UNITARY GRASSMANNIANS OF DIVISION ALGEBRAS

NIKITA A. KARPENKO

Abstract. We consider a central division algebra over a separable quadratic extension of a base field endowed with a unitary involution and prove 2-incompressibility of certain varieties of isotropic right ideals of the algebra. The remaining related projective homogeneous varieties are shown to be 2-compressible in general. Together with [17], where a similar issue for orthogonal and symplectic involutions has been treated, the present paper completes the study of Grassmannians of isotropic right ideals of division algebras.

Contents

1. Introduction

Let F be a field, K/F a separable quadratic field extension, n an integer ≥ 1, and D a central division K-algebra of degree 2^n endowed with a K/F-unitary involution σ. For definitions as well as for basic facts about involutions on central simple algebras, we refer to [20].

For any integer i, we write X_i for the F-variety of *isotropic* (with respect to σ) right ideals in D of reduced dimension i. (The reduced dimension of a right ideal in D is its dimension over K divided by $\deg D := \sqrt{\dim_K D}$.) For any i, the variety X_i is smooth and projective. It is nonempty if and only if $0 \leq i \leq 2^{n-1}$ (X_0 is simply $\operatorname{Spec} F$) in which case it is geometrically irreducible and has dimension

$$\dim X_i = i \cdot (2 \deg D - 3i).$$

For any i, the variety X_i is a closed subvariety of the Weil transfer $\mathcal{R}_{K/F} \operatorname{SB}_i D$, where $\operatorname{SB}_i D$ is the ith generalized Severi-Brauer variety of D – the K-variety of all right ideals in D of reduced dimension i. We recall that according to [15] (see [19] for a more recent

Date: 22 August 2014. Revised: 31 December 2014.

Key words and phrases. Algebraic groups, quadratic forms, projective homogeneous varieties, Chow groups and motives. *Mathematical Subject Classification (2010):* 14L17; 14C25.

The author acknowledges a partial support of the French Agence Nationale de la Recherche (ANR) under reference ANR-12-BL01-0005; his work has also been supported by the start-up grant of the University of Alberta and a Discovery Grant from the National Science and Engineering Board of Canada.

and simple proof), for any $r = 0, 1, \dots, n-1$, the variety $\mathcal{R}_{K/F}\,\mathrm{SB}_{2^r} D$ is 2-*incompressible*. This means, roughly speaking, that any self-correspondence

$$\mathcal{R}_{K/F}\,\mathrm{SB}_{2^r} D \rightsquigarrow \mathcal{R}_{K/F}\,\mathrm{SB}_{2^r} D$$

of odd multiplicity is dominant. In particular, any rational self-map

$$\mathcal{R}_{K/F}\,\mathrm{SB}_{2^r} D \dashrightarrow \mathcal{R}_{K/F}\,\mathrm{SB}_{2^r} D$$

is dominant.

The following theorem is the main result of this note. It extends to the unitary setting the results on orthogonal and symplectic involutions obtained in [17].

Theorem 1.1 (Main Theorem)**.** *For any $r = 0, 1, \dots, n-1$, the variety X_{2^r} is 2-incompressible.*

The proof will be given in Section 3, right after some preparation work made in Section 2 and in the beginning of Section 3. It extensively uses the notion of *upper motives* introduced in [18] and [13]. In our exposition, we go along the lines of [17] undertaking the necessary modifications.

Examples 4.3 and 4.5, given in Section 4, show that Theorem 1.1 precisely detects the types of those nontrivial (i.e., $\neq \operatorname{Spec} F$) projective homogeneous varieties under the projective unitary group $\mathrm{Aut}(D, \sigma)$ of a *division* algebra of arbitrary given degree, which are 2-incompressible *in general*, i.e., for any F, K, D and σ.

Note that $\mathrm{Aut}(D, \sigma)$ is an absolutely simple adjoint affine algebraic group of outer type $\mathcal{A}_{2^n-1}$. The varieties X_i for $i = 1, \dots, 2^{n-1} - 1$ correspond to the pairs of vertices of the Dynkin diagram exchanged by the action of $\mathrm{Gal}(K/F)$; the variety $X_{2^{n-1}}$ corresponds to the unique $\mathrm{Gal}(K/F)$-stable vertex.

Acknowledgements. I am grateful to Jean-Pierre Tignol for useful comments. Many thanks to two anonymous referees whose suggestions improved the exposition essentially. This work has been accomplished during my stay at the Universität Duisburg-Essen (research group of Marc Levine), under ideal work conditions.

2. Quadric-like behavior

In this section we establish some results on Grassmannians of isotropic right ideals which are very close (in the statement as well as in the proof) to results on projective quadrics in the spirit of [27].

Let F be a field, K/F a separable quadratic field extension, A a central simple K-algebra endowed with a K/F-unitary involution σ.

For a right ideal $J \subset A$, its orthogonal complement $J^\perp$ is defined as the (right) annihilator of the left ideal $\sigma(J)$. This is a right ideal of reduced dimension $\operatorname{rdim} J^\perp = \deg A - \operatorname{rdim} J$, [20, Proposition 6.2].

A right ideal J is *nondegenerate* if $J \cap J^\perp = 0$.

Construction 2.1. Given a nondegenerate right ideal $J \subset A$, the right A-module A is a direct sum of the submodules J and $J^\perp$. The image $e \in J$ of $1 \in A$ with respect to the projection $A \to J$ is a symmetric (with respect to the involution σ) idempotent generating J: $\sigma(e) = e$, $e^2 = e$, and $J = eA$. The K-algebra $\operatorname{End}_A J$ is identified with the subalgebra

eAe of A (see [20, Corollary 1.13]) stable under the involution σ. (Note that the unit of the algebra eAe is the element e which may differ from the unit 1 of A so that the unital algebra eAe is, in general, not a unital subalgebra of A.) The restriction of σ to eAe is a K/F-unitary involution. Note that the degree of the algebra eAe is equal to the reduced dimension of the right ideal J.

In contrast to [20], we define the (Witt) index $\operatorname{ind}\sigma$ of σ as the maximum of reduced dimension of an isotropic right ideal in A. The information given by the Witt index of σ in the sense of [20], or equivalently by the Tits index of the algebraic group $\operatorname{Aut}(A,\sigma)$, is equivalent to the information given by $\operatorname{ind}\sigma$ and $\operatorname{ind}A$.

Construction 2.2. Given an isotropic right ideal I in A, we have $I \subset I^\perp$. Let us choose a right ideal $J \subset I^\perp$ such that $I^\perp = I \oplus J$. The right ideal J is nondegenerate so that, using Construction 2.1, we get the algebra eAe with restriction of σ. Note that $\deg(eAe) = \operatorname{rdim} J = \deg A - 2\operatorname{rdim} I$. The (Witt) index of this restriction is equal to $\operatorname{ind}\sigma - \operatorname{rdim} I$. (Note that Construction 2.1 applied to the right ideal $J^\perp$ produces an algebra with *hyperbolic* unitary involution.)

We are working with Chow groups and Chow motives with coefficients in $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$. In particular, *multiplicities of correspondences*, [7, §75], take values in $\mathbb{F}_2$.

A variety is called *anisotropic* here if every its closed point has even degree. According to [18, Lemma 2.21], any direct summand of the motive of an anisotropic projective homogeneous variety has even rank.

The following statement is an analogue of the motivic decomposition [7, Proposition 70.1] of smooth projective quadrics, observed originally by M. Rost. It is also the unitary analogue of [17, Lemma A.3] which contains a mistake in the statement: the motive of Y occurs in the decomposition of $M(X)$ with the shift $(\operatorname{ind}A)^2$, not with the shift $2\operatorname{ind}A$ as claimed there. The shift we have in the unitary setting is $2(\operatorname{ind}A)^2$:

Lemma 2.3. *Assume that* $\operatorname{ind}A$ *is a power of* 2. *Let* I *be an isotropic right ideal of reduced dimension* $\operatorname{ind}A$ *in* A. *Let* X *be the variety of isotropic right ideals of reduced dimension* $\operatorname{ind}A$ *in* A. *Let* B *be an algebra* eAe *given by Construction 2.2. Let* Y *be the variety of isotropic right ideals of reduced dimension* $\operatorname{ind}A = \operatorname{ind}B$ *in* B *(*Y *is nonempty iff* $\deg A \geq 4\operatorname{ind}A$*). Then there exists a motivic decomposition of* X *with summands* $\mathbb{F}_2$, $\mathbb{F}_2(\dim X)$, *and – in the case of nonempty* Y *–*

$$M(Y)(2(\operatorname{ind}A)^2) = M(Y)((\dim X - \dim Y)/2)$$

such that each of the remaining summands of the decomposition is the motive of an anisotropic variety.

Proof. See [14, Lemma 2.3]. In order to determine the shift of $M(Y)$, one may use [5]. □

For any integer i, we write X_i for the variety of isotropic right ideals in A of reduced dimension i. The variety X_i is nonempty if and only if $0 \leq i \leq (\deg A)/2$.

Proposition 2.4 and Proposition 2.7 below are analogues of computation of canonical 2-dimension of smooth projective quadrics [7, Theorem 90.2]. We refer to [11] for definition and basic properties of canonical dimension.

Proposition 2.4. *For some $r \geq 0$ with $2^{r+1} \leq \deg A$ and 2^r dividing $\operatorname{ind} A$, assume that the variety $X := X_{2^r}$ is anisotropic and has no multiplicity 1 correspondence to $X_{2^{r+1}}$. Then the variety X is 2-incompressible.*

Proof. We recall (see [7, Page 328]) that a correspondence $Y \rightsquigarrow Z$ of multiplicity 1 from a smooth projective irreducible variety Y to a smooth projective Z exists if and only if $Z_{F(Y)}$ has a closed point of odd degree. (A rational map $Y \dashrightarrow Z$ exists if and only if $Z_{F(Y)}$ has a rational point.) In particular, a correspondence $Y \rightsquigarrow X_i$ of multiplicity 1 for some integer i exists if and only there exists a finite extension $E/F(Y)$ of odd degree such that the algebra $A \otimes_F E$ possesses an isotropic right ideal of reduced dimension i.

By the index reduction formula as in Lemma 3.1, we see that the index of $A_{F(X)}$ is 2^r. The $F(X)$-variety Y as in Lemma 2.3 is anisotropic (because of absence of a multiplicity 1 correspondence $X \rightsquigarrow X_{2^{r+1}}$). It follows that all summands of the complete motivic decomposition of the variety $X_{F(X)}$ but $\mathbb{F}_2$ and $\mathbb{F}_2(\dim X)$ have even ranks. On the other hand, since the variety X is anisotropic, its upper motive $U(X)$ is also of even rank. It follows that $U(X)_{F(X)}$ contains $\mathbb{F}_2(\dim X)$. Therefore X is 2-incompressible (see [11, Theorem 5.1]). □

Lemma 2.5. *Let n be a positive integer, let k be a positive divisor of n, and let $l := n/k$. Let V be an n-dimensional vector space over K and let X be the Grassmannian of k-dimensional subspaces in V. Let W be a subspace in V of codimension k. Let h be the class in the integral Chow group $\operatorname{CH} X$ of the closed subvariety in X given by the Grassmannian of k-dimensional subspaces in W. Then for any $r = 1, \dots, l-1$ and any subspace $W_r \subset V$ of codimension rk, h^r is equal to the class in $\operatorname{CH} X$ of the Grassmannian of k-dimensional subspaces in W_r. In particular, h^{l-1} is the class of a rational point on X.*

Proof. Note that h does not depend on the choice of W. Computing h^r, i.e., the product of r copies of h, we may choose for each copy a different subspace W the way that the intersection W_r of all these subspaces has codimension kr. The intersection of the corresponding subvarieties in X is then the Grassmannian of W_r. We are done by [7, Corollary 57.22]. □

Remark 2.6. In the settings of Lemma 2.5, let us consider the element

$$\mathcal{R}_{K/F} h \in \operatorname{CH} \mathcal{R}_{K/F} X.$$

(See [10] for the definition of $\mathcal{R}_{K/F} h$.) Then $(\mathcal{R}_{K/F} h)^r = \mathcal{R}_{K/F}(h^r)$. In particular, $(\mathcal{R}_{K/F} h)^{l-1}$ is the class of a rational point.

Proposition 2.7. *Assume that $\operatorname{ind} A = 2^r$ for some $r \geq 0$. Assume that the variety $X := X_{2^r}$ is anisotropic. Let i be the maximal integer such that there exists a multiplicity 1 correspondence $X \rightsquigarrow X_{(i+1)\cdot 2^r}$. Then the canonical 2-dimension $\operatorname{cdim}_2 X$ of X is equal to $\dim X - i \cdot 2^{2r+1}$ and the Tate motive $\mathbb{F}_2(\dim X - i \cdot 2^{2r+1})$ is a summand of $U(X)_{F(X)}$.*

Proof. Note that the case of $i = 0$ follows by Proposition 2.4.

Since there exists a multiplicity 1 correspondence $X \rightsquigarrow X_{(i+1)\cdot 2^r}$ and i is maximal with this property, we may find a finite field extension $E/F(X)$ of odd degree such that the Witt index of the involution σ_E is $(i+1) \cdot 2^r$.

Let us prove the following claim:

$$U(X)(2^{2r+1}),\ U(X)(2\cdot 2^{2r+1}),\ \dots,\ U(X)(i\cdot 2^{2r+1})$$

are summands of $M(X)$.

Let us observe that $U(X)(j)$ is a summand of $M(X)$ for a given integer j, if there exist $a\in \mathrm{CH}_j X_E$ and $b\in \mathrm{CH}^j X$ with odd $\deg(a\cdot b_E)$. Indeed, replacing a by its image under the norm map $\mathrm{CH}_j X_E\to \mathrm{CH}_j X_{F(X)}$, we come to the case with $E=F(X)$, where the statement is similar to and can be proved as [8, Corollary 4.11]. (Unfortunately, [8] deals with projective homogeneous varieties under semisimple algebraic groups of *inner* type only.) Namely, let f be a preimage of a under the pull-back epimorphism

$$f\in \mathrm{CH}_{\dim X+j}(X\times X)\to \mathrm{CH}_j X_{F(X)}\ni a$$

given by the generic point of the first factor of $X\times X$. Let g be the product

$$g:=(b\times[X])\cdot\Delta_X\in \mathrm{CH}_{\dim X-j}(X\times X),$$

where Δ_X is the class of the diagonal. The correspondences f and $g: X\rightsquigarrow X$ produce morphisms of motives $M(X)\to M(X)(-j)$ and $M(X)(-j)\to M(X)$. The composition of correspondences $g\circ f$ has odd multiplicity. It follows by [18, Lemma 2.14] that $U(X)$ is a summand of $M(X)(-j)$, i.e., $U(X)(j)$ is a summand of $M(X)$. This proves the observation.

We apply the above observation to the case, where j is a multiple of 2^{2r+1} so that we write $j\cdot 2^{2r+1}$ instead of j. We are only interested in $j=1,\dots,i$.

Let J be an isotropic right ideal in A_E of reduced dimension $(j+1)\cdot 2^r$. We take for a the class of the closed subvariety in X_E of the isotropic right ideals of reduced dimension 2^r contained in J.

The variety X is a closed subvariety of $\mathcal{R}_{K/F}\,\mathrm{SB}_{2^r} A$. Let H be a right ideal in A of reduced dimension $\deg A-j\cdot 2^r$. We take for $b\in \mathrm{CH}\,X$ the pull-back of the class in $\mathrm{CH}\,\mathcal{R}_{K/F}\,\mathrm{SB}_{2^r} A$ of the closed subvariety of the right ideals contained in H. Using Lemma 2.5 with Remark 2.6 and applying the projection formula [7] to the closed imbedding $X\hookrightarrow \mathcal{R}_{K/F}\,\mathrm{SB}_{2^r} A$, we get that $\deg(a\cdot b_E)=1$. Thus our claim is proved.

It follows that $\dim U(X)\le d_X-i\cdot 2^{2r+1}$, where $d_X:=\dim X$. Recall that $\mathrm{cdim}_2 X=\dim U(X)$ by [11, Theorem 5.1]. Therefore, to finish the proof of Proposition 2.7 it suffices to show that $U(X)_E$ contains the Tate summand $\mathbb{F}_2(d_X-i\cdot 2^{2r+1})$.

Applying Lemma 2.3 $i+1$ times, we get a motivic decomposition of the variety X_E consisting of motives of anisotropic varieties along with the $2(i+1)$ Tate motives

$$\mathbb{F}_2(j\cdot 2^{2r+1})\quad\text{and}\quad \mathbb{F}_2(d_X-j\cdot 2^{2r+1}),\ \ j=0,1,\dots,i.$$

Note that $i\cdot 2^{2r+1}\le d_X-i\cdot 2^{2r+1}$.

By definition of $U(X)$, $U(X)_E$ contains the Tate summand $\mathbb{F}_2$. Since the rank of $U(X)$ is even, $U(X)_E$ contains at least one more Tate summand. By the above claim, it can be nothing but $\mathbb{F}_2(d_X-i\cdot 2^{2r+1})$. □

3. Proof of Main Theorem

Now we return to notation of Section 1 and continue the preparation for the proof of Theorem 1.1. Note that for any $i=0,1,\dots,2^{n-1}$, the tensor product $K\otimes_F F(X_i)$ is a field (namely, the field $K(X_i)$) and the tensor product $D \otimes_F F(X_i)$ is a central simple $K(X_i)$-algebra.

Lemma 3.1. *For any $r = 0, 1, \ldots, n-1$, the Schur index of the central simple $K(X_{2^r})$-algebra $D \otimes_F F(X_{2^r})$ is equal to 2^r.*

Proof. First of all, although $X := X_{2^r}$ is an F-variety, the center of D is K, not F. Therefore we do not need the index reduction formula [23, (9.29)] for the F-variety X here, we rather need an index reduction formula for the K-variety X_K. The variety X_K is isomorphic to the variety of flags of right ideals in D of reduced dimensions 2^r and $2^n - 2^r$. The generic fiber of the projection of the flag variety to $(\mathrm{SB}_{2^r} D)_K$ is rational and the index of a central simple algebra does not change under a purely transcendental extension of the base field. Therefore the desired result on

$$\operatorname{ind}(D \otimes_F F(X)) = \operatorname{ind}(D \otimes_K K(X))$$

is contained in [26]. It is also a consequence of the index reduction formula for the generalized Severi-Brauer varieties [4] (see also [22, (5.11)]). □

Lemma 3.2. *Theorem 1.1 holds for $r = n-1$.*

Proof. This is a particular case of Proposition 2.4. □

We recall that we are working with Chow groups modulo 2. In particular, *multiplicities of correspondences*, [7, §75], take values in $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$. We also refer the reader to the comments on existence of multiplicity 1 correspondences given in the beginning of the proof of Proposition 2.4.

Lemma 3.3. *Assume that for some $r = 0, 1, \ldots, n-2$ there is no multiplicity 1 correspondence $X_{2^r} \rightsquigarrow X_{2^{r+1}}$. Then the variety X_{2^r} is 2-incompressible.*

Proof. This is a particular case of Proposition 2.4. □

Before proving the general case of Theorem 1.1, as a warm up, we prove the case of maximal r among yet unproved ones:

Proposition 3.4. *Theorem 1.1 holds for $r = n-2$.*

Proof. By Lemma 3.3, we may assume that there exists a multiplicity 1 correspondence $X_{2^{n-2}} \rightsquigarrow X_{2^{n-1}}$. We set $T := X_{2^{n-1}}$. The involution $\sigma_{F(T)}$ is hyperbolic. Besides, by Lemma 3.1, the Schur index of the algebra $D \otimes_F F(T)$ is 2^{n-1}.

For $X := X_{2^{n-2}}$, we have $\operatorname{ind} D \otimes_F F(X) = 2^{n-2}$ by Lemma 3.1. The existence of a multiplicity 1 correspondence $X \rightsquigarrow T$ means that $\operatorname{ind} \sigma_E = 2^{n-1}$ for some finite field extension $E/F(X)$ of odd degree. (By [1, Theorem 1.4], which includes the case of characteristic 2, the index relation holds already for $E = F(X)$, but let us ignore this fact as a preparation for the proof of Theorem 1.1, where an appropriate generalization of this fact will not be available.) Since $\operatorname{ind} D$ is a power of 2 and $[E : F]$ is odd, $\operatorname{ind} D \otimes_F E = \operatorname{ind} D \otimes_F F(X) = 2^{n-2}$.

By Lemma 2.3 (applied twice), the complete motivic decomposition of X_E contains four Tate summands: $\mathbb{F}_2$, $\mathbb{F}_2(2^{2n-3})$, $\mathbb{F}_2(\dim X - 2^{2n-3})$, $\mathbb{F}_2(\dim X)$. Note that $\dim X = 2^{2n-4} \cdot 5$, so that $2^{2n-3} < \dim X - 2^{2n-3}$, showing that the four Tate summands have pairwise different shifts.

Each of the remaining summands of the complete motivic decomposition of X_E is of even rank because by Lemma 2.3 it is a summand of the motive of an anisotropic variety. (The definition of anisotropic variety is given right before Lemma 2.3; the rank of any summand of the motive of an anisotropic variety is even by [18, Lemma 2.21]). For the upper motive $U(X)$, we are going to show that $U(X)_E$ contains all the 4 Tate summands; this will imply that X is 2-incompressible, cf. [11, Theorem 5.1].

By definition of $U(X)$, $U(X)_E$ contains the Tate summand $\mathbb{F}_2$.

We claim that by Proposition 2.7, $U(X)_E$ contains the Tate summand $\mathbb{F}_2(\dim X - 2^{2n-3})$. Indeed, let us apply Proposition 2.7 to $r = n-2$, to the function field $F' := F(\mathcal{R}_{K/F}\,\mathrm{SB}_{2^{n-2}}\,D)$ in the role of the base field, and the central simple F'-algebra $D_{F'}$ with the induced involution. The variety $X_{F'}$ is anisotropic by [11, Example 4.5]. (Indeed, if $X_{F'}$ would be isotropic, $X_{F''}$ with $F'' := F(\mathcal{R}_{K/F}\,\mathrm{SB}_1\,D)$ would be also isotropic, contradicting [11, Example 4.5].) So, the conclusion of Proposition 2.7 tells us that $U(X)_{F'(X)}$ contains the desired Tate summand. Enlarging first the field $F'(X)$ to $F'\otimes_F E$ and then noticing that the field extension $F'\otimes_F E/E$ is purely transcendental (since $E \supset F(X)$, the projective homogeneous variety $(\mathcal{R}_{K/F}\,\mathrm{SB}_{2^{n-2}}\,D)_E$ has a rational point and is therefore a rational variety), we get the desired Tate summand over E.

Let C be a central division $K(T)$-algebra (of degree 2^{n-1}) Brauer-equivalent to

$$D\otimes_F F(T) = D\otimes_K K(T).$$

Since there exist multiplicity 1 correspondences

$$X_{F(T)} \leftrightsquigarrow \mathcal{R}_{K(T)/F(T)}\,\mathrm{SB}_{2^{n-2}}\,C,$$

the upper motive of the variety $X_{F(T)}$ is isomorphic to the upper motive of the variety $\mathcal{R}_{K(T)/F(T)}\,\mathrm{SB}_{2^{n-2}}\,C$, [18, Corollary 2.15]. Since the latter variety is 2-incompressible and has dimension

$$[K(T):F(T)]\cdot\dim\mathrm{SB}_{2^{n-2}}\,C = 2^{2n-3},$$

$U(X_{F(T)})_{F(T)(X)}$ contains the Tate summand $\mathbb{F}_2(2^{2n-3})$. In particular, $U(X)_{E(T)}$ contains this Tate summand. Since the field extension $E(T)/E$ is purely transcendental (the projective homogeneous variety T_E is rational because it has a rational point), $U(X)_E$ contains the Tate summand $\mathbb{F}_2(2^{2n-3})$.

Finally, since $U(X)$ has even rank, $U(X)_E$ contains the remaining (fourth) Tate summand $\mathbb{F}_2(\dim X)$. □

For the general case of Theorem 1.1 we need one more observation:

Lemma 3.5. *For some $r = 0,1,\dots,n-1$, let us consider the biggest i such that there exists a multiplicity 1 correspondence $X_{2^r} \rightsquigarrow X_i$. (In particular, $X_i \neq \emptyset$, so that $i \le 2^{n-1}$.) Then $i = 2^s$ for some $s \in \{r, r+1,\dots,n-1\}$.*

Proof. Assuming that $i > 2^s$ for some $s = r, r+1,\dots,n-2$, we show that $i \ge 2^{s+1}$. Since $\operatorname{ind} D\otimes_F F(X_{2^r}) = 2^r$ by Lemma 3.1, it is a priori clear that $i \ge 2^s + 2^r$.

Note that $\operatorname{ind} D\otimes_F F(T) = 2^s$ for $T := X_{2^s}$ by Lemma 3.1. Let I be the isotropic right ideal of reduced dimension 2^s in $D\otimes_F F(T)$ corresponding to the generic point of the variety T. Let

$$C := \operatorname{End}_{D\otimes_F F(T)} I$$

so that C is a central division $K(T)$-algebra of degree 2^s Brauer-equivalent to $D \otimes_F F(T) = D \otimes_K K(T)$. Let A be a central simple $K(T)$-algebra with a $K(T)/F(T)$-unitary involution (which we are going to denote by τ) obtained out of I by Construction 2.2. Let X be the variety of isotropic right ideals in A of reduced dimension 2^r. The upper motives of X and of $\mathcal{R}_{K(T)/F(T)} \mathrm{SB}_{2^r} C$ are isomorphic. Since $\mathcal{R}_{K(T)/F(T)} \mathrm{SB}_{2^r} C$ is 2-incompressible and has dimension

$$d := \dim \mathcal{R}_{K(T)/F(T)} \mathrm{SB}_{2^r} C = [K(T) : F(T)] \cdot \dim \mathrm{SB}_{2^r} C = 2^{r+1}(2^s - 2^r),$$

the motive $U(X)_{F(T)(X)}$ contains the Tate motive $\mathbb{F}_2(d)$ as a summand. (Note that $F(T)$ is the field of definition of the variety X.) In particular, for any extension field $E/F(T)(X)$, the motive of the variety X_E contains the Tate summand $\mathbb{F}_2(d)$

Now we claim: it follows by Lemma 2.3 that the maximum of the Witt index of the unitary involution τ_E on $A \otimes_{F(T)} E$ for E running over finite field extensions of $F(T)(X)$ of odd degree is at least 2^s. Indeed, let us take such E for which the Witt index $\operatorname{ind} \tau_E$ is maximal and assume that $\operatorname{ind} \tau_E < 2^s$. Since $\operatorname{ind} \tau_E$ is a multiple of 2^r, it is equal to $2^r j$ for some j with $1 \leq j \leq 2^{s-r} - 1$. Applying Lemma 2.3 j times, starting with the algebra A_E, we get that the complete motivic decomposition of X_E consists of the $2j$ Tate summands

$$\begin{array}{lllll} \mathbb{F}_2, & \mathbb{F}_2(2^{2r+1}), & \mathbb{F}_2(2 \cdot 2^{2r+1}), & \dots, & \mathbb{F}_2((j-1) \cdot 2^{2r+1}), \\ \mathbb{F}_2(l), & \mathbb{F}_2(l - 2^{2r+1}), & \mathbb{F}_2(l - 2 \cdot 2^{2r+1}), & \dots, & \mathbb{F}_2(l - (j-1) \cdot 2^{2r+1}), \end{array}$$

where $l := \dim X$, and motives of anisotropic varieties. Since

$$(j-1)2^{2r+1} < d < l - (j-1)2^{2r+1},$$

we get a contradiction proving the claim.

It follows that the Witt index of σ_E is at least $2^s + 2^s = 2^{s+1}$ for some finite field extension $E/F(T)(X)$ of odd degree. Since $F(T)(X)$ is the function field of the F-variety Z of flags of right isotropic ideals in D of reduced dimensions 2^s and $2^s + 2^r$, it follows that there exists a multiplicity 1 correspondence $Z \rightsquigarrow X_{2^{s+1}}$. At the same time, according to our assumptions, there exists a multiplicity 1 correspondence $X_{2^r} \rightsquigarrow Z$. Taking the composition and using the computation [9, Corollary 1.7] of the multiplicity of the composition, we obtain a multiplicity 1 correspondence $X_{2^r} \rightsquigarrow X_{2^{s+1}}$. Therefore $i \geq 2^{s+1}$. □

Proof of Theorem 1.1. We set $X := X_{2^r}$. Let i be the maximal integer such that there exists a multiplicity 1 correspondence $X \rightsquigarrow X_i$. By Lemma 3.5, $i = 2^s$ for some $s \in \{r, r+1, \dots, n-1\}$.

By Lemma 3.1, $\operatorname{ind} D \otimes_F F(X) = 2^r$. Let $E/F(X)$ be a finite field extension of odd degree such that $\operatorname{ind} \sigma_E = 2^s$. By Lemma 2.3 (applied 2^{s-r} times), the complete motivic decomposition of the variety X_E contains the Tate summands with the shifts $j2^{2r+1}$ and $\dim X - j2^{2r+1}$ for $j = 0, 1, \dots, 2^{s-r} - 1$ (precisely one Tate summand for each shifting number). Note that $(2^{s-r} - 1)2^{2r+1} < \dim X - (2^{s-r} - 1)2^{2r+1}$ so that the shifting numbers are pairwise different. Each of the remaining summands in the complete motivic decomposition of X_E is of even rank. For the upper motive $U(X)$ it suffices to show that $U(X)_E$ contains the Tate summand $\mathbb{F}_2(\dim X)$.

We claim that by Proposition 2.7, $U(X)_E$ contains the Tate summand

$$\mathbb{F}_2(\dim X - (2^{s-r} - 1)2^{2r+1}).$$

Indeed, let us apply Proposition 2.7 to the function field $F' := F(\mathcal{R}_{K/F}\,\mathrm{SB}_{2^r} D)$ in the role of the base field, and the central simple F'-algebra $D_{F'}$ with the induced involution. The variety $X_{F'}$ is anisotropic by [11, Example 4.5]. So, the conclusion of Proposition 2.7 tells us that $U(X)_{F'(X)}$ contains the desired Tate summand. Enlarging first the field $F'(X)$ to $F' \otimes_F E$ and then noticing that the field extension $F' \otimes_F E/E$ is purely transcendental, we get the desired Tate summand over E.

By Lemma 3.1, $\operatorname{ind} D \otimes_F F(T) = 2^s$, where $T := X_{2^s}$. Let C be a central division $K(T)$-algebra of degree 2^s Brauer-equivalent to $D \otimes_F F(T)$. The upper motives of the varieties $X_{F(T)}$ and $S := \mathcal{R}_{K(T)/F(T)}\,\mathrm{SB}_{2^r} C$ are isomorphic. Passing to the dual motives and shifting, we get that

$$U(X_{F(T)})^*(\dim X) \simeq U(S)^*(\dim X).$$

Since the variety S is 2-incompressible, the motive $U(S)_{F(T)(X)}$ contains the Tate summands $\mathbb{F}_2$ and $\mathbb{F}_2(\dim S)$. Consequently, $U(S)^*_{F(T)(X)}(\dim X)$ contains the Tate summands $\mathbb{F}_2(\dim X)$ and $\mathbb{F}_2(\dim X - \dim S)$. In particular, $U(X)^*_{E(T)}(\dim X)$ contains both of these Tate summands. Since the field extension $E(T)/E$ is purely transcendental, $U(X)^*_E(\dim X)$ contains both of these Tate summands. Note that

$$\dim S = (2^{s-r} - 1)2^{2r+1}$$

and $U(X)^*(\dim X)$ is an indecomposable summand of $M(X)$. Since $U(X)_E$ also contains the Tate summand $\mathbb{F}_2(\dim X - \dim S)$, the Krull-Schmidt principle of [6] (see also [13]) tells us that $U(X) \simeq U(X)^*(\dim X)$ and therefore $U(X)_E$ contains $\mathbb{F}_2(\dim X)$ as desired. □

4. Some examples

Example 4.3, produced below, shows that for $G = \mathrm{Aut}(D, \sigma)$, the varieties listed in Theorem 1.1 are the only projective G-homogeneous varieties which are 2-incompressible in general (i.e., for any field F, any separable quadratic field extension K/F, and any central division K-algebra D of degree 2^n endowed with a K/F-unitary involution σ). We recall that an arbitrary (different from $\operatorname{Spec} F$) projective G-homogeneous variety is isomorphic to the variety $X_{l_1 \dots l_k}$ of flags of isotropic right ideals in D of some fixed reduced dimensions $1 \le l_1 < \dots < l_k \le 2^{n-1}$ with some $k \ge 1$.

As a preparation for Example 4.3, we need the following

Lemma 4.1. *For any given $n \ge 2$, let F be a field of characteristic $\neq 2$, admitting a central division F-algebra D' of degree 2^n endowed with an orthogonal or a symplectic involution σ' such that (D', σ') is the tensor product of n quaternion algebras with involutions. Let K/F be a separable quadratic field extension. We define a K/F-unitary involution σ on $D := D' \otimes_F K$ as the tensor product of σ' by the nontrivial automorphism of K/F. Then for any field extension L/F, the unitary involution on $D \otimes_F L$ given by σ is either anisotropic or hyperbolic.*

Proof. We only need to consider the case where $K \otimes_F L$ is a field (because the involution is hyperbolic otherwise). We assume that the involution is isotropic for such an L and we want to show that it is hyperbolic. Replacing F by L, we simply assume that σ (over F) is isotropic and we want to show that σ is hyperbolic.

By [12, Theorem A.2] (see also [14]), we may replace F by the function field of the F-variety $\mathcal{R}_{K/F}\,\mathrm{SB}_1\,D$. Since now the F-algebra D' splits over the quadratic extension K/F, it is equivalent to a quaternion F-algebra Q. It follows by [3] that the algebra D' together with the (orthogonal or symplectic) involution σ' is isomorphic to a tensor product of n quaternions algebras with involutions, where the first quaternion algebra is Q and the remaining $n-1$ quaternion algebras are split. Since the quadratic extension K/F splits Q, our algebra D together with the unitary involution σ is the tensor product (over K) of n split quaternion K-algebras with K/F-unitary involutions.

Let h be a K/F-hermitian form on an n-dimensional K-vector space V such that σ is adjoint to h. Because of the tensor decomposition we have for (D,σ), the quadratic form $q : v \mapsto h(v,v)$ on V over F is similar to an $(n+1)$-fold Pfister form. Isotropy of σ implies isotropy of h, which implies isotropy of q, which implies hyperbolicity of q, which in its turn implies hyperbolicity of h and of σ (cf. [16, Lemma 9.1]). □

Remark 4.2. A field F satisfying the requirement of Lemma 4.1 exists. One may, for instance, take an arbitrary field k of characteristic $\neq 2$, set $F := k(x_1, y_1, \dots, x_n, y_n)$ with indeterminates $x_1, y_1, \dots, x_n, y_n$, take for D' the tensor product of n quaternion algebras $(x_1, y_1), \dots, (x_n, y_n)$, and take for σ' the tensor product of their canonical (symplectic) involutions.

Example 4.3. Let F, K, D, and σ be as in Lemma 4.1 and assume that D is a division algebra. An arbitrary nontrivial projective G-homogeneous variety is isomorphic to the variety $X_{l_1 \dots l_k}$ with some $k \geq 1$ and some $1 \leq l_1 < \cdots < l_k \leq 2^{n-1}$. By Lemma 4.1, this variety is equivalent (in the sense of existing of rational maps in both directions) to the variety X_{2^r}, where 2^r is the largest 2-power dividing $l_1, \dots, l_k$. In particular, canonical 2-dimensions of these two varieties coincide (see e.g. [19, Lemma 3.3]). Hence $\{l_1, \dots, l_k\} = \{2^r\}$ (i.e., $k = 1$ and $l_1 = 2^r$) if the variety $X_{l_1 \dots l_k}$ is 2-incompressible (because $\dim X_{l_1 \dots l_k} > \dim X_{2^r}$ otherwise).

Example 4.5, produced below with a help of Lemma 4.4, shows that in the case when $\deg D$ is not a power of 2, i.e., when $\deg D = 2^n \cdot m$ with $n \geq 0$ and with odd $m \geq 3$, none (but $\operatorname{Spec} F$) of the projective homogeneous varieties under $G = \operatorname{Aut}(D,\sigma)$ is 2-incompressible in general.

Lemma 4.4. *For any odd integer $m \geq 3$, there exist a field F, a separable quadratic field extension K/F, a degree m central division K-algebra D, a K/F-unitary involution σ on D, and a finite field extension L/F of odd degree such that σ is isotropic over L.*

Proof. For prime $m \geq 5$, such examples can be derived from [25] and [24]. For general m, one may proceed by a generic construction as follows.

Let $m \geq 3$ be an odd integer and let D be a central division algebra of degree m (over an appropriate field K) possessing a unitary involution σ. Let $F \subset K$ be the subfield of the σ-invariant elements. There exists a finite field extension L/F of odd degree such that the algebra $D \otimes_F L$ is split (cf. [2, 3.3.1] or [14, Page 938]). Let h be a hermitian form such that the involution σ_L on the split algebra $D \otimes_F L$ is adjoint to h. Let q be the quadratic form over L (of dimension $2m$) given by h. Let X be the projective quadric of q. We are going to replace the base field F by the function field of the Weil transfer $\mathcal{R}_{L/F}X$. The quadratic form q (and therefore h and σ_L as well) will then become isotropic. We only need to check that D will remain a division algebra.

We have:

$$\operatorname{ind}(D \otimes_F F(\mathcal{R}_{L/F}X)) = \operatorname{ind}(D \otimes_K K(\mathcal{R}_{L/F}X)).$$

Since $(\mathcal{R}_{L/F}X)_K = \mathcal{R}_{(L\otimes_F K)/K}(X_{L\otimes_F K})$ and $X(L \otimes_F K) \neq \emptyset$ (the quadratic form q becomes hyperbolic over $L \otimes_F K$), the projective homogeneous K-variety $(\mathcal{R}_{L/F}X)_K$ is rational so that its function field does not change the index. □

Example 4.5. Let n and m be integers with $n \geq 0$ and with odd $m \geq 3$. Let F, K, D, and σ be as in Lemma 4.4 with the above m. Changing notation, we write D_m and σ_m for these D and σ. Replacing F by a purely transcendental extension field of sufficiently large transcendence degree, we may find a central division K-algebra D' of degree 2^n with a K/F-unitary involution σ'. (For instance, one may take the degree $2n$ purely transcendental field extension $F(x_1, y_1, \ldots, x_n, y_n)$, consider the tensor product of quaternion algebras $(x_1, y_1) \otimes \cdots \otimes (x_n, y_n)$ with the tensor product of their canonical involutions, and then take the induced unitary involution over $K(x_1, y_1, \ldots, x_n, y_n)$.) We define the central division K-algebra D as the tensor product $D' \otimes_K D_m$ and we define a K/F-unitary involution σ on D as the tensor product of σ' and σ_m. We claim that none of the nontrivial projective homogeneous varieties given by (D, σ) is 2-incompressible. Indeed, canonical 2-dimension of any such variety X remains the same over any finite field extension of F of odd degree, [21, Proposition 1.5]. Replacing F by a field extension L/F of odd degree given by Lemma 4.4, we come to the situation where σ_m is isotropic. Therefore σ is isotropic and it follows that X is not 2-incompressible.

To elucidate the very last conclusion, we pick up a nonzero isotropic right ideal I in D. Let l be the reduced dimension of I. Since $I \neq 0$, we have $l > 0$. Applying Construction 2.2, we get a K-algebra eDe with unitary involution such that

$$\deg(eDe) = \deg D - 2l < \deg D$$

and our variety X, which is equal to $X_{l_1 \ldots l_k}$ for some $k \geq 0$ and some $1 \leq l_1 \leq \cdots \leq l_k \leq 2^{n-1}m$, is equivalent to the variety Y of flags of isotropic right ideals in eDe of reduced dimensions $l_i - l, l_{i+1} - l, \ldots, l_k - l$, where $i \geq 1$ is the least integer with $l_i \geq l$. (We have $Y = \operatorname{Spec} F$ if $l_k < l$.) Since $\dim Y < \dim X$, the conclusion follows. Note that making the suggested above concrete choice of D' and σ' and assuming 2-incompressibility of X, we ensure that $k = 1$, simplifying computation and comparison of the dimensions.

References

[1] Bayer-Fluckiger, E. Self-dual normal bases. *Nederl. Akad. Wetensch. Indag. Math. 51*, 4 (1989), 379–383.

[2] Bayer-Fluckiger, E., and Parimala, R. Galois cohomology of the classical groups over fields of cohomological dimension ≤ 2. *Invent. Math. 122*, 2 (1995), 195–229.

[3] Becher, K. J. A proof of the Pfister factor conjecture. *Invent. Math. 173*, 1 (2008), 1–6.

[4] Blanchet, A. Function fields of generalized Brauer-Severi varieties. *Comm. Algebra 19*, 1 (1991), 97–118.

[5] Chernousov, V., Gille, S., and Merkurjev, A. Motivic decomposition of isotropic projective homogeneous varieties. *Duke Math. J. 126*, 1 (2005), 137–159.
[6] Chernousov, V., and Merkurjev, A. Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem. *Transform. Groups 11*, 3 (2006), 371–386.
[7] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.
[8] Garibaldi, S., Petrov, V., and Semenov, N. Shells of twisted flag varieties and non-decomposibility of the Rost invariant. arXiv:1012.2451v4 [math.AG] (5 Dec 2014), 39 pages.
[9] Karpenko, N. A. On anisotropy of orthogonal involutions. *J. Ramanujan Math. Soc. 15*, 1 (2000), 1–22.
[10] Karpenko, N. A. Weil transfer of algebraic cycles. *Indag. Math. (N.S.) 11*, 1 (2000), 73–86.
[11] Karpenko, N. A. Canonical dimension. In *Proceedings of the International Congress of Mathematicians. Volume II* (New Delhi, 2010), Hindustan Book Agency, pp. 146–161.
[12] Karpenko, N. A. Hyperbolicity of orthogonal involutions. *Doc. Math. Extra Volume: Andrei A. Suslin's Sixtieth Birthday* (2010), 371–392 (electronic). With an Appendix by Jean-Pierre Tignol.
[13] Karpenko, N. A. Upper motives of outer algebraic groups. In *Quadratic forms, linear algebraic groups, and cohomology*, vol. 18 of *Dev. Math.* Springer, New York, 2010, pp. 249–258.
[14] Karpenko, N. A. Hyperbolicity of unitary involutions. *Sci. China Math. 55*, 5 (2012), 937–945.
[15] Karpenko, N. A. Incompressibility of quadratic Weil transfer of generalized Severi-Brauer varieties. *J. Inst. Math. Jussieu 11*, 1 (2012), 119–131.
[16] Karpenko, N. A. Unitary Grassmannians. *J. Pure Appl. Algebra 216*, 12 (2012), 2586–2600.
[17] Karpenko, N. A. Orthogonal and symplectic Grassmannians of division algebras. *J. Ramanujan Math. Soc. 28*, 2 (2013), 213–222.
[18] Karpenko, N. A. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. *J. Reine Angew. Math. 677* (2013), 179–198.
[19] Karpenko, N. A., and Reichstein, Z. A numerical invariant for linear representations of finite groups. Linear Algebraic Groups and Related Structures (preprint server) 534 (2014, May 15, revised: 2014, June 18), 24 pages.
[20] Knus, M.-A., Merkurjev, A., Rost, M., and Tignol, J.-P. *The book of involutions*, vol. 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.
[21] Merkurjev, A. S. Essential dimension. In *Quadratic Forms – Algebra, Arithmetic, and Geometry*, vol. 493 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2009, pp. 299–326.
[22] Merkurjev, A. S., Panin, I. A., and Wadsworth, A. R. Index reduction formulas for twisted flag varieties. I. *K-Theory 10*, 6 (1996), 517–596.
[23] Merkurjev, A. S., Panin, I. A., and Wadsworth, A. R. Index reduction formulas for twisted flag varieties. II. *K-Theory 14*, 2 (1998), 101–196.
[24] Parimala, R. Homogeneous varieties—zero-cycles of degree one versus rational points. *Asian J. Math. 9*, 2 (2005), 251–256.
[25] Parimala, R., Sridharan, R., and Suresh, V. Hermitian analogue of a theorem of Springer. *J. Algebra 243*, 2 (2001), 780–789.
[26] Schofield, A., and Van den Bergh, M. The index of a Brauer class on a Brauer-Severi variety. *Trans. Amer. Math. Soc. 333*, 2 (1992), 729–739.
[27] Vishik, A. Motives of quadrics with applications to the theory of quadratic forms. In *Geometric methods in the algebraic theory of quadratic forms*, vol. 1835 of *Lecture Notes in Math.* Springer, Berlin, 2004, pp. 25–101.

Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
E-mail address: karpenko *at* ualberta.ca, *web page*: www.ualberta.ca/~karpenko