \times U_{(\tilde{\eta}'_{F(x)}=cx^2+b)})$$

is zero. This cokernel is evidently equal to the Chow group $\mathrm{CH}_0(U_{(\chi=a)} \times U_{(\tilde{\eta}' \perp \langle -c \rangle = b)})$ (use the spectral sequence converging to $\mathrm{CH}_0(U_{(\chi=a)} \times U_{(\tilde{\eta}' \perp \langle -c \rangle = b)})$ associated with the appropriate morphism $U_{(\chi=a)} \times U_{(\tilde{\eta}' \perp \langle -c \rangle = b)} \to \mathbb{A}^1$), which is zero by Lemma A.8, because the form $\tilde{\eta}' \perp \langle -c \rangle = \eta' \perp \langle c \rangle \perp \langle -c \rangle$ is isotropic.

We have finished the proof of Proposition A.6. □

Theorem A.4 is proved. □

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