WEIL TRANSFER OF ALGEBRAIC CYCLES

NIKITA A. KARPENKO

Abstract. Let L/F be a finite separable field extension of degree n, X a smooth quasi-projective L-scheme, and $\mathcal{R}(X)$ the Weil transfer of X with respect to L/F. The map $Z \mapsto \mathcal{R}(Z)$ of the set of simple cycles $Z \subset X$ extends in a natural way to a map $\mathcal{Z}(X) \to \mathcal{Z}(\mathcal{R}(X))$ on the whole group of algebraic cycles $\mathcal{Z}(X)$. This map factors through the rational equivalence of cycles and induces this way a map of the Chow groups $\mathrm{CH}(X) \to \mathrm{CH}(\mathcal{R}(X))$, which, in its turn, produces a natural functor of the categories of Chow correspondences $\mathcal{CV}(L) \to \mathcal{CV}(F)$. Restricting to the graded components, one has a map $\mathcal{Z}_*(X) \to \mathcal{Z}_{n\cdot *}(\mathcal{R}(X))$, which produces a functor of the categories of degree 0 Chow correspondences $\mathcal{CV}^0(L) \to \mathcal{CV}^0(F)$, a functor of the categories of the Grothendieck Chow-motives $\mathcal{M}(L) \to \mathcal{M}(F)$, as well as functors of several other classical motivic categories.

Contents

0. Introduction

Let L/F be a finite separable field extension. We write $\mathcal{V}(L)$ for the category of smooth projective L-varieties. We consider the functor $\mathcal{V}(L) \to \mathcal{V}(F)$ given by the Weil transfer of L-varieties with respect to L/F. It is a very natural aspiration to extend this functor to the category of Chow-correspondences in order to get a functor $\mathcal{CV}(L) \to \mathcal{CV}(F)$. Of course, one may also consider different variations of $\mathcal{CV}$ like $\mathcal{CV}^0$ and $\mathcal{CV}^*$ or the Grothendieck motivic categories $\mathcal{M}^{\mathrm{eff}}$ and $\mathcal{M}$ (see §4 for the definitions). It is doubtless useful to have such functors: various known motivic isomorphisms of certain L-varieties would be transferred via these functors to motivic isomorphisms of their Weil transfers. So, it is really surprising why this sort of questions was not considered till the very recent time.

In a recent work [7], V. Joukhovitski constructed the Weil transfer functor for the category of K_0-motives introduced in [12] (see also [11]). The essential part of such construction is a definition of a map from K_0 of a variety to K_0 of its Weil transfer. For this, a method of the so called *polynomial maps*, based on some ideas of [4], was developed in [7]. Although this method can be applied to the Chow groups as well, there is another, simpler approach we use here (see Definition 2.2). In contrast to this, there is a small additional difficulty with the Chow-motives (at least with the more subtle versions of them) absent in the K_0-case: the necessity to control the degrees of the Chow-correspondences.

After recalling some basic properties of the Weil transfer of schemes (§1), we start with a definition of the Weil transfer on the group of algebraic cycles of a scheme (§2). In the next section (§3), we show that this map factors through the rational equivalence of cycles and therefore determines a map of Chow groups. In the last section (§4), we consider certain five more or less standard motivic categories based on the Chow-correspondences and show in every case that the Weil transfer gives a functor.

As an application, we compute the motive of the Weil transfer of a projective bundle (Propositin 4.6). One obviously may generalize this computation to the case of a Grassmanian bundle or to the case of a split projective homogeneous variety using the motivic decompositions of [9].

Using the motivic decompositions of isotropic flag varieties, obtained in [8], one also may get certain decompositions of the motives of the Weil transfers of such varieties.

1. Weil transfer of schemes

By a *scheme* we always mean a scheme of finite type over a field.

Let L/F be a finite extension of arbitrary fields. If the Weil transfer of an L-scheme X with respect to L/F exists, it will be denoted by $\mathcal{R}(X)$. Recall that $\mathcal{R}(X)$ is the F-scheme representing the functor $R \mapsto X(R \otimes_F L)$, where R is a commutative F-algebra. If $f\colon X \to Y$ is a morphism of L-schemes such that the Weil transfers $\mathcal{R}(X)$ and $\mathcal{R}(Y)$ exist, we write $\mathcal{R}(f)$ for the morphism $\mathcal{R}(X) \to \mathcal{R}(Y)$ induced by f.

Some known properties of the Weil transfer of schemes, which are essential for our purposes, are collected in the following

Proposition 1.1 ([2, §7.6] or [14, §4]). *The Weil transfer $\mathcal{R}(X)$ exists for any quasi-projective L-scheme X. If X is a smooth L-scheme, then $\mathcal{R}(X)$ (if exists) is a smooth F-scheme. If X_1 and X_2 are L-schemes whose Weil transfers exist, then the direct product of the F-schemes $\mathcal{R}(X_1) \times \mathcal{R}(X_2)$ is the Weil transfer of the direct product of the L-schemes $X_1 \times X_2$. If $f\colon X \to Y$ is a closed (resp., an open) imbedding of L-schemes, then $\mathcal{R}(f)\colon \mathcal{R}(X) \to \mathcal{R}(Y)$ is also a closed (resp., open) imbedding.*

Lemma 1.2. *For the affine space $\mathbb{A}_L(V)$, given by a finite-dimensional vector space V over L, one has $\mathcal{R}(\mathbb{A}_L(V)) = \mathbb{A}_F(V)$ (on the left-hand side, V is considered as a vector space over F).*

Proof. Recall that $\mathbb{A}_L(V)(S) := V \otimes_L S$ for any commutative L-algebra S. So, for any commutative F-algebra R, one has

$$\mathcal{R}(\mathbb{A}_L(V))(R) = \mathbb{A}_L(V)(R \otimes_F L) = V \otimes_L (R \otimes_F L) = V \otimes_F R = \mathbb{A}_F(V)(R)\ .$$

Therefore $\mathcal{R}(\mathbb{A}_L(V)) = \mathbb{A}_F(V)$. □

Lemma 1.3. *If the field extension L/F is separable, then the Weil transfer of the (full) grassmanian $\mathbb{\Gamma}_L(V)$ of a finite-dimensional L-vector space V is a closed subscheme of the grassmanian $\mathbb{\Gamma}_F(V)$ of the F-vector space V.*

Proof. For any commutative L-algebra R, the set $\mathbb{\Gamma}_L(V)(R)$ of all R-points of $\mathbb{\Gamma}_L(V)$ consists of the R-submodules $N \subset V \otimes_L R$ such that the quotient $(V \otimes_L R)/N$ is a projective R-module (cf. [3]). Consequently, for any commutative F-algebra R, the set $\mathcal{R}(\mathbb{\Gamma}_L(V))(R)$ consists of the R_L-submodules $N \subset V \otimes_F R$ such that the quotient $(V \otimes_F R)/N$ is a projective R_L-module. Since a projective R_L module is also projective over R, we have an imbedding of sets $\mathcal{R}(\mathbb{\Gamma}_L(V))(R) \hookrightarrow \mathbb{\Gamma}_F(V)(R)$. Moreover, an R-module $M \in \mathbb{\Gamma}_F(V)(R)$ is an element of $\mathcal{R}(\mathbb{\Gamma}_L(V))(R)$ iff $L \cdot M \subset M$ (it is essential here, that the extension L/F is separable and hence an R_L-module, which is projective over R, is projective over R_L as well). Therefore, $\mathcal{R}(\mathbb{\Gamma}_L(V))$ is a closed subscheme of $\mathbb{\Gamma}_F(V)$ (see [8, cor. 10.4]). □

Corollary 1.4. *If L/F is separable, then the Weil transfer of a quasi-projective L-scheme is a quasi-projective F-scheme and the Weil transfer of a projective L-scheme is a projective F-scheme.* □

Starting from this point, all schemes are assumed to be **quasi-projective** (although this assumption may be almost always replaced by the assumption that their Weil transfers exist).

From now on, L/F will be a finite **separable** extension of fields and n will denote its degree. We write E for the normal closure of L in a fixed separable closure of F. The Galois group of the finite Galois extension E/F is denoted by G. We write H for the subgroup of G corresponding to L. [1]

For any $\sigma \in G$ and any E-scheme X, the *conjugation* ${}^\sigma X$ of X by σ is the E-scheme, obtained from X by the base change $\sigma\colon \operatorname{Spec} E \to \operatorname{Spec} E$ (cf. [1, §2.4]).

Let G/H be the set of left cosets of G modulo H. If now X is an L-scheme, the conjugated scheme ${}^\sigma(X_E)$ depends (up to a canonical isomorphism) only on the left coset of σ modulo H; thus ${}^\sigma X_E$ is defined for $\sigma \in G/H$. Let us write $X_E^{G/H}$ for the direct product of E-schemes $\prod_{\sigma \in G/H} {}^\sigma X_E$. As an abstract scheme, $X_E^{G/H}$ has a left action of G (the action of $\tau \in G$ on $X_E^{G/H}$ is determined by the morphisms $\tau\colon {}^\sigma X_E \to {}^{\tau\sigma} X_E$). This action is compatible with the action of G on E.

Note that the action of G on E determines an action of G on Y_E for any F-scheme Y.

Lemma 1.5 ([1, §2.8])**.** *For any L-scheme X, there is a canonical isomorphism of E-schemes $\mathcal{R}(X)_E \simeq X_E^{G/H}$ respecting the action of G; moreover, the F-scheme $\mathcal{R}(X)$ is characterized by this property. If $f\colon X \to Y$ is a morphisms of L-schemes, then $\mathcal{R}(f)_E = f_E^{G/H}$.*

2. Weil transfer of cycles

Let X be an L-scheme. We consider the group $\mathcal{Z}(X)$ of algebraic cycles on X. This is by definition the free abelian group on the closed irreducible subsets of (the topological space of) X (which are called the *simple cycles* in this context). For any closed subscheme $Z \subset X$, the cycle $[Z] \in \mathcal{Z}(X)$ of Z is defined (see [5, §1.5]) as a certain linear combination (with integral coefficients) of the irreducible components of Z.

The group $\mathcal{Z}(X)$ has a gradation $\mathcal{Z}(X) = \bigoplus_{i=0}^{\dim X} \mathcal{Z}_k(X)$, where $\mathcal{Z}_k(X)$ is the subgroup generated by the simple cycles of dimension k.

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[1]For many applications it suffices to consider the case, where the extension L/F is Galois. Assuming this, one gets a little bit simpler situation, where $E = L$ and $H = \{1\}$.

A scheme is called equidimensional, if the dimensions of its irreducible components are equal. We say that a scheme is *quasi-equidimensional*, if its connected components are equidimensional. Note that a scheme smooth over a field is always quasi-equidimensional (because its connected components are irreducible).

For a quasi-equidimensional scheme X, one may define another gradation on $\mathcal{Z}(X)$, the gradation $\mathcal{Z}^*(X)$ by the codimension of cycles: since a simple cycle on X is contained in a unique connected component of X, its codimension is defined in the evident way, and $\mathcal{Z}^k(X)$ is the subgroup generated by all simple cycles of codimension k.

One says that two simple cycles Z_1 and Z_2 on a smooth scheme X meet properly, if each irreducible component of their intersection has the codimension $\operatorname{codim} Z_1 + \operatorname{codim} Z_2$. Two cycles α and β on X are said to meet properly, if each simple cycle contained in α meets properly each simple cycle contained in β; in this situation the product $\alpha \cdot \beta \in \mathcal{Z}(X)$ is defined (e.g., via Serre's formula, [16]).

Now let $f : X \to Y$ be a morphism of a quasi-equidimensional scheme X to a smooth scheme Y. One says that a simple cycle $Z \subset Y$ has a proper preimage (with respect to f), if the codimension in X of each irreducible component of the preimage $f^{-1}(Z)$ equals $\operatorname{codim}_Y Z$. A cycle $\alpha \in \mathcal{Z}(Y)$ has a proper preimage, if every simple cycle contained in Y has; in this situation the pull-back $f^*(\alpha) \in \mathcal{Z}(X)$ is defined (e.g., via Serre's formula, [16]).

We shall also speak of the *flat pull-back* and *proper push-forward* of algebraic cycles defined in [5, §1.7 and §1.4].

Our aim in this section is to define the Weil transfer map $\mathcal{R}\colon \mathcal{Z}(X) \to \mathcal{Z}(\mathcal{R}(X))$ and to establish some of its basic properties. This will be a map (not a homomorphism!) such that $\mathcal{R}([Z]) = [\mathcal{R}(Z)]$ for any closed subscheme $Z \subset X$; however this formula is insufficient for a definition of the map: at least the cycles with negative coefficients are not covered by it.

Our key tool is the following simple

Lemma 2.1. *For any F-scheme Y, the restriction homomorphism $\operatorname{res}_{E/F}\colon \mathcal{Z}(Y) \to \mathcal{Z}(Y_E)$ is injective; its image coincides with the subgroup of G-invariant cycles in $\mathcal{Z}(Y_E)$.* □

Now we give a definition of the Weil transfer map $\mathcal{R}\colon \mathcal{Z}(X) \to \mathcal{Z}(\mathcal{R}(X))$. For any cycle $\alpha \in \mathcal{Z}(X)$ and any $\sigma \in G/H$, let us write ${}^\sigma\alpha_E$ for α_E considered as a cycle on ${}^\sigma X_E$ with the help of the isomorphisms of abstract schemes $\sigma : {}^\sigma X \to X$. We write then $\alpha_E^{G/H}$ for the exterior product of cycles (defined as in [5, §1.10]) $\prod_{\sigma\in G/H} {}^\sigma\alpha_E \in \mathcal{Z}(X_E^{G/H}) = \mathcal{Z}(\mathcal{R}(X)_E)$. Since the cycle $\alpha_E^{G/H}$ is G-invariant, we conclude by Lemma 2.1 that $\alpha_E^{G/H} = \mathcal{R}(\alpha)_E$ for a unique cycle $\mathcal{R}(\alpha) \in \mathcal{Z}(\mathcal{R}(X))$.

Definition 2.2. The Weil transfer of cycles is the map $\mathcal{R} : \mathcal{Z}(X) \to \mathcal{Z}(\mathcal{R}(X))$ defined by the formula $\alpha \mapsto \mathcal{R}(\alpha)$, where $\mathcal{R}(\alpha)$ is the cycle in $\mathcal{Z}(\mathcal{R}(X))$ such that $\mathcal{R}(\alpha)_E = \alpha_E^{G/H} \in \mathcal{Z}(\mathcal{R}(X)_E)$.

Remark 2.3. Restricting the map $\mathcal{R}\colon \mathcal{Z}(X) \to \mathcal{Z}(\mathcal{R}(X))$ to the gradation components of $\mathcal{Z}(X)$, one gets maps $\mathcal{R}\colon \mathcal{Z}_k(X) \to \mathcal{Z}_{nk}(\mathcal{R}(X))$ for all k. Note that these maps do not determine the original map, since we are not dealing with the group homomorphisms.

In view of Lemma 1.5, the following fact is obvious:

Lemma 2.4. *For any closed subscheme $Z \subset X$, one has $\mathcal{R}([Z]) = [\mathcal{R}(Z)]$.* □

Now we list some basic properties of this Weil transfer map.

Proposition 2.5. *The Weil transfer of cycles commutes with the flat pull-backs, smooth pull-backs, proper push-forwards, products of cycles (interior and exterior ones), and with the diagonals. In more detail,*

- **(flat pull-back)** *for any flat morphism of L-schemes $f\colon X \to Y$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{Z}(X) & \xleftarrow{f^*} & \mathcal{Z}(Y) \\ \mathcal{R}\downarrow & & \downarrow\mathcal{R} \\ \mathcal{Z}(\mathcal{R}(X)) & \xleftarrow{\mathcal{R}(f)^*} & \mathcal{Z}(\mathcal{R}(Y)) \end{array} ;$$

- **(smooth pull-back)** *for any morphism of L-schemes $f\colon X \to Y$ with smooth Y and quasi-equidimensional X, and for any cycle $\alpha \in \mathcal{Z}(Y)$ with proper $f^{-1}(\alpha)$, the cycle $\mathcal{R}(\alpha) \in \mathcal{Z}(\mathcal{R}(Y))$ has proper $\mathcal{R}(f)^{-1}(\mathcal{R}(\alpha))$ and $\mathcal{R}(f^*(\alpha)) = \mathcal{R}(f)^*(\mathcal{R}(\alpha))$;*
- **(proper push-forward)** *for any proper morphism of L-schemes $f\colon X \to Y$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{Z}(X) & \xrightarrow{f_*} & \mathcal{Z}(Y) \\ \mathcal{R}\downarrow & & \downarrow\mathcal{R} \\ \mathcal{Z}(\mathcal{R}(X)) & \xrightarrow{\mathcal{R}(f)_*} & \mathcal{Z}(\mathcal{R}(Y)) \end{array} ;$$

- **(interior product)** *for any smooth L-scheme X and for any properly meeting cycles $\alpha, \beta \in \mathcal{Z}(X)$, the cycles $\mathcal{R}(\alpha), \mathcal{R}(\beta) \in \mathcal{Z}(\mathcal{R}(X))$ meet also proper and $\mathcal{R}(\alpha) \cdot \mathcal{R}(\beta) = \mathcal{R}(\alpha \cdot \beta)$;*
- **(exterior product)** *for any L-schemes X and Y and for any cycles $\alpha \in \mathcal{Z}(X)$ and $\beta \in \mathcal{Z}(Y)$, one has $\mathcal{R}(\alpha) \times \mathcal{R}(\beta) = \mathcal{R}(\alpha \times \beta)$ under the identification $\mathcal{R}(X) \times \mathcal{R}(Y) = \mathcal{R}(X \times Y)$;*
- **(diagonal)** *for any separated L-scheme X, one has $\mathcal{R}(\Delta_X) = \Delta_{\mathcal{R}(X)}$, where Δ_X stays for the cycle of the diagonal of X.*

Proof. The proof of the first five properties is reduced to the proof that two certain cycles α and β in $\mathcal{Z}(\mathcal{R}(X))$ coincide, where X is a certain L-scheme. By Lemma 2.1, we may restrict the scalars to E and work with the cycles α_E and β_E on $X_E^{G/H}$. Now these statements become completely evident.

The assertion on the diagonal follows from Lemma 2.4 and the fact that the Weil transfer of the diagonal morphism $X \hookrightarrow X \times X$ is the diagonal morphism $\mathcal{R}(X) \hookrightarrow \mathcal{R}(X) \times \mathcal{R}(X)$. □

3. Weil transfer on Chow groups

The Chow group $\mathrm{CH}(X)$ of a scheme X is defined as the group of cycles on X modulo rational equivalence (see [5, §1.3]). Since the group of cycles rationally equivalent to zero is a homogeneous subgroup of $\mathcal{Z}_*(X)$, the gradation of $\mathcal{Z}(X)$ is inherited by $\mathrm{CH}(X)$. If X is quasi-equidimensional, then the group of cycles rational equivalent to zero is also homogeneous with respect to the gradation $\mathcal{Z}^*(X)$, therefore the gradation by codimension of cycles is also inherited by $\mathrm{CH}(X)$ in this case.

Let X be a *smooth* (as always, quasi-projective) L-scheme. The aim of this section is to show that the Weil transfer of cycles on X induces a map $\mathrm{CH}(X) \to \mathrm{CH}(\mathcal{R}(X))$.

The following statement, which is an alternative definition of the rational equivalence, is well-known (cf. [5, §1.6]):

Lemma 3.1. *Let X be a smooth L-scheme and let $\alpha_0, \alpha_1 \in \mathcal{Z}(X)$. The cycles α_0 and α_1 are rationally equivalent if and only if there exists a cycle $\beta \in \mathcal{Z}(X \times \mathbb{A}^1_L)$ such that*

- *β has a proper preimage with respect to both $i_0, i_1 \colon X \hookrightarrow X \times \mathbb{A}^1_L$, where i_0 and i_1 are the closed imbeddings given by the rational points 0 and 1 of $\mathbb{A}^1_L = \mathbb{A}_L(L)$ respectively;*
- *$i_0^*(\beta) = \alpha_0$ and $i_1^*(\beta) = \alpha_1$.*

Proposition 3.2. *In the conditions of Lemma 3.1, if the cycles $\alpha_0, \alpha_1 \in \mathcal{Z}(X)$ are rationally equivalent, then the cycles $\mathcal{R}(\alpha_0), \mathcal{R}(\alpha_1) \in \mathcal{Z}(\mathcal{R}(X))$ are rationally equivalent as well.*

Proof. We take a cycle $\beta \in \mathcal{Z}(X \times \mathbb{A}^1_L)$ as in Lemma 3.1. By Lemma 2.5, the cycle $\mathcal{R}(\beta)$ has a proper preimage with respect to $\mathcal{R}(i_0)$ and $\mathcal{R}(i_1)$, and its pull-backs are $\mathcal{R}(\alpha_0)$ and $\mathcal{R}(\alpha_1)$.

The morphisms $\mathcal{R}(i_0), \mathcal{R}(i_1) : \mathcal{R}(X) \to \mathcal{R}(X \times \mathbb{A}^1_L) = \mathcal{R}(X) \times \mathbb{A}_F(L)$ are the closed imbeddings given by the rational points $0, 1$ of $\mathbb{A}_F(L)$. They are sections of the projection $pr \colon \mathcal{R}(X) \times \mathbb{A}_F(L) \to \mathcal{R}(X)$. Since the pull-back pr^* is an isomorphism of the Chow groups ([5, thm. 3.3.a]), it follows that the cycles $\mathcal{R}(i_0)^*(\beta)$ and $\mathcal{R}(i_1)^*(\beta)$ are rationally equivalent. □

Corollary 3.3. *For any finite separable field extension L/F and any smooth quasi-projective L-scheme X, the Weil transfer map $\mathcal{Z}(X) \to \mathcal{Z}(\mathcal{R}(X))$ factors through the rational equivalence giving a map $\mathrm{CH}(X) \to \mathrm{CH}(\mathcal{R}(X))$.* □

Passing to the Chow group, we obtain from Proposition 2.5 the following

Proposition 3.4. *Let X and Y be smooth L-schemes. Then*

- **(pull-back)** *for any morphism $f \colon X \to Y$, the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{CH}(X) & \xleftarrow{f^*} & \mathrm{CH}(Y) \\ {\scriptstyle\mathcal{R}}\downarrow & & \downarrow{\scriptstyle\mathcal{R}} \\ \mathrm{CH}(\mathcal{R}(X)) & \xleftarrow{\mathcal{R}(f)^*} & \mathrm{CH}(\mathcal{R}(Y)) \end{array} ;$$

- **(proper push-forward)** *for any proper morphism $f \colon X \to Y$, the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{CH}(X) & \xrightarrow{f_*} & \mathrm{CH}(Y) \\ {\scriptstyle\mathcal{R}}\downarrow & & \downarrow{\scriptstyle\mathcal{R}} \\ \mathrm{CH}(\mathcal{R}(X)) & \xrightarrow{\mathcal{R}(f)_*} & \mathrm{CH}(\mathcal{R}(Y)) \end{array} ;$$

- **(interior product)** *for any $\alpha, \beta \in \mathrm{CH}(X)$,*

$$\mathcal{R}(\alpha \cdot \beta) = \mathcal{R}(\alpha) \cdot \mathcal{R}(\beta) \in \mathrm{CH}(\mathcal{R}(X)) ;$$

- **(exterior product)** *for any* $\alpha \in \mathrm{CH}(X)$ *and* $\beta \in \mathrm{CH}(Y)$*, one has*

$$\mathcal{R}(\alpha \times \beta) = \mathcal{R}(\alpha) \times \mathcal{R}(\beta) \in \mathrm{CH}(\mathcal{R}(X \times Y)) .$$

□

4. Weil transfer of motives

We write $\mathcal{V}(F)$ for the category of smooth projective F-schemes. The objects of $\mathcal{V}(F)$ will be called (F-)varieties for short. For $X, Y \in \mathcal{V}(F)$, the group $\mathrm{CH}(X \times Y)$ of correspondences from X to Y will be denoted $\mathrm{Corr}(X, Y)$ (the elements of $\mathrm{Corr}(X, Y)$ are called correspondences; in what follows, we use the classical notion of the composition for correspondences, see [5, def. 16.1.1]).

Recall that for any $r \in \mathbb{Z}$, any $X \in \mathcal{V}(F)$, and any equidimensional $Y \in \mathcal{V}(F)$, the correspondences in $\mathrm{Corr}(X, Y)$ given by the elements of the group $\mathrm{CH}^{\dim Y + r}(X \times Y)$ are called the correspondences of degree r. The group of all correspondences of degree r is denoted by $\mathrm{Corr}^r(X, Y)$. In the case of arbitrary $Y \in \mathcal{V}(F)$, one sets $\mathrm{Corr}^r(X, Y) := \bigoplus_i \mathrm{Corr}^r(X, Y_i)$, where Y_i are the components of Y. Note that the degrees of correspondences are added when the correspondences are composed ([5, example 16.1.1]). In particular, the composition of degree 0 correspondences is a degree 0 correspondence as well.

Remark 4.1. The definition of the degree of correspondences we use coincides with that of [5, example 16.1.1] and *does not* coincides with that of [6], [10], and [15]. By consequence, our motivic categories, introduced below, are *dual* to the corresponding motivic categories of [6], [10], and [15] (this is not important for $\mathcal{CV}(F)$, which is self-dual, but *is* important for $\mathcal{CV}^0(F)$, $\mathcal{CV}^*(F)$, $\mathcal{M}^{\mathrm{eff}}(F)$, and $\mathcal{M}(F)$). The choice of one of these two possible definitions of the degree is a question of taste; essentially, it depends on whether one prefers to have a *covariant* or *contravariant* canonical functor from $\mathcal{V}(F)$ to the motivic categories (this functor is covariant in our setting).

We are going to consider several classical motivic categories. The first one is the full category of correspondences $\mathcal{CV}(F)$. This is an additive category with the same objects as $\mathcal{V}(F)$ and with $\mathrm{Hom}(X, Y) := \mathrm{Corr}(X \times Y)$.

The second one is the category $\mathcal{CV}^0(F)$ of the degree 0 correspondences. Recall that this is a category with the same objects as $\mathcal{V}(F)$ and with $\mathrm{Hom}(X, Y) := \mathrm{Corr}^0(X, Y)$.

The next one is the category $\mathcal{M}^{\mathrm{eff}}(F)$ of the effective Grothendieck Chow-motives. This category is by definition the pseudo-abelian completion of $\mathcal{CV}^0(F)$. In more detail, the objects of $\mathcal{M}^{\mathrm{eff}}(F)$ are pairs (X, p), where $X \in \mathcal{V}(F)$ and p is a projector on X, i.e., an idempotent of the ring $\mathrm{Corr}^0(X, X)$; the group $\mathrm{Hom}((X, p), (Y, q))$ is defined as $q \circ \mathrm{Corr}^0(X, Y) \circ p \subset \mathrm{Corr}^0(X, Y)$.

The category $\mathcal{M}^{\mathrm{eff}}(F)$ can be enlarged by adding a formal twist. The result is denoted by $\mathcal{M}(F)$ and called the category of Grothendieck's Chow-motives. The following very simple formal definition (equivalent to the original one) of $\mathcal{M}(F)$ (as well as the definition of $\mathcal{M}^{\mathrm{eff}}(F)$ given above) is due to Jannsen, [6] (see also [15, §1.4]): the objects of $\mathcal{M}(F)$ are the triples (X, p, l), where $X \in \mathcal{V}(F)$, p is a projector on X, and $l \in \mathbb{Z}$; the group $\mathrm{Hom}((X, p, l), (Y, q, m))$ is defined as

$$q \circ \mathrm{Corr}^{m-l}(X, Y) \circ p \subset \mathrm{Corr}^{m-l}(X, Y) .$$

Let us accept the usual agreement that in the notation (X, p, l), one may omit p, if $p = id_X$; also one may omit l, if $l = 0$.

There is an important subcategory of $\mathcal{M}(F)$: the full subcategory of finite direct sums $(X_1, l_1) \oplus \cdots \oplus (X_k, l_k)$ of twists of varieties. Let us denote this subcategory by $\mathcal{CV}^*(F)$. Obviously, one may give a direct definition of $\mathcal{CV}^*(F)$: the objects are the formal direct sums $(X_1, l_1) \oplus \cdots \oplus (X_k, l_k)$ with $k \geq 0$ and $\operatorname{Hom}((X, l), (Y, m)) := \operatorname{Corr}^{m-l}(X, Y)$.

All these categories (namely, $\mathcal{CV}(F)$, $\mathcal{CV}^0(F)$, $\mathcal{M}^{\text{eff}}(F)$, $\mathcal{M}(F)$, and $\mathcal{CV}^*(F)$) are additive (for $\mathcal{CV}(F)$ and $\mathcal{CV}^0(F)$, one should add the zero-object formally or agree that $\emptyset \in \mathcal{V}(F)$). The only not completely obvious part of this statement is the fact that the finite direct sums exist in $\mathcal{M}(F)$; this fact is a consequence of the following known lemma (which will be needed below also for other purposes). We recall that the category $\mathcal{M}(F)$ (as well as the other four motivic categories) has a tensor structure, defined by the formula $(X, p, l) \otimes (Y, q, m) := (X \times Y, p \otimes q, l + m)$ (cf. [15, §1.9]). We write $\mathbf{pt} \in \mathcal{V}(F)$ for $\operatorname{Spec} F$.

Lemma 4.2 (cf. [10, §6] with Remark 4.1)**.** *Let Y be a connected F-variety with a rational point y (e.g., a product of projective spaces). Then $[y \times Y]$ is a projector on Y and in $\mathcal{M}(F)$ there is an isomorphism*

$$(Y, [y \times Y]) \simeq (\mathbf{pt}, l) \ ,$$

where $l = \dim Y$.

Proof. The mutually inverse isomorphisms are given by the correspondences $[y \times \mathbf{pt}] \in \operatorname{Corr}^l(Y, \mathbf{pt})$ and $[\mathbf{pt} \times Y] \in \operatorname{Corr}^{-l}(\mathbf{pt}, Y)$. □

Corollary 4.3. *The finite direct sums exist in $\mathcal{M}(F)$.*

Proof. We have to show that the sum $(X, p, l) \oplus (Y, q, m)$ exists. We may assume that $l \geq m$. Since $(\mathbf{pt}, l - m) = (X', p')$ for some $X' \in \mathcal{V}(F)$ and some projector p' by Lemma 4.2 (e.g., one may take as X' the $(l-m)$-dimensional projective space or the product of $l - m$ projective lines), we have

$$(X, p, l) = (X, p, m) \otimes (\mathbf{pt}, l - m) = (X, p, m) \otimes (X', p') = (X \times X', p \otimes p', m) \ .$$

Therefore we may assume in the very beginning that $l = m$. Then the direct sum is given by the object $(X \coprod Y, p \coprod q, m)$, where $X \coprod Y$ stays for the disjoint union of the varieties X and Y. □

The five motivic categories and the category of varieties are related by some evident functors shown in the following (commutative) diagram:

$$\begin{array}{ccccccc}
\mathcal{V}(F) & \longrightarrow & \mathcal{CV}^0(F) & \longrightarrow & \mathcal{CV}^*(F) & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{M}^{\text{eff}}(F) & \longrightarrow & \mathcal{M}(F) & \longrightarrow & \mathcal{CV}(F)
\end{array}$$

(we recall that the functor $\mathcal{V}(F) \to \mathcal{CV}^0(F)$ takes the graphs of the morphisms of varieties; the functor $\mathcal{M}(F) \to \mathcal{CV}(F)$ is a sort of forgetful one: the object (X, p, l) is send to X). In particular, we have a functor from $\mathcal{V}(F)$ to any of the motivic categories, so that we may speak about extending the Weil transfer functor $\mathcal{R} \colon \mathcal{V}(L) \to \mathcal{V}(F)$ to every of the motivic categories (note that $\mathcal{R}(X) \in \mathcal{V}(F)$ for any $X \in \mathcal{V}(L)$: the scheme $\mathcal{R}(X)$ is smooth by Proposition 1.1 and projective by Corollary 1.4).

Here is the main assertion of the paper:

Theorem 4.4. *Let L/F be a finite separable field extension of degree n. The Weil transfer functor $\mathcal{R}\colon \mathcal{V}(L) \to \mathcal{V}(F)$ can be extended to the functors (non-additive but commuting with the tensor products)*

$$\mathcal{CV}(L) \to \mathcal{CV}(F),\ \ \mathcal{CV}^0(L) \to \mathcal{CV}^0(F),\ \ \mathcal{CV}^*(L) \to \mathcal{CV}^*(F),$$
$$\mathcal{M}^{\mathrm{eff}}(L) \to \mathcal{M}^{\mathrm{eff}}(F), \quad \textit{and} \quad \mathcal{M}(L) \to \mathcal{M}(F)\ ,$$

where the last functor on the objects is $(X,p,l) \mapsto (\mathcal{R}(X), \mathcal{R}(p), nl)$ (this formula determines also all the other functors on the objects). On the morphisms, each of these functors is given by the Weil transfer of the correspondences.

Proof. We consider the five motivic categories one by one.

$\underline{\mathcal{CV}(L) \to \mathcal{CV}(F).}$ The definition is: $X \mapsto \mathcal{R}(X)$ on the objects and $\alpha \mapsto \mathcal{R}(\alpha)$ on the morphisms.

Since the Weil transfer commutes with direct products (Proposition 1.1), $\mathcal{R}(\alpha)$ is really an element of $\mathrm{Hom}(\mathcal{R}(X), \mathcal{R}(Y))$ for $\alpha \in \mathrm{Hom}(X,Y)$. Since the identities in $\mathcal{CV}(L)$ are given by the diagonal classes, which are preserved by the Weil transfer (Proposition 2.5), one has $\mathcal{R}(id_X) = id_{\mathcal{R}(X)}$ for any X.

The composition of correspondences is defined by applying pull-back, push-forward and by taking products of cycles. Since the Weil transfer commutes with these operations (Proposition 3.4), it also commutes with the composition of correspondences.

Finally, it is clear from the direct product part of Proposition 1.1 and from the exterior product part of Proposition 3.4 that the constructed functor commutes with the tensor products.

$\underline{\mathcal{CV}^0(L) \to \mathcal{CV}^0(F).}$ We have to look at the degrees of correspondences under the Weil transfer:

Lemma 4.5. *For any $r \in \mathbb{Z}$ and any $X, Y \in \mathcal{CV}(L)$, one has $\mathcal{R}(\mathrm{Corr}^r(X,Y)) \subset \mathrm{Corr}^{nr}(\mathcal{R}(X), \mathcal{R}(Y))$.*

Proof. For $X, Y \in \mathcal{V}(L)$, let us denote the group $\mathcal{Z}(X \times Y)$ by $\mathcal{Z}(X,Y)$ and define $\mathcal{Z}^r(X,Y) \subset \mathcal{Z}(X,Y)$ (for any $r \in \mathbb{Z}$) the way similar to as $\mathrm{Corr}^r(X,Y)$ was defined. Take a cycle $\alpha \in \mathcal{Z}^r(X,Y)$. To prove the lemma, it suffices to show that $\mathcal{R}(\alpha) \in \mathcal{Z}^{nr}(\mathcal{R}(X), \mathcal{R}(Y))$. For this it suffices to show that $\mathcal{R}(\alpha)_E \in \mathcal{Z}^{nr}(\mathcal{R}(X)_E, \mathcal{R}(Y)_E)$ (the last reduction is not possible on the level of Chow groups, but is possible on the level of cycles). Since $\mathcal{R}(\alpha)_E = \alpha_E^{G/H}$, the last inclusion is already obvious. □

In particular, the Weil transfer of a degree 0 correspondence is once again a correspondence of degree 0. So, restricting the functor $\mathcal{R}\colon \mathcal{CV}(L) \to \mathcal{CV}(F)$ to the subcategory $\mathcal{CV}^0(L) \subset \mathcal{CV}(L)$, one obtains the required functor $\mathcal{CV}^0(L) \to \mathcal{CV}^0(F)$.

$\underline{\mathcal{M}^{\mathrm{eff}}(L) \to \mathcal{M}^{\mathrm{eff}}(F).}$ The definition on the object is $(X,p) \mapsto (\mathcal{R}(X), \mathcal{R}(p))$. Since $\mathcal{R}(p)$ is really a projector on $\mathcal{R}(X)$, the definition on the objects is correct. If $\alpha \in \mathrm{Hom}((X,p),(Y,q))$, then $\alpha = q \circ \alpha \circ p$, and we have $\mathcal{R}(\alpha) = \mathcal{R}(q) \circ \mathcal{R}(\alpha) \circ \mathcal{R}(p) \in \mathrm{Hom}(\mathcal{R}(X,p), \mathcal{R}(Y,q))$.

$\underline{\mathcal{M}(L) \to \mathcal{M}(F).}$ To check that the definition on morphisms is correct, one uses Lemma 4.5. The tensor structure is respected by the same reason.

$\mathcal{CV}^*(L) \to \mathcal{CV}^*(F)$. We claim that restricting the functor $\mathcal{R} : \mathcal{M}(L) \to \mathcal{M}(F)$ to the subcategory $\mathcal{CV}^*(L) \subset \mathcal{M}(L)$, one gets a functor $\mathcal{CV}^*(L) \to \mathcal{CV}^*(F)$. More precisely, for any $X_1, \dots, X_k \in \mathcal{V}(L)$ and any integers $l_1, \dots, l_k$, the Weil transfer of the motive $(X_1, l_1) \oplus \dots \oplus (X_k, l_k)$ is the direct sum of certain twists of the motives of the components of the variety $\mathcal{R}(X)$, where $X := X_1 \coprod \dots \coprod X_k$.

Let us prove this. Since the Weil transfer on $\mathcal{M}$ commutes with tensor products, we may assume that all the integers $l_1, \dots, l_k$ are non-negative. For any non-negative integer j, let P_j be the product of j projective lines. Then, by Lemma 4.2,

$$(X_1, l_1) \oplus \dots \oplus (X_k, l_k) \simeq \left(\coprod_{r=1}^{k} X_r \times P_{l_r}, \coprod_{r=1}^{k} id_{X_r} \otimes [p_r \times P_{l_r}] \right) ,$$

where $p_r \in P_{l_r}$ is a rational point. The right-hand side of this formula has the advantage that the definition of $\mathcal{R}$ can be applied to it directly.

Set $T_r := X_r \times P_{l_r}$ and $T := \coprod_{r=1}^{k} T_r$. The E-scheme $T_E^{G/H}$ is the disjoint union of the products $T_i := \prod_{\sigma \in G/H} {}^{\sigma}T_{i(\sigma)}$, where i is a map $G/H \to \{1, \dots, k\}$. Let us introduce the notation $X_i := \prod_{\sigma \in G/H} {}^{\sigma}X_{i(\sigma)}$. Then obviously $T_i = X_i \times P_{|i|}$, where $|i| := \sum_{\sigma \in G/H} l_{i(\sigma)}$.

Acting on G/H, the Galois group G acts on the set of the maps i. Acting on $T_E^{G/H}$, G interchanges T_i in the following way: $\tau(T_i) = T_{\tau(i)}$. Every orbit S of this action determines an F-scheme T_S; $\mathcal{R}(T)$ is the disjoint union of these T_S. Acting on $X_E^{G/H}$, G interchanges X_i in the same way: $\tau(X_i) = X_{\tau(i)}$. Every orbit S of this action determines an F-scheme X_S; $\mathcal{R}(X)$ is the disjoint union of these X_S. Moreover, $T_S = X_S \times P_{|S|}$ (here $P_{|S|}$ is the product of projective lines *over* F), if we define $|S|$ as $|i|$, where $i \in S$ (clearly, $|i|$ are equal for all i from the same orbit S).

Finally, computing the Weil transfer of the projector $\coprod_{r=1}^{k} id_{X_r} \otimes [p_r \times P_{l_r}]$, we get $\coprod_S X_S \otimes [p_S \times P_{|S|}]$, where p_S is a rational point on $P_{|S|}$. Applying Lemma 4.2 once again, we see that

$$\mathcal{R}((X_1, l_1) \oplus \dots \oplus (X_k, l_k)) = \bigoplus_S (X_S, |S|) .$$

We have proven that the restriction of the functor $\mathcal{R} : \mathcal{M}(L) \to \mathcal{M}(F)$ to the subcategory $\mathcal{CV}^*(L) \subset \mathcal{M}(L)$ gives a functor $\mathcal{CV}^*(L) \to \mathcal{CV}^*(F)$. The proof of Theorem 4.4 is complete. □

Let us explain an explicit way to compute the Weil transfer $\mathcal{CV}^*(L) \to \mathcal{CV}^*(F)$ which comes out from the proof of Theorem 4.4 and can be used for a direct definition of this functor. Take an object $(X_1, l_1) \oplus \dots \oplus (X_k, l_k)$ of $\mathcal{CV}^*(L)$. Set $X := X_1 \coprod \dots \coprod X_k \in \mathcal{V}(L)$. The E-scheme $X_E^{G/H}$ is a disjoint union of the products $X_i := \prod_{\sigma \in G/H} {}^{\sigma}X_{i(\sigma)}$, where i is a map $G/H \to \{1, \dots, k\}$. Acting on G/H, the Galois group G acts on the set of the maps i. Acting on $X_E^{G/H}$, G interchanges X_i in the following way: $\tau(X_i) = X_{\tau(i)}$. Every orbit S of this action determines an F-scheme X_S; $\mathcal{R}(X)$ is the disjoint union of these X_S. As can be seen from the proof of Theorem 4.4, one has

$$\mathcal{R}((X_1, l_1) \oplus \dots \oplus (X_k, l_k)) = \bigoplus_S (X_S, |S|) ,$$

where $|S| := |i|$ for any $i \in S$ and $|i| := \sum_{\sigma \in G/H} l_{i(\sigma)}$.

We conclude by the computation of the Weil transfer of the motive of a projective bundle.

Proposition 4.6. *Let $X \in \mathcal{V}(L)$ and let $\mathcal{E}$ be a rank k vector bundle over X. The motive of the Weil transfer of the projective bundle $\mathbb{P}_X(\mathcal{E})$ can be described as follows. Consider the action of the Galois group G on the set of maps $G/H \to \{0, \ldots, k\}$. For every orbit S of this action, we fix an element $i_S \in S$ and write $H_S \subset G$ for the stabilizer of i_S, $L_S \subset E$ for the subfield of the elements fixed by H_S, and $|S|$ for $|i_S| = \sum_{\sigma \in G/H} i_S(\sigma)$. Then the motive of $\mathcal{R}(\mathbb{P}_X(\mathcal{E}))$ (in the category $\mathcal{CV}^*(F)$ as well as in the category $\mathcal{M}(F)$) is isomorphic to the direct sum of the motives $\bigoplus_S (\mathcal{R}(X)_{L_S}, |S|)$ taken over the all orbits S.*

Remark 4.7. For a given orbit S, the subfield $L_S \subset E$ depends of course on the choice of $i_S \in S$. However two different choices give conjugated subfields, therefore the isomorphism class of L_S over F does not depend on the choice of i_S (if the extension E/F is abelian as in the examples below, then of course L_S is uniquely determined as a subfield of E). As discussed above, the integer $|S|$ never depends on the choice of i_S.

Proof of Proposition 4.6. First of all, using the motivic isomorphism $(\mathbb{P}_X(\mathcal{E})) \simeq (X) \otimes (\mathbb{P}^k_L)$ (cf. [10, §7]), we reduce the proof to the case where $X = \operatorname{Spec} L$ and $\mathbb{P}_X(\mathcal{E}) = \mathbb{P}^k_L$. Then we use the classical motivic decomposition of the projective space $(\mathbb{P}^k_L) \simeq \operatorname{Spec}(L) \oplus (\operatorname{Spec}(L), 1) \oplus \cdots \oplus (\operatorname{Spec}(L), k)$ (cf. [10, §7]) and the algorithm formulated before Proposition 4.6. □

Examples 4.8. We apply Proposition 4.6 to the projective line $\mathbb{P}^1_L$ in the case of some particular Galois extensions L/F of small degrees. For a finite separable field extension K/F, let us agree to write simply K for the F-variety $\operatorname{Spec} K \in \mathcal{V}(F)$.

1. Let L/F be a separable quadratic extension (so, $G = \mathbb{Z}/2$). Then

$$(\mathcal{R}(\mathbb{P}^1_L)) \simeq (F) \oplus (L, 1) \oplus (F, 2) .$$

(Suppose that $\operatorname{char} F \neq 2$ and $L = F(\sqrt{d})$. Then the variety $\mathcal{R}(\mathbb{P}^1_L)$ is isomorphic to the projective quadric over F determined by an isotropic 4-dimensional quadratic form of the discriminant d, and the motivic decomposition given above coincides with Rost's motivic decomposition of isotropic quadrics [13, prop. 2] written down for this particular quadric).

2. If L/F is a cubic Galois extension ($G = \mathbb{Z}/3$), then

$$(\mathcal{R}(\mathbb{P}^1_L)) \simeq (F) \oplus (L, 1) \oplus (L, 2) \oplus (F, 3) .$$

3. If L/F is a separable biquadratic extension ($G = \mathbb{Z}/2 \oplus \mathbb{Z}/2$), then

$$(\mathcal{R}(\mathbb{P}^1_L)) \simeq (F) \oplus (L, 1) \oplus (K_1, 2) \oplus (K_2, 2) \oplus (K_3, 2) \oplus (L, 3) \oplus (F, 4) ,$$

where K_1, K_2, K_3 are the three different quadratic subextensions of L/F.

4. If L/F is a cyclic extension of degree 4 ($G = \mathbb{Z}/4$), then

$$(\mathcal{R}(\mathbb{P}^1_L)) \simeq (F) \oplus (L, 1) \oplus (L, 2) \oplus (K, 2) \oplus (L, 3) \oplus (F, 4) ,$$

where K is the quadratic subextension of L/F.

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Westfälische Wilhelms-Universität, Mathematisches Institut, Einsteinstr. 62, D-48149 Münster, Germany

E-mail address: `karpenk@math.uni-muenster.de`