Grothendieck Chow-motives of Severi-Brauer varieties

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Abstract

For any central simple algebra, the Grothendieck Chow-motive of the corresponding Severi-Brauer variety is decomposed in a direct sum where each summand is a twisted motive of the Severi-Brauer variety corresponding to the underlying division algebra. It leads to decompositions in other theories (for instance, of K-cohomologies) because of the universal property of the Chow-motives.

In the second part, it is shown that the Chow-motive of a Severi-Brauer variety corresponding to a division algebra is indecomposable as an object in the category of motives.

We fix a basefield F, a central simple algebra D over F, put $r = \deg D$ and X be the Severi-Brauer variety $\mathrm{SB}\,(D)$ corresponding to D [1].

Moreover, put $X^n = \mathrm{SB}\,(M_n(D))$ where $M_n(D)$ is the F-algebra of $n\times n$-matrices over D.

In the first part, we decompose the Grothendieck Chow-motive $\widetilde{X^n}$ of the variety X^n [5] in the direct sum $\bigoplus_{i=0}^{n-1} \widetilde{X}(ir)$ of twisted motives of X (1.3.2) (note that in the trivial case $D = F$ it is the well-known decomposition of the motive of the projective space [5]). Hence in any "geometrical" cohomology theory H we have:

$$H(X^n) = \bigoplus_{i=0}^{n-1} H(X)$$

with some twistings of gradations in the graded case. A list of examples is given in (1.3.2).

So, computation of cohomology groups of Severi-Brauer varieties is reduced to the non-splitted case, i.e. to the case when D is a division algebra. However no further reduction can be obtained on the motivic level: as shown in the second part, if D is a division algebra then the motive $\widetilde{X}$ is indecomposable as an object in the category of motives.

1 Decomposition

For any D-module V of finite dimension over F one can consider the Severi-Brauer variety $\mathrm{SB}\,(\mathrm{End}_D V)$ which we will denote by $X(V)$. Our aim is to show that any direct decomposition of the D-module V, say $V = W_1 \oplus W_2$, produces a direct decomposition in the category of Chow-motives

$$\widetilde{X(V)} = \widetilde{X(W_1)} \oplus \widetilde{X(W_2)}(\mathrm{rk}\, W_1)$$

where the second summand is the motive of $X(W_2)$ twisted by rank $\mathrm{rk}\, W_1 = \dim_F W_1/r$ of the module W_1. In (1.1), we construct following [6] a closed imbedding $X(W) \hookrightarrow X(V)$ and a "projection" $X(V) \setminus X(W) \to X(W')$ for any exact sequence of D-modules

$$0 \longrightarrow W \longrightarrow V \longrightarrow W' \longrightarrow 0\ .$$

In (1.2), this allows after some general remarks about the language and properties of correspondences and motives (some important notations are introduced there) to define correspondences in and pr which play the crucial role in the further. In (1.3), we state the main theorem and deduce some consequences. Lastly, we prove the theorem in (1.4).

1.1 Varieties $X(V)$ and morphisms

As noted in [6], there exist a natural (with respect to base field extensions) bijection between the set of rational points of the variety $X(V)$ (understanding as the set of right ideals in $\mathrm{End}_D V$ of rank 1) and the set of right D-submodules in V of rank 1 which sends a submodule $N \subset V$ to the ideal $\mathrm{Hom}_D(V, N) \subset \mathrm{End}_D V$.

Now giving an exact sequence of D-modules

$$0 \longrightarrow W \longrightarrow V \longrightarrow W' \longrightarrow 0$$

we can consider each D-submodule of W (of rank 1) as a submodule in V. This map gives rise to a closed imbedding

$$In : X(W) \hookrightarrow X(V)\ .$$

Further, if $N \subset V$ is a D-submodule of rank 1 then its image in W' has rank 1 too if $N \not\subset W$. So, we get a morphism

$$Pr : X(V) \setminus X(W) \longrightarrow X(W')\ .$$

It is clear that Pr is a flat morphism with a fiber over $N' \subset W'$ isomorphic to the affine space $\mathrm{Hom}_D(N', W)$.

Now fix a direct sum decomposition $V = W \oplus W'$. We get morphisms $In : X(W) \hookrightarrow X(V)$ and $Pr : X(V) \setminus X(W') \to X(W)$ and it is clear from the definitions above that the composition $Pr \circ In$ is an identity.

1.2 Correspondences, motives

We are going to work in the categories of Chow-correspondences and Chow-motives [5]. So, we start from the category $\mathcal{V}(F)$ of smooth projective (or more generally – complete) varieties over F; consider the category of correspondences $C\mathcal{V}(F)$ which has the same objects but $\mathrm{Hom}\,(Y_1, Y_2)$ being the Chow group $\mathrm{Ch}^*(Y_1 \times Y_2)$ of cycles on $Y_1 \times Y_2$ modulo rational equivalence (the notion of composition for correspondences needed here is the standard one [5, 2]); then consider a category $C\mathcal{V}^\circ(F)$ of degree-0-correspondences with $\mathrm{Hom}\,(Y_1, Y_2) = \mathrm{Ch}^k(Y_1 \times Y_2)$ the Chow group of cycles of codimension k for an irreducible k-dimensional variety Y_1.

To distinguish usual morphisms of varieties and correspondences we will write all names of morphisms starting with a capital letter. We denote for instance by Id the identity morphism of a variety while by id the identity correspondence on it.

Considering the graph of a morphism $\Phi : Y_1 \to Y_2$ as a cycle φ on $Y_2 \times Y_1$ (not on $Y_1 \times Y_2$!) one gets a (contravariant) functor $\mathcal{V}(F) \to C\mathcal{V}(F)$ and also a functor $\mathcal{V}(F) \to C\mathcal{V}^\circ(F)$ since graphs have the appropriate codimension. In particular, the graph of a composition $\Psi \circ \Phi$ where $\Psi : Y_2 \to Y_3$ has a graph ψ coincides with a composition of correspondences $\varphi \circ \psi$.

The category $C\mathcal{V}(F)$ is self-dual. If τ is a correspondence from Y_1 to Y_2 we put τ^t be the determined by τ correspondence in the opposite direction.

One extends the Chow functor on $\mathcal{V}(F)$ to a covariant functor Ch on $C\mathcal{V}(F)$. Note that if φ is the graph of a morphism Φ then the homomorphism $\mathrm{Ch}(\varphi)$ coincides with the pull-back Φ^* while $\mathrm{Ch}(\varphi^t)$ is the same as the push-forward Φ_*.

The category of (Chow-)motives is by definition a pseudo-abelian completion of $C\mathcal{V}^\circ(F)$. In particular, a motive is a pair (Y, p) where $Y \in \mathcal{V}(F)$ and p is a projector on Y, i.e. a correspondence with $p \circ p = p$. The motive of a variety Y is a pair (Y, id) denoted by $\widetilde{Y}$; L is the Tate motive. There is a notion of tensor product for motives [5]; L^i is the i-th tensor power of L and for any motive M a notation for $M \otimes L^i$ is $M(i)$. [1] To formulate (1.3.1) we need the following easy computation.

Lemma 1.2.1 ([5]) *Let Y_1 and Y_2 be varieties (from $\mathcal{V}(F)$), Y_1 irreducible and k-dimensional. Then*

1. $\mathrm{Hom}\,(\widetilde{Y_1}(i), \widetilde{Y_2}) = \mathrm{Ch}^{k+i}(Y_1 \times Y_2)$ *;*
2. $\mathrm{Hom}\,(\widetilde{Y_1}, \widetilde{Y_2}(i)) = \mathrm{Ch}^{k-i}(Y_1 \times Y_2)$.

Now return to $X(V)$, fix a direct sum decomposition $V = W \oplus W'$ and put $n = \mathrm{rk}\, V$, $m = \mathrm{rk}\, W$. Define a correspondence $in : X(V) \to X(W)$ as the graph of the morphism $In : X(W) \to X(V)$ from (1.1). We want to construct also a correspondence $pr : X(W) \to X(V)$ with a help of $Pr : X(V) \setminus X(W') \to X(W)$.

Lemma 1.2.2 *The homomorphism*

$$(Id \times In)^* : \mathrm{Ch}^{m-1}(X(W) \times X(V)) \longrightarrow \mathrm{Ch}^{m-1}(X(W) \times X(W))$$

[1] We follow here to the terminology and notations of [5] which are partially wrong. The right name of the motive L is Lefschetz motive (the real Tate motive is the "inverse"to L) and the right notation for $M \otimes L^i$ is $M(-i)$.

is bijective.

Proof We have an exact sequence

$$\begin{array}{ccc} \mathrm{Ch}^{-1}(X(W)\times X(W')) & \longrightarrow & \mathrm{Ch}^{m-1}(X(W)\times X(V)) \longrightarrow \\ \| & & \\ 0 & & \mathrm{Ch}^{m-1}(X(W)\times X(V)\setminus X(W')) \longrightarrow 0 \end{array}$$

and a homomorphism

$$(Id\times Pr)^* : \mathrm{Ch}^{m-1}(X(W)\times X(W)) \longrightarrow \mathrm{Ch}^{m-1}(X(W)\times X(V)\setminus X(W'))$$

which is bijective because of properties of Pr stated in (1.1). The last to note is that the inverse to $(Id\times Pr)^*$ is the pull-back of the imbedding

$$X(W)\times X(W) \hookrightarrow X(W)\times X(V)\setminus X(W')$$

since this imbedding splits the "projection" $Id\times Pr$. □

Now we define the correspondence pr as a cycle on $X(W)\times X(V)$ which corresponds to the diagonal on $X(W)\times X(W)$ under the isomorphism just stated.

Lemma 1.2.3 *The composition of correspondences $in\circ pr$ is an identity.*

Proof By definition of pr we have: $(Id\times In)^*(pr) = id$. From the other hand $(Id\times In)^* = \mathrm{Ch}(id\otimes in)$ and by (1.4.4) $\mathrm{Ch}(id\otimes in)(pr) = in\circ pr$. □

Remark The correspondence pr can be also defined as the closure of the graph of $Pr : X(V)\setminus X(W') \to X(W)$ in $X(W)\times X(V)$ (we don't prove it since we don't need it). Although pr is of course not a graph of a morphism (in general) one can hence imagine it as a graph of a "many-valued morphism" $X(V)\to X(W)$ which coincides with Pr on $X(V)\setminus X(W')$. Then the statement of (1.2.3) is not a surprise since we know that $Pr\circ In = Id$.

1.3 Main theorem

Theorem 1.3.1 *Let $V = W_1\oplus W_2$ be a direct sum of D-modules, in_j and pr_j $(j=1,2)$ the constructed above correspondences between $X(W_j)$ and $X(V)$. Then*

$$\widetilde{X(V)} = \widetilde{X(W_1)} \oplus \widetilde{X(W_2)}(rkW_1)$$

where the morphisms of inclusions and projections are

$$\begin{array}{ccc} & \widetilde{X(V)} & \\ in_1 \swarrow\nearrow pr_1 & & in_2^t \nwarrow\searrow pr_2^t \\ \widetilde{X(W_1)} & & \widetilde{X(W_2)}(rkW_2)\ . \end{array}$$

Remark The correspondences in_2^t and pr_2^t determine some morphisms between the motives $\widetilde{X(V)}$ and $\widetilde{X(W_2)}(\operatorname{rk} W_1)$ according to (1.2.1).

Corollary 1.3.2 *Put* $X^n = X(D^n) = \mathrm{SB}\,(M_n(D))$ *where* $M_n(D)$ *is the matrix algebra. One has a motivic decomposition* $\widetilde{X^n} = \bigoplus_{i=0}^{n-1} \widetilde{X}(ir)$ *and therefore*

1. *a decomposition of (Quillen's or Milnor's) K-cohomologies* [4]*:*

$$H^p(X^n, K_q) = \bigoplus_{i=0}^{n-1} H^{p-ir}(X, K_{q-ir})$$

(note that at most one of the summands from the right is non-trivial); in particular $\mathrm{Ch}^p(X^n) = \bigoplus_{i=0}^{n-1} \mathrm{Ch}^{p-ir}(X)$;

2. *a decomposition of factorgroups of the topological filtration on K-groups* [9]*:*

$$K_q(X^n)^{(p/p+1)} = \bigoplus_{i=0}^{n-1} K_q(X)^{(p-ir/p-ir+1)} \ ;$$

3. *a decomposition of the étale cohomology groups (with coefficients in $\mu_l^{\otimes q}$)* [7]*:*

$$H^p(X^n, q) = \bigoplus_{i=0}^{n-1} H^{p-2ir}(X, q-ir) \ ;$$

4. *and so on.*

Remark The decomposition of the Chow groups from *1* is obtained in [6].

1.4 Proof of main theorem

We put $n = \operatorname{rk} V$ and $m_j = \operatorname{rk} W_j \ \ (j = 1, 2)$.

Proposition 1.4.1 *The Chow group* $\mathrm{Ch}^{n-1}(X(V) \times X(V))$ *is a direct sum of*

$$\mathrm{Ch}^{m_j-1}(X(W_j) \times X(W_j)) \ \ (j = 1, 2)$$

with the inclusions and projections given below:

$$\begin{array}{ccc} & \mathrm{Ch}^{n-1}(X(V) \times X(V)) & \\ \mathrm{Ch}(pr_1^t \otimes in_1) \swarrow \nearrow \mathrm{Ch}(in_1^t \otimes pr_1) & & \mathrm{Ch}(pr_2 \otimes in_2^t) \nwarrow \searrow \mathrm{Ch}(in_2 \otimes pr_2^t) \\ \mathrm{Ch}^{m_1-1}(X(W_1) \times X(W_1)) & & \mathrm{Ch}^{m_2-1}(X(W_2) \times X(W_2)) \end{array}$$

Proof We have an exact row

$$\mathrm{Ch}^{m_2-1}(X(V)\times X(W_2)) \to \ \mathrm{Ch}^{n-1}(X(V)\times X(V)) \to$$
$$\mathrm{Ch}^{n-1}(X(V)\times X(V)\setminus X(W_2)) \to 0$$

where the first arrow is $(Id \times In_2)_* = \mathrm{Ch}(id\otimes in_2^t)$ and the second one can be included in a commutative triangle

$$\begin{array}{ccc} \mathrm{Ch}^{n-1}(X(V)\times X(V)) & \to & \mathrm{Ch}^{n-1}(X(V)\times X(V)\setminus X(W_2)) \\ & \searrow & \downarrow \\ & & \mathrm{Ch}^{n-1}(X(V)\times X(W_1)) \end{array}$$

formed by the pull-backs with respect to the imbeddings (the vertical arrow is bijective since it splits the isomorphism $(Id\times Pr_1)^*$). So, we obtain an exact sequence

$$\mathrm{Ch}^{m_2-1}(X(V)\times X(W_2)) \xrightarrow{\mathrm{Ch}(id\otimes in_2^t)} \mathrm{Ch}^{n-1}(X(V)\times X(V)) \xrightarrow{\mathrm{Ch}(id\otimes in_1)}$$
$$\mathrm{Ch}^{n-1}(X(V)\times X(W_1)) \longrightarrow 0\ .$$

The homomorphism $\mathrm{Ch}(id\otimes in_2^t)$ has a splitting from the left $\mathrm{Ch}(id\otimes pr_2^t)$ since

$$\mathrm{Ch}(id\otimes pr_2^t)\circ \mathrm{Ch}(id\otimes in_2^t) = \mathrm{Ch}((id\otimes pr_2^t)\circ(id\otimes in_2^t)) =$$
$$\mathrm{Ch}((id\circ id)\otimes(pr_2^t\circ in_2^t)) = \mathrm{Ch}(id\otimes(in_2\circ pr_2)^t) =$$
$$\mathrm{Ch}(id\otimes id) \ = \ \mathrm{Ch}(id) = Id\ .$$

Hence the first arrow is a splitted monomorphism and our sequence turns out to be a splitted short exact one. A splitting for the epimorphism $\mathrm{Ch}(id\otimes in_1)$ is $\mathrm{Ch}(id\otimes pr_1)$ because of $in_1\circ pr_1 = id$. But we also need to know that the both splittings agree, i.e. that their composition is trivial. It happens due to

Lemma 1.4.2 *The composition of correspondences $pr_2^t\circ pr_1$ is trivial.*

Proof The statement holds by a very simple cause. One needs just to look at dimensions. The codimension of the cycle $pr_1\times X(W_2)$ on $X(W_1)\times X(V)\times X(W_2)$ equals m_1-1, the codimension of $X(W_1)\times pr_2^t$ is m_2-1. Thus their product has codimension $n-2$ and lands after the push-forward to $X(W_1)\times X(W_2)$ in codimension -1. □

Return to the proof of the proposition. For $X(V)\times X(W_2)$ one can draw an exact sequence:

$$\begin{array}{ccc} \mathrm{Ch}^{-1}(X(W_1)\times X(W_2)) & \longrightarrow & \mathrm{Ch}^{m_2-1}(X(V)\times X(W_2)) \longrightarrow \\ \| & & \\ 0 & & \mathrm{Ch}^{m_2-1}(X(V)\setminus X(W_1)\times X(W_2)) \longrightarrow 0 \end{array}$$

and a commutative triangle

$$\begin{array}{ccc} \mathrm{Ch}^{m_2-1}(X(V)\times X(W_2)) & \longrightarrow & \mathrm{Ch}^{m_2-1}(X(V)\setminus X(W_1)\times X(W_2)) \\ & \searrow & \downarrow \\ & & \mathrm{Ch}^{m_2-1}(X(W_2)\times X(W_2)) \end{array}$$

(with vertical isomorphism) to see that

$$(In_2\times Id)^* = \mathrm{Ch}(in_2\otimes id) : \mathrm{Ch}^{m_2-1}(X(V)\times X(W_2)) \to \mathrm{Ch}^{m_2-1}(X(W_2)\times X(W_2))$$

is an isomorphism. The relation $in_2\circ pr_2 = id$ shows that $\mathrm{Ch}(pr_2\otimes id)$ is the inverse isomorphism. Finally, the exact sequence

$$\mathrm{Ch}^{m_1-1}(X(W_1)\times X(W_1)) \xrightarrow{(In_1\times Id)_*} \mathrm{Ch}^{n-1}(X(V)\times X(W_1)) \longrightarrow$$
$$\mathrm{Ch}^{n-1}(X(V)\setminus X(W_1)\times X(W_1)) \longrightarrow 0$$

where

$$\mathrm{Ch}^{n-1}(X(V)\setminus X(W_1)\times X(W_1)) \simeq \mathrm{Ch}^{n-1}(X(W_2)\times X(W_1)) = 0$$

shows that $(In_1\times Id)_* = \mathrm{Ch}(in_1^t\otimes id)$ is a surjection and it is moreover a bijection since $\mathrm{Ch}(pr_1^t\otimes id)$ splits it from the left.

We have got the picture:

$$\begin{array}{ccc} & \mathrm{Ch}^{n-1}(X(V)\times X(V)) & \\ \mathrm{Ch}(id\otimes in_1)\swarrow\nearrow\mathrm{Ch}(id\otimes pr_1) & & \mathrm{Ch}(id\otimes in_2^t)\nwarrow\searrow\mathrm{Ch}(id\otimes pr_2^t) \\ \mathrm{Ch}^{n-1}(X(V)\times X(W_1)) & & \mathrm{Ch}^{m_2-1}(X(V)\times X(W_2)) \\ \mathrm{Ch}(pr_1^t\otimes id)\swarrow\nearrow\mathrm{Ch}(in_1^t\otimes id) & & \mathrm{Ch}(pr_2\otimes id)\nwarrow\searrow\mathrm{Ch}(in_2\otimes id) \\ \mathrm{Ch}^{m_1-1}(X(W_1)\times X(W_1)) & & \mathrm{Ch}^{m_2-1}(X(W_2)\times X(W_2)) \end{array}$$

(with isomorphisms on the lower level). Computing the compositions one gets what is required. □

Proposition 1.4.3 *The correspondences $p_1 = pr_1\circ in_1$ and $p_2 = (pr_2\circ in_2)^t$ are projectors on $X(V)$ with the sum id.*

Proof One sees that the both correspondences are projectors immediately using the relations $in_j\circ pr_j = id_{X(W_j)}$. We need to prove only the assertion on their sum.

We can apply (1.4.1) to obtain a decomposition of the cycle

$$id \in \mathrm{Ch}^{n-1}(X(V)\times X(V))$$

into two summands:

$$id = \mathrm{Ch}(in_2^t \otimes pr_1) \circ \mathrm{Ch}(pr_1^t \otimes in_2)(id) + \mathrm{Ch}(pr_2 \otimes in_2^t) \circ \mathrm{Ch}(in_2 \otimes pr_2^t)(id) =$$

$$= \mathrm{Ch}(p_1^t \otimes p_1)(id) + \mathrm{Ch}(p_2^t \otimes p_2)(id) \ .$$

We will show now that the first summand is the cycle p_1 and the second one is p_2.

Lemma 1.4.4 ([5]) *Let Y_1, Y_2, Y_3 be varieties and $f_j \in \mathrm{Ch}^*(Y_j \times Y_{j+1})$ for $j = 1, 2$. Then $\mathrm{Ch}(id_1 \otimes f_2)(f_1) = f_2 \circ f_1$.*

A dual variation of (1.4.4) is

Lemma 1.4.5 *In the conditions of (1.4.4), the following equality holds:*

$$\mathrm{Ch}(f_1^t \otimes id_3)(f_2) = f_2 \circ f_1 \ .$$

Corollary 1.4.6 *Let Y_1, Y_2, Y_3, Y_4 be varieties and $f_j \in \mathrm{Ch}^*(Y_j \times Y_{j+1})$ for $j = 1, 2, 3$. Then $\mathrm{Ch}(f_1^t \otimes f_3)(f_2) = f_3 \circ f_2 \circ f_1$.*

Proof One has:

$$\mathrm{Ch}(f_1^t \otimes f_3)(f_2) = \mathrm{Ch}(f_1^t \otimes id_4) \circ \mathrm{Ch}(id_2 \otimes f_3)(f_2) \overset{\text{by (1.4.4)}}{=\!=}$$

$$\mathrm{Ch}(f_1^t \otimes id_4)(f_3 \circ f_2) \overset{\text{by (1.4.5)}}{=\!=} f_3 \circ f_2 \circ f_1 \ .$$

□

To finish the proof of (1.4.3) we compute the both summands of the decomposition of id by using (1.4.6):

$$\mathrm{Ch}(p_j^t \otimes p_j)(id) = p_j \circ id \circ p_j = p_j \quad (j = 1, 2)$$

where the last equality holds since p_j is a projector. □

Corollary 1.4.7 *The following motivic decomposition holds:*

$$\widetilde{X(V)} = (X(V), p_1) \oplus (X(V), p_2) \ .$$

The last step in the prove of (1.3.1) is

Proposition 1.4.8 *There exist isomorphisms of motives:*

1. $(X(V), p_1) \simeq X(\widetilde{W_1})$;
2. $(X(V), p_2) \simeq X(\widetilde{W_2})(m_1)$.

Proof *1.* Morphisms of the motives in the both directions are defined in the diagram:

$$\begin{array}{ccc} X(V) & \underset{pr_1}{\overset{in_1}{\rightleftarrows}} & X(W_1) \\ \downarrow{\scriptstyle p_1 = pr_1 \circ in_1} & & \downarrow{\scriptstyle id} \\ X(V) & \underset{pr_1}{\overset{in_1}{\rightleftarrows}} & X(W_1)\ . \end{array}$$

Both the morphisms are well defined since

$$in_1 \circ (pr_1 \circ in_1) = (in_1 \circ pr_1) \circ in_1 = id \circ in_1$$

and

$$(pr_1 \circ in_1) \circ pr_1 = pr_1 \circ (in_1 \circ pr_1) = pr_1 \circ id\ .$$

Composing the morphism determined by in_1 with the other one determined by pr_1 we get what is determined by $pr_1 \circ in_1 = p_1$ and coincides with the identity on the motive $(X(V), p_1)$. The composition in the inverse order is an identity already on the level of correspondences.
2. Let Y be an l-dimensional variety having a closed rational point y. Then $L^l = (Y, Y \times y)$ [5] (by definition, L^l is such a pair with $Y = \mathbf{P}^1 \times \ldots \times \mathbf{P}^1$). Denote by I the morphism of varieties $\operatorname{Spec} F \to Y$ given by the point y and by P the structure morphism $Y \to \operatorname{Spec} F$. If i and p are their graphs then $i \circ p = id$ and $L^l = (Y, (p \circ i)^t)$. Now we have:

$$X(\widetilde{W_2})(l) = X(\widetilde{W_2}) \otimes L^l = (X(W_2) \times Y, id \otimes (p \circ i)^t)\ .$$

Taking $l = m_1$ and identifying $(X(V), p_2)$ with $(X(V), p_2) \otimes \widetilde{\operatorname{Spec} F} = (X(V) \times \operatorname{Spec} F, p_2 \otimes id)$ for a technical convenience we describe the isomorphisms required as follows:

$$\begin{array}{ccc} X(V) \times \operatorname{Spec} F & \underset{in_2^t \otimes p^t}{\overset{pr_2^t \otimes i^t}{\rightleftarrows}} & X(W_2) \times Y \\ \downarrow{\scriptstyle p_2 \otimes id} & & \downarrow{\scriptstyle id \otimes (p \circ i)^t} \\ X(V) \times \operatorname{Spec} F & \underset{in_2^t \otimes p^t}{\overset{pr_2^t \otimes i^t}{\rightleftarrows}} & X(W_2) \times Y \end{array}$$

Note firstly that the correspondences $pr_2^t \otimes i^t$ and $in_2^t \otimes p^t$ have the right degrees. To verify that the both squares are commutative and that the both compositions are identities one needs just to use the relations $in_2 \circ pr_2 = id$ and $i \circ p = id$ several times. □

2 Indecomposability

Let D be a central simple algebra of degree r over F, $M_n(D)$ the algebra of matrices $n \times n$ over D and $X^n = \operatorname{SB}(M_n(D))$ the Severi-Brauer variety corresponding to $M_n(D)$.

As was shown in the first part, the Chow-motive $\widetilde{X^n}$ has the following direct decomposition:

$$\widetilde{X^n} = \bigoplus_{i=0}^{n-1} \widetilde{X}(ir)$$

where $X = X^1$ and $\widetilde{X}(ir)$ are twistings of $\widetilde{X}$.

In this part we show that if D is a division algebra then the motive $\widetilde{X}$ is indecomposable (i.e. has no non-trivial decomposition in a direct sum). It means (since twistings obviously preserve indecomposability) that all summands in the decomposition above are also indecomposable in the case.

2.1 Degrees of cycles

Let as above D be a central simple algebra over the fixed field F and $X = \mathrm{SB}\,(D)$ the Severi-Brauer variety corresponding to D. For any $k, 0 \leq k \leq \dim X$, consider a homomorphism $\deg : \mathrm{Ch}^k(X) \longrightarrow \mathbf{Z}$ which value $\deg Z$ on a simple cycle $Z \subset X$ is by definition the degree of $Z_{F(X)}$ as a subvariety of the projective space $X_{F(X)}$ [3] where $F(X)$ is the function field of X. In other words, deg is a composition of the restriction map $\mathrm{Ch}^k(X) \longrightarrow \mathrm{Ch}^k(X_{F(X)})$ and the canonical isomorphism $\mathrm{Ch}^k(X_{F(X)}) \xrightarrow{\sim} \mathbf{Z}$. [2]

Proposition 2.1.1 *Let $X = \mathrm{SB}\,(D)$ where D is a central simple algebra. If $1 \in \deg \mathrm{Ch}^k(X)$ for some k then $k \,\vdots\, ind D$. In particular, if D is a division algebra then the degree of any cycle of a positive codimension is not 1.*

Remark. It is shown in [1] that if a *simple* cycle on X of codimension k and degree 1 exists then $k \,\vdots\, \mathrm{ind}\, D$. But this statement is weaker than (2.1.1).

Proof. Suppose that the statement is proved for all algebras (over all fields) of a p-primary index (for all primes p). If we are given an algebra D of an arbitrary index and such k that $1 \in \deg \mathrm{Ch}^k(X)$ we can go to the maximal algebraic extension F_p/F of degree prime to p and see (since still $1 \in \deg \mathrm{Ch}^k(X_{F_p})$) that the p-part of $\mathrm{ind}\, D$ which coincides with the index of D_{F_p} divides k. Since we can do it for each prime p the index of D should divide k.

So, it suffices to consider only the case $\mathrm{ind}\, D = p^n$.

Let $K(X) = K_0'(X) = K_0(X)$ be the Grothendieck group of X (which is moreover a ring) [3, 9] and $K(X)^{(0)} \supset K(X)^{(1)} \supset \ldots$ the topological filtration. The canonical epimorphisms $\mathrm{Ch}^i(X) \twoheadrightarrow K(X)^{(i/i+1)}$ becomes to be bijective after the restriction of scalars to $F(X)$; moreover the ring $K(X_{F(X)})$ is generated by the class of a hyperplane h being subject of the only relation: $h^{\deg D} = 0$, and the topological filtration on $K(X_{F(X)})$ coincides with the filtration by degrees of h [3].

[2]The field F(X) in this section can be replaced if one likes by any extension E/F which spits D (i.e. the algebra D becomes to be isomorphic to a matrix algebra over E).

Hence the statement (2.1.1) is equivalent to the following one: if $h^k + \ldots \in K(X)^{(k)}$ where the dots denote a linear combination of h^j with $j > k$ then $k \,\vdots\, p^n$. [3] It is difficult to compute the topological filtration on $K(X)$ but the point is that the last assertion remains true after replacing $K(X)^{(k)}$ just by $K(X)$. To prove it in this form is our goal now.

Consider in a polynomial ring $\mathbf{Z}[\xi]$ a subgroup $G(D)$ (in fact a subring) generated by all monomials $(\operatorname{ind}(D^{\otimes i})) \cdot \xi^i$. According to [9], $G(D)/(h^{\deg D})$ where $h = 1 - \xi$ is isomorphic to $K(X)$ (note that $h^{\deg D} \in G(D)$ since $\deg D \,\vdots\, p^n$). So, it is enough to show that if $h^k + \ldots \in G(D)$ then $k \,\vdots\, p^n$. Note that if D' is an algebra having index p^n and exponent p then $G(D') \supset G(D)$. Consequently, it suffices to consider algebras of exponent p only. We denote the group $G(D)$ corresponding to an algebra D of index p^n and exponent p by $G(n)$. It is generated by monomials $a_i \xi^i$ where

$$a_i = \begin{cases} 1 & \text{if } i \,\vdots\, p \\ p^n & \text{otherwise} \end{cases}$$

and (2.1.1) has been reduced to the following elementary

Lemma 2.1.2 *Let $G(n)$ be the subgroup of the polynomial ring $\mathbf{Z}[\xi]$ defined above, $h = 1 - \xi$. If $h^k + \ldots \in G(n)$ for some k then $k \,\vdots\, p^n$.*

We will deduce it from the

Sublemma 2.1.3 *If $bh^{k-1} + \ldots \in G(n)$ for some k with $k \,\vdots\, p^n$ and some integer b then $b \,\vdots\, p^n$.*

Elementary Proof. Note that the factorgroup $G(n)/(h^k)$ is generated by only $a_i \xi^i$ with $i < k$ (without relations). It holds because $\xi^k = \xi^k - (1-\xi)^k$ is a linear combination of $a_i \xi^i$ with $i < k$ in $G(n)/(h^k)$. Consequently, if $bh^{k-1} \in G(n)/(h^k)$ then $bh^{k-1} = \sum_{i=0}^{k-1} b_i a_i (1-h)^i$. The coefficient by h^{k-1} on the right equals $b_{k-1} a_{k-1}$. Hence $b = b_{k-1} a_{k-1}$ and is divisible by p^n (remember that $a_{k-1} = p^n$).

Algebro-geometrical Proof. The statement is equivalent to the fact that the degree of any closed point on a Severi-Brauer variety is divisible by the index of the algebra. It holds because the residue field of a point splits the algebra.

Proof of (2.1.2). Consider a homomorphism $\phi : G(n+1) \longrightarrow G(n)$ mapping $f(\xi) \in G(n+1)$ to the polynomial $\frac{\xi f'(\xi)}{p}$ (where $f'(\xi)$ is the derivative with respect to ξ). It is easy to check that ϕ maps the generators of $G(n+1)$ to elements of $G(n)$ so all values of ϕ indeed lie in $G(n)$.

We proof (2.1.2) using an induction by n (starting from $n = 0$ when there is nothing to prove). Let $h^k + \ldots \in G(n+1)$. Since $G(n+1) \subset G(n)$ we know from the induction that $k \,\vdots\, p^n$. Applying ϕ to the polynomial $h^k + \ldots$ we get $\frac{k}{p} h^{k-1} + \ldots \in G(n)$ and hence (by (2.1.3)) $\frac{k}{p} \,\vdots\, p^n$. Thus $k \,\vdots\, p^{n+1}$.

[3] Since the homomorphism $K(X) \longrightarrow K(X_{F(X)})$ is injective [9] we may identify $K(X)$ with a subgroup of $K(X_{F(X)})$.

2.2 Indecomposability

Theorem 2.2.1 *The Chow-motive $\widetilde{X}$ of the Severi-Brauer variety $X = \mathrm{SB}(D)$ corresponding to a division algebra D is indecomposable as an object in the category of motives.*

Proof. To prove that an object in an additive category is indecomposable it is enough to show that the ring of endomorphisms of the object does not contain non-trivial idempotetnts. The ring $End(\widetilde{X})$ is $\mathrm{Ch}^d(X \times X)$ where $d = \dim X$ and the cycles are multiplied as correspondences. The restriction of scalars

$$\mathrm{Ch}^d(X \times X) \longrightarrow \mathrm{Ch}^d(X_{F(X)} \times X_{F(X)})$$

is a ring homomorphism. The theorem will be proved by showing that neither the kernel nor the image of the restriction contain a non-trivial idempotent — see (2.2.3) and (2.2.4).

Definition. Let $f : T \to Y$ be a morphism of (irreducible) varieties. The filtration $\mathcal{F}^0 \supset \mathcal{F}^1 \supset \dots$ on the group $\mathrm{Ch}^*(T)$ where $\mathcal{F}^i\mathrm{Ch}^*(T)$ is the subgroup generated by all simple cycles $Z \subset T$ with $\mathrm{codim}_Y \overline{f(Z)} \geq i$ will be called the *filtration defined by f*.

Lemma 2.2.2 *Let $f : T \to Y$ be a flat morphism and $\mathcal{F}^0 \supset \mathcal{F}^1 \supset \dots$ the filtration defined by f.*

1. *The sequence $0 \to \mathcal{F}^1\mathrm{Ch}^*(T) \to \mathrm{Ch}^*(T) \to \mathrm{Ch}^*(T_\theta) \to 0$ where T_θ is the fiber of f over the generic point $\theta \in Y$ is exact.*

2. *If each fiber of f is isomorphic to a projective space then for each p and n there exist a canonical epimorphism $\mathrm{Ch}^p(Y) \twoheadrightarrow \mathcal{F}^{p/p+1}\mathrm{Ch}^n(T)$.*

   For the rest of the lemma all varieties are supposed to be smooth.

3. *If T over Y is a projective space bundle and if $t \in \mathrm{Ch}^1(T)$ is the canonical element for which $1, t, \dots, t^l$ is a basis of $\mathrm{Ch}^*(T)$ as a module over $\mathrm{Ch}^*(Y)$ (where $l = \dim T - \dim Y$ and $\mathrm{Ch}^*(Y)$ acts on $\mathrm{Ch}^*(T)$ via the pull-back) [3] then*
$$\mathcal{F}^p\mathrm{Ch}^*(T) = \bigoplus_{j=0}^{l} \left(\bigoplus_{i \geq p} \mathrm{Ch}^i(Y) \right) \cdot t^j.$$

4. *If T over Y is a fibred product of two projective space bundles over Y and if $t_1, t_2 \in \mathrm{Ch}^1(T)$ are the canonical elements for which $\{t_1^{j_1} \cdot t_2^{j_2}\}$ where $j_1 = 0, 1, \dots, l_1$ and $j_2 = 0, 1, \dots, l_2$ is a basis of $\mathrm{Ch}^*(T)$ over $\mathrm{Ch}^*(Y)$ then*
$$\mathcal{F}^p\mathrm{Ch}^*(T) = \bigoplus_{j_1=0}^{l_1} \bigoplus_{j_2=0}^{l_2} \left(\bigoplus_{i \geq p} \mathrm{Ch}^i(Y) \right) \cdot t_1^{j_1} t_2^{j_2}.$$

5. *In conditions of 3 or 4 the filtration on* $\mathrm{Ch}^*(T)$ *defined by* f *is compatible with the multiplication of cycles, i.e.* $\mathcal{F}^i\mathcal{F}^j \subset \mathcal{F}^{i+j}$.

Proof. *1.* Consider (for each n) the spectral sequence

$$E_1^{p,q} = \coprod_{y \in Y^p} H^q(T_y, K_{n-p}) \Rightarrow H^{p+q}(T, K_n)$$

associated with f [4]. Since there are no differentials starting or finishing at $E_s^{0,n}$ and $E_1^{0,n} = H^n(T_\theta, K_n) = \mathrm{Ch}^n(T_\theta)$ we obtain an isomorphism $\mathrm{Ch}^n(T_\theta) \simeq \mathcal{F}^{0/1}\mathrm{Ch}^n(T)$ which proves *1.*
2. If each fiber T_y of f is isomorphic to a projective space then $H^q(T_y, K_{n-p}) \simeq K_{n-p-q}(F(y))$ [10, 11] where $F(y)$ is the residue field of a point y and K_m denotes the m-th Quillen's K-group (or 0 if $m < 0$). Hence $E_2^{p,n-p}$ is equal to $\mathrm{Ch}^p(Y)$ or to 0. Since there are no differentials starting from $E_2^{p,n-p}$ the spectral sequence defines an epimorphism $E_2^{p,n-p} \twoheadrightarrow \mathcal{F}^{p/p+1}\mathrm{Ch}^n(T)$.
3. Consider a filtration $\mathcal{F}'^0 \supset \mathcal{F}'^1 \supset \ldots$ on $\mathrm{Ch}^*(T)$ defined by the formula:

$$\mathcal{F}'^p\mathrm{Ch}^*(T) = \bigoplus_{j=0}^{l} \left(\bigoplus_{i \geq p} \mathrm{Ch}^i(Y) \right) \cdot t^j.$$

It is obviously that $\mathcal{F}'^p\mathrm{Ch}^*(T) \subset \mathcal{F}^p\mathrm{Ch}^*(T)$ for each p. Now consider the both filtration on each gradation component of $\mathrm{Ch}^*(T)$ separately. The composition

$$\mathrm{Ch}^p(Y) \xrightarrow{\cdot t^{n-p}} \mathcal{F}'^{p/p+1}\mathrm{Ch}^n(T) \longrightarrow \mathcal{F}^{p/p+1}\mathrm{Ch}^n(T)$$

is surjective by *2*, hence the second homomorphism is surjective (for each p), hence the both filtrations coincide.
4. The proof is completely analogous to *3.*
5. It is an obvious consequence of *3* and *4.*

Proposition 2.2.3 *The kernel of the restriction*

$$\mathrm{Ch}^d(X \times X) \longrightarrow \mathrm{Ch}^d(X_{F(X)} \times X_{F(X)})$$

is nilpotent.

Proof. Let α be a cycle from this kernel. We will show using an induction on i that $\alpha^{\circ i} = \alpha \circ \alpha \circ \ldots \circ \alpha$ belongs to $\mathcal{F}^i\mathrm{Ch}^d(X \times X)$ where the filtration $\mathcal{F}^i$ is defined by the first projection $X \times X \longrightarrow X$. It will prove (2.2.3) because $\mathcal{F}^{d+1}\mathrm{Ch}^d(X \times X) = 0$.

We start the induction from $i = 0$ when there is nothing to prove.

Consider the filtration on $\mathrm{Ch}^*(X \times X \times X)$ defined by the first projection and the cycle $X \times \alpha$ on $X \times X \times X$. According to (2.2.2) we have the following exact sequence:

$$0 \to \mathcal{F}^1\mathrm{Ch}^*(X \times X \times X) \to \mathrm{Ch}^*(X \times X \times X) \to \mathrm{Ch}^*(X_{F(X)} \times X_{F(X)}) \to 0.$$

Since the image of $X \times \alpha$ in the right term equals $\alpha_{F(X)} = 0$ we conclude that

$$X \times \alpha \in \mathcal{F}^1 \mathrm{Ch}^*(X \times X \times X).$$

Suppose that $\alpha^{\circ(i-1)} \in \mathcal{F}^{i-1}\mathrm{Ch}^d(X \times X)$. Then we obviously have: $\alpha^{\circ(i-1)} \times X \in \mathcal{F}^{i-1}\mathrm{Ch}^*(X \times X \times X)$. To conclude that the product of cycles $(X \times \alpha)(\alpha^{\circ(i-1)} \times X)$ lies in $\mathcal{F}^i\mathrm{Ch}^*(X \times X \times X)$ we apply (2.2.2) taking in account that $X \times X$ over X is a projective space bundle [8] and hence $X \times X \times X$ over X is a fibred product of two projective space bundles. Finally, push-forward of the latter product with respect to the projection of $X \times X \times X$ on the first and the last factors (which is $\alpha^{\circ i} = \alpha \circ \alpha^{\circ(i-1)}$ by the definition of how to compose correspondences) lies consequently in $\mathcal{F}^i\mathrm{Ch}^d(X \times X)$.

Proposition 2.2.4 *The image oh the restriction*

$$\mathrm{Ch}^d(X \times X) \longrightarrow \mathrm{Ch}^d(X_{F(X)} \times X_{F(X)})$$

does not contain non-trivial idempotents.

Proof. The additive group of the ring $\mathrm{Ch}^d(X_{F(X)} \times X_{F(X)})$ is a free abelian group generated by the products $h^i \times h^{d-i}$, where $i = 0, 1, \dots, d$ and h^i is an i-codimensional linear subspace of the projective space $X_{F(X)}$. The multiplicative structure is described by saying that all $h^i \times h^{d-i}$ are orthogonal idempotents [5] . Hence an arbitrary idempotent here is a sum of some $h^i \times h^{d-i}$ (without repetitions).

Suppose that it exists a non-trivial idempotent e in the image of the restriction. Replacing if needed e by $1 - e$ (note that $1 = \sum_{i=0}^{d} h^i \times h^{d-i}$) we may assume that $e = h^p \times h^{d-p} + \dots$ with $p > 0$ where the dots denote a sum of some $h^i \times h^{d-i}$ with $i > p$. Then e generates the factorgroup $\mathcal{F}^{p/p+1}\mathrm{Ch}^d(X_{F(X)} \times X_{F(X)})$ of the filtration defined by the first projection since for each p the subgroup $\mathcal{F}^p$ is generated by all $h^i \times h^{d-i}$ with $i \geq p$.

Now let us take once more in account that $X \times X$ considered over X (via the first projection) is a projective space bundle [8]. The element $t \in \mathrm{Ch}^1(X \times X)$ as in (2.2.2) defines an isomorphism $\mathrm{Ch}^d(X \times X) \simeq \mathrm{Ch}^*(X)$. Restricting it to $F(X)$ we get an isomorphism

$$\mathrm{Ch}^d(X_{F(X)} \times X_{F(X)}) \simeq \bigoplus_{i=0}^{d} \mathbf{Z} \cdot x_i$$

where $x_i = t_{F(X)}^{d-i} \cdot (h^i \times 1)$ (note that it is another system of generators as the one we had in the beginning of the proof). So, our restriction map

$$\mathrm{Ch}^d(X \times X) \longrightarrow \mathrm{Ch}^d(X_{F(X)} \times X_{F(X)})$$

is the same as

$$\bigoplus_{i=0}^{d} \deg : \bigoplus_{i=0}^{d} \mathrm{Ch}^i(X) \longrightarrow \bigoplus_{i=0}^{d} \mathbf{Z} \cdot x_i \ .$$

The filtration on the latter group looks according to (2.2.2) as follows:

$$\mathcal{F}^p = \bigoplus_{i=p}^{d} \mathbf{Z} \cdot x_i \ .$$

Since the idempotent e generates $\mathcal{F}^{p/p+1}$ it has the kind $e = \pm x_p + \ldots$ where the dots denote a linear combination of other x_i. Since we have supposed that e comes from $\mathrm{Ch}^*(X)$ it implies that $1 \in \deg \mathrm{Ch}^p(X)$. This is a contradiction to (2.1.1).

Together with this proof we finished the proof of (2.2.1).

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