A COUNTER-EXAMPLE BY YAGITA

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ABSTRACT. According to a 2018 preprint by Nobuaki Yagita, the conjecture on a relationship between K- and Chow theories for a generically twisted flag variety of a split semisimple algebraic group G, due to the author, fails for G the spinor group $\mathrm{Spin}(17)$. Yagita's tools include a Brown-Peterson version of algebraic cobordism, ordinary and connective Morava K-theories, as well as Grothendieck motives related to various cohomology theories over fields of characteristic 0. We provide a proof using only the K- and Chow theories themselves and extend the (slightly modified) example to arbitrary characteristic.

CONTENTS

1. INTRODUCTION

For a smooth algebraic variety X over an arbitrary field F, we write $\mathrm{CH}(X)$ for the Chow ring of X and $K(X)$ for the Grothendieck ring of vector bundles on X. There is a canonical surjective homomorphism of graded rings $\varphi_X \colon \mathrm{CH}(X) \to \mathrm{Chow}\,K(X)$ onto the ring associated with the Chow filtration on $K(X)$, [12, §3]. The elements in the kernel of φ_X are known to be of finite order.

Let G be a split semisimple algebraic group over F and let X be the variety of its Borel subgroups, twisted by a generic G-torsor. It has been conjectured in [8, Conjecture 1.1] that φ_X is an isomorphism. This conjecture appears already in [9, §1] in form of a question. The answer depends only on G and does not depend on the choice of a generic G-torsor. For any given G, the variety of Borel subgroups can be replaced by the variety of *special* (i.e., not admitting non-trivial torsors over fields) parabolic subgroups of any given type: an equivalent version of the conjecture/question is produced this way.

For any odd integer $l \geq 1$, the cases $G = \mathrm{Spin}(l)$ and $G = \mathrm{Spin}(l+1)$ of this conjecture are equivalent and mean that φ_X is an isomorphism for X the highest orthogonal grassmannian of a generic l-dimensional quadratic form of trivial Clifford invariant. The *index* $\mathrm{ind}(X)$ (= the greatest common divisor of the degrees of closed points on X) has been computed for such X by Burt Totaro in [16]. In particular, one has $\mathrm{ind}(X) = 2^4$ in the case of $l = 17$.

According to Nobuaki Yagita the above conjecture fails for $G = \mathrm{Spin}(17)$. The following statement appears in [19] over a field of characteristic 0 and is extended here to arbitrary characteristic:

Theorem 1.1 (cf. [19]). *Let X be the highest orthogonal grassmannian of an arbitrary non-degenerate 17-dimensional quadratic form over F of trivial Clifford invariant. Assume that $\mathrm{ind}\, X = 2^4$. Then φ_X is not an isomorphism.*

To provide more details, let us write $c_k^{\mathrm{CH}} \in \mathrm{CH}^k(X)$, $k = 1, \ldots, 8$, for the Chern classes of the dual to the (rank-8) tautological vector bundle on X. Since the Clifford invariant of the quadratic form is trivial, the group $\mathrm{CH}^1(X)$ is generated by an element e^{CH} satisfying the relation $2e^{\mathrm{CH}} = c_1^{\mathrm{CH}}$. By Corollary 3.4 and Proposition 4.4, proved below, the product

$$c_2^{\mathrm{CH}} c_3^{\mathrm{CH}} c_6^{\mathrm{CH}} c_7^{\mathrm{CH}} (e^{\mathrm{CH}})^{15} \in \mathrm{CH}^{33}(X)$$

is not divisible by 2, but its image under φ_X is.

The proof of the non-divisibility is given in §3 and makes use of the Steenrod operations on $\mathrm{Ch}(X) := \mathrm{CH}(X)/2\,\mathrm{CH}(X)$. Cohomological operations in algebraic geometry have been introduced by Vladimir Voevodsky in the context of motivic cohomology, [18]. The Steenrod operations on the Chow groups modulo a prime p have been constructed over fields of characteristic $\neq p$ by Patrick Brosnan in [1]. Very recently, they've been constructed over fields of characteristic p by Eric Primozic, [15].

The computations in §3 are quite mysterious. Not only because the Steenrod operations themselves are so, but also because we do not know much about $\mathrm{CH}(X)$. Still it is possible to determine the values of the operations on certain "easy" part of $\mathrm{CH}(X)$: the subring generated by the Chern classes of the tautological vector bundle on X. This subring is explicitly known and very rigid: it does not change under extensions of the base field (see, e.g., [11, Theorem 2.1]). On the other hand, when taken modulo 2, it vanishes over an algebraic closure. We finish by computing over an algebraic closure, where the whole integral Chow ring is known. But instead of computing modulo 2, we work modulo a higher 2-power, despite of the fact that the operations themselves are only defined modulo 2...

The proof of the divisibility is given in §4 and makes use of an imitation $CK(X)$ of the connective K-theory of X, built from the terms of the Chow filtration on $K(X)$. One could expect this part to be a boring computation because $K(X)$ is computable as a ring. In the generic case (as in the conjecture), which is the only case we are really interested in, the Chow filtration on $K(X)$ is also computable. However, making a direct computation turns out to be extremely complicated and has been not worked out. The idea of the computation actually made is borrowed from [19]. It looks like it only uses the structure of the modulo-2 Chow ring of $\bar{X}$ (which is X over an algebraic closure of the base field). This structure has been beautifully described by Alexander Vishik in [17] (see also [4,

§86]). It is simpler than that of the integral Chow ring $\mathrm{CH}(\bar{X})$ and much more simpler than that of $K(X) = K(\bar{X})$.

Our proofs, at the same time, are guided by and differ in many ways from [19], where, e.g., a Brown-Peterson version of algebraic cobordism, ordinary and connective Morava K-theories, as well as Grothendieck motives related to various theories are involved, in the arguments as well as in the statements. None of these appear here. Also it seems, some arguments in [19] need justification (or modification). For instance, multiplication in various additive groups, related to motives, is constantly used there.

Let us conclude by remark that the conjecture we are discussing has been verified for many groups including the spinor ones up to $\mathrm{Spin}(12)$, see [10]. The cases of $\mathrm{Spin}(13/14)$ and $\mathrm{Spin}(15/16)$ are still open.

2. Basic notation

Throughout this note, n is an integer ≥ 1, q is a non-degenerate quadratic form of dimension $2n + 1$ over a field F, and X is the highest orthogonal grassmannian of q, i.e., its variety of n-dimensional totally isotropic subspaces.

For every integer k, let $\mathrm{CH}^k(X)$ be the codimension-k component of the graded Chow ring $\mathrm{CH}(X)$. (On some rare occasions we write $\mathrm{CH}_k(X)$ for the dimension-k component $\mathrm{CH}^{\dim X - k}(X)$.) For $k \geq 0$, let $c_k^{\mathrm{CH}} \in \mathrm{CH}^k(X)$ be the kth Chern class of the dual to the (rank-n) tautological vector bundle $\mathcal{T}$ on X. In particular, $c_0^{\mathrm{CH}} = 1$, $c_k^{\mathrm{CH}} = 0$ for $k > n$, and $(-1)^k c_k^{\mathrm{CH}}$ is the kth Chern class of $\mathcal{T}$ for any k. Note that by [4, Proposition 86.13], c_k^{CH} itself (without the sign correction) is the kth Segre class of $\mathcal{T}$.

We also consider the Chow ring $\mathrm{Ch}(X) := \mathrm{CH}(X)/2\,\mathrm{CH}(X)$ with coefficients in $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$ and write c_k for the class in $\mathrm{Ch}^k(X)$ of the element $c_k^{\mathrm{CH}} \in \mathrm{CH}^k(X)$.

Remaining notation will be introduced on the go.

3. Steenrod operations

We consider the cohomological Steenrod operation $S_X \colon \mathrm{Ch}(X) \to \mathrm{Ch}(X)$, as in [4, Chapter XI]. We do not exclude the characteristic 2 (excluded in [4, Chapter XI]), where the reference is [15]. All the properties of S_X we need are available in arbitrary characteristic. For characteristic 2 they are obtained in Corollary 2.4 (commutation with pull-backs), Proposition 5.1 (Cartan formula), Corollary 6.3 (relation with squaring), Corollary 6.5 (vanishing), and Proposition 7.1 (relation with push-forwards) of [15].

For any $i \geq 0$, we write S_X^i for the ith homogeneous component $S_X^i \colon \mathrm{Ch}(X) \to \mathrm{Ch}^{+i}(X)$ of the total operation S_X.

Proposition 3.1. *For any $k \geq 0$ and any $i \geq 0$, one has*

$$S_X^i(c_k) = \binom{k-1}{i} c_{k+i}.$$

Proof. We adapt the proof of [4, Theorem 89.1]. The formula [4, (86.5)] is replaced by

$$(3.2) \qquad\qquad c_k = (s_* \circ t^*)(h^{n-1+k}),$$

where $h \in \mathrm{Ch}^1(X_1)$ is the class of a hyperplane section of the quadric X_1 of q, and s_*, t^* are as in [4, 86.5]: for $X_n := X$, t is the projection of the flag variety $X_{1 \subset n} \subset X_1 \times X_n$

onto X_1 and s is the other projection $X_{1 \subset n} \to X_n$. Since s is the projective bundle given by the tautological vector bundle on X_n and h is the Euler class of the canonical line bundle on X_1, formula (3.2) follows from the definition of Segre classes as in [5, Chapter 3] or [4, §58.A].

Instead of the formula for the value of the Steenrod operation on the elements in $\mathrm{Ch}(X_1)$ which appear in the proof of [4, Theorem 89.1], we apply the formula

$$S_{X_1}(h^k) = (1 + h)^k h^k$$

which holds for any $k \geq 0$ as a consequence of the formula

$$S_{X_1}(h) = S^0_{X_1}(h) + S^1_{X_1}(h) = h + h^2$$

(for characteristic $\neq 2$ see [4, Theorem 61.13 and Example 61.16]) and Cartan formula ([4, Corollary 61.15] for characteristic $\neq 2$). $\qquad\square$

We now specify the situation we are considering. First we assume that the Clifford invariant of the quadratic form q is trivial. This implies that $c_1^{\mathrm{CH}} = 2e^{\mathrm{CH}}$ for some $e^{\mathrm{CH}} \in \mathrm{CH}^1(X)$ (see, e.g., [4, Exercise 88.14(1)]). We write $e \in \mathrm{Ch}^1(X)$ for the modulo-2 class of e^{CH}.

Then we additionally assume that $n = 8$ and the index of X is 16. In this case we have a homomorphism $(\deg /16) \colon \mathrm{Ch}(X) \to \mathbb{F}_2$ mapping the class of any closed point on X to the class modulo 2 of its degree divided by 16 (and mapping to 0 the classes of positive dimensions).

Proposition 3.3. $(\deg /16)\big(S_X^3(c_2 c_3 c_6 c_7 e^{15})\big) \neq 0$.

Proof. By Proposition 3.1 we have: $S_X(c_2) = c_2 + c_3$, $S_X(c_3) = c_3 + c_5$, $S_X(c_6) = c_6 + c_7$, and $S_X(c_7) = c_7$. Besides, $S_X(e) = e + e^2$. It follows that

$$S_X(c_2 c_3 c_6 c_7 e^{15}) = S_X(c_2) S_X(c_3) S_X(c_6) S_X(c_7) S_X(e)^{15} =$$
$$(c_2 + c_3)(c_3 + c_5)(c_6 + c_7) c_7 (e + e^2)^{15}.$$

Let $\bar{X}$ be X over an algebraic closure of F. The ring $\mathrm{CH}(\bar{X})$ has been computed (for arbitrary n) by Alexander Vishik in [17] (see also [4, §86]). The group $\mathrm{CH}(\bar{X})$ is free of finite rank. For any $k > 0$, the image of c_k^{CH} in $\mathrm{CH}(\bar{X})$ is $2e_k^{\mathrm{CH}}$ for certain $e_k^{\mathrm{CH}} \in \mathrm{CH}^k(\bar{X})$. (In particular, e_1^{CH} is the image of e^{CH}.) The products $e_J^{\mathrm{CH}} := \prod_{k \in J} e_k^{\mathrm{CH}}$, where J runs over the subsets of $\{1, \ldots, n\}$, form an additive basis of $\mathrm{CH}(\bar{X})$. In particular, the product $e_1^{\mathrm{CH}} e_2^{\mathrm{CH}} \ldots e_n^{\mathrm{CH}} \in \mathrm{CH}^{\dim X}(\bar{X}) = \mathrm{CH}_0(\bar{X})$ is the class of a rational point. Note that $\dim X = n(n+1)/2$. In our special case of $n = 8$ we have $\dim X = 36$.

The product

$$(c_2^{\mathrm{CH}} + c_3^{\mathrm{CH}})(c_3^{\mathrm{CH}} + c_5^{\mathrm{CH}})(c_6^{\mathrm{CH}} + c_7^{\mathrm{CH}}) c_7^{\mathrm{CH}} \big(e^{\mathrm{CH}} + (e^{\mathrm{CH}})^2\big)^{15}$$

is an integral representative of $S_X(c_2 c_3 c_6 c_7 e^{15})$. The degree-36 part of this product is an integral representative of $S_X^3(c_2 c_3 c_6 c_7 e^{15})$. This degree-36 part is a sum, where each summand is a product of 4 elements c_k^{CH} (with various k) and a power of e^{CH}. One of them is

$$c_3^{\mathrm{CH}} c_5^{\mathrm{CH}} c_6^{\mathrm{CH}} c_7^{\mathrm{CH}} (e^{\mathrm{CH}})^{15},$$

an integral representative of $S^1(c_2)S^2(c_3)c_6c_7e^{15}$, which is an element of degree 16 modulo 32. Indeed, the image in $CH(\bar{X})$ of $c_3^{CH}c_5^{CH}c_6^{CH}c_7^{CH}$ is $16e_3^{CH}e_5^{CH}e_6^{CH}e_7^{CH}$ and the image of $(e^{CH})^{15}$ is $(e_1^{CH})^{15} = e_1^{CH}(e_1^{CH})^2(e_1^{CH})^4(e_1^{CH})^8$ which is congruent modulo 2 to $e_1^{CH}e_2^{CH}e_4^{CH}e_8^{CH}$ because of the modulo-2 relation $(e_k^{CH})^2 \equiv e_{2k}^{CH}$ (which holds for all n and all k).

We claim that the degree of every other summand in the degree-36 part is divisible by 32. Indeed, since $(e_1^{CH})^{16}$ is 0 modulo 2, the degree of every summand containing a power of e^{CH} greater than 15 is divisible by 32. The remaining summand is $c_2^{CH}c_5^{CH}c_7^{CH}c_7^{CH}(e^{CH})^{15}$. Because of the formula

$$(c_k^{CH})^2 = 2\big(c_{k-1}^{CH}c_{k+1}^{CH} - c_{k-2}^{CH}c_{k+2}^{CH} + \cdots - (-1)^k c_0^{CH}c_{2k}^{CH}\big) \in CH(X)$$

(which holds again for all n and k, but this time over F, see [4, §86] or [11, Theorem 2.1]) its image in $CH(\bar{X})$ is divisible by 32 . $\qquad\square$

Corollary 3.4. *Let X be the highest orthogonal grassmannian of a non-degenerate 17-dimensional quadratic form of trivial Clifford invariant over an arbitrary field. Assume that $\mathrm{ind}(X) = 16$. Then the element $c_2^{CH}c_3^{CH}c_6^{CH}c_7^{CH}(e^{CH})^{15} \in CH^{33}(X)$ is not divisible by 2.* $\qquad\square$

4. "CONNECTIVE K-THEORY"

In this section we work with the Grothendieck ring $K(X)$. By [13], $K(X)$ is identified with a subring in $K(\bar{X})$ (where X is as in §2; we do not assume yet that the Clifford invariant of q is trivial). We consider the Chow filtration on $K(X)$ and write $K(X)^{(i)}$ for its ith term. The index i is allowed to be any integer: $K(X)^{(i)} = K(X)^{(0)} = K(X)$ for negative i.

Note that $K(X)^{(i)} \subset K(\bar{X})^{(i)}$ for any i.

For $k \geq 0$, we write $c_k^K \in K(X)^{(k)}$ for the K-theoretic Chern class of the dual to the tautological vector bundle on X, defined as in [12, Example 2.3]. For $k \geq 1$, we also define the element $e_k^K \in K(\bar{X})^{(k)}$ as $e_k^K := 0$ for $k > n$ and $e_k^K := (s_* \circ t^*)(l_{n-k}^K)$ for $k \leq n$, where s, t are as in the proof of Proposition 3.1 and $l_{n-k}^K \in K(\bar{X}_1)^{n+k-1}$ is the class of a linear projective $(n-k)$-subspace on the quadric X_1. (Note that $\dim X_1 = \dim q - 2 = 2n - 1$.)

Lemma 4.1. *For any $k \geq 1$, the element $2e_k^K + e_{k+1}^K$ of $K(\bar{X})^{(k)}$ actually belongs to $K(X)^{(k)}$ and is congruent modulo $K(X)^{(k+1)}$ to c_k^K. More precisely, the difference*

$$(2e_k^K + e_{k+1}^K) - c_k^K$$

is a sum of monomials in $c_1^K, \ldots, c_n^K$ of degree greater than k (where for any $r \geq 0$, the degree of c_r^K is defined as r).

Proof. Let $h^K \in K(X_1)^{(1)}$ be the class of a linear section of the quadric X_1. By [7, §3] (see [6, §2] for more details), $(h^K)^{n+k-1} = 2l_{n-k}^K + l_{n-k-1}^K \in K(\bar{X}_1)$ for any $k \geq 1$. Therefore, to prove Lemma 4.1 is to prove that $(s_* \circ t^*)((h^K)^{n+k-1})$ equals c_k^K plus a sum of monomials in $c_1^K, \ldots, c_n^K$ of degree greater than k.

The modulo-2 formula (3.2) actually holds integrally. The homomorphism

$$\varphi_- \colon CH(-) \to ChowK(-)$$

commutes with pull-backs and push-forwards and respects Chern classes. Therefore the element $(s_* \circ t^*)((h^K)^{n+k-1})$ is c_k^K modulo $K(X)^{(k+1)}$.

To prove the more precise statement of Lemma 4.1, let us consider the generic $(2n+1)$-dimensional quadratic form q^{gen} as in [11, §3]: q^{gen} is the quadratic form over the field F^{gen} of rational functions over F in $\dim q^{\text{gen}}(1 + \dim q^{\text{gen}})/2 = (2n+1)(n+1)$ variables, whose coefficients are the variables. For the maximal orthogonal grassmannian X^{gen} of q^{gen}, since the ring $\text{CH}(X^{\text{gen}})$ is generated by $c_1^{\text{CH}}, \ldots, c_n^{\text{CH}}$ (see, [14] or [11, Theorem 6.1]), the group $K(X^{\text{gen}})^{(k+1)}$ is generated by monomials in $c_1^K, \ldots, c_n^K$ of degree greater than k. Therefore, the statement we want for q holds for q^{gen}. Since over an algebraic closure $\bar{F}^{\text{gen}}$ of F^{gen} the forms q^{gen} and q become isomorphic, the statement holds for $q_{\bar{F}^{\text{gen}}}$. Since the change of fields homomorphism $K(X) \to K(X_{\bar{F}^{\text{gen}}})$ is injective, the statement holds for q as well. $\qquad\square$

We consider the following imitation of the connective K-theory of X (for definition of the connective K-theory itself see [2] or [3]). Let $CK(X)$ be the $\mathbb{Z}$-graded ring with $CK^i(X) := K(X)^{(i)}$ for any i. We write $v \in CK^{-1}(X)$ for the element $1 \in K(X)^{(-1)}$. So, v is an element of degree -1 and $CK(X)$ is the subring of the Laurent polynomial ring $K(X)[v, v^{-1}]$ consisting of the polynomials $\sum_i a_i v^i$ satisfying the condition $a_i \in K(X)^{(-i)}$ for any i. In particular, $a_i = 0$ for $i < -\dim X$.

Example 4.2. The graded ring $CK(\operatorname{Spec} F)$ is the polynomial ring $\mathbb{Z}[v]$.

We write c_k^{CK} and e_k^{CK} for $c_k^K v^{-k} \in CK^k(X)$ and $e_k^K v^{-k} \in CK^k(\bar{X})$.
Let $I(X) \subset CK(X)$ be the ideal generated by 2 and v. Note that

$$CK(X)/v\,CK(X) = \operatorname{Chow}K(X) \quad \text{and} \quad CK(X)/I = \operatorname{Chow}K(X)/2\operatorname{Chow}K(X).$$

Lemma 4.3. *For any $k \geq 1$ one has:* $(e_k^{CK})^2 \equiv e_{2k}^{CK} \pmod{I(\bar{X})}$.

Proof. For any $k \geq 1$, the element e_k^{CK} has been defined the way that its class modulo $v\,CK(\bar{X})$ coincides with the image of $e_k^{\text{CH}} \in \text{CH}^k(\bar{X})$ under the homomorphism $\text{CH}(\bar{X}) \to CK(\bar{X})/v\,CK(\bar{X})$ (see [4, Formula (86.5)]). In particular, the class of e_k^{CK} modulo $I(\bar{X})$ coincides with the image of $e_k \in \text{Ch}^k(\bar{X})$ under the homomorphism $\text{Ch}(\bar{X}) \to CK(\bar{X})/I(\bar{X})$. The congruence follows from the relation $e_k^2 = e_{2k}$ we have in $\text{Ch}(\bar{X})$. $\qquad\square$

Below we are assuming that the Clifford invariant of the quadratic form q is trivial. Therefore the subring $K(X)$ of $K(\bar{X})$ coincides with $K(\bar{X})$. The Chow filtrations on $K(X)$ and $K(\bar{X})$ may still differ. However $K(X)^{(1)} = K(\bar{X})^{(1)}$. In particular, $e^K := e_1^K \in K(X)^{(1)}$.

Proposition 4.4. *For $n = 8$ (dim $q = 17$), assume that the Clifford invariant of q is trivial. Then the class of the element $c_2^K c_3^K c_6^K c_7^K (e^K)^{15} \in K(X)^{(33)}$ in the quotient $\operatorname{Chow}^{33}K(X) = K(X)^{(33)}/K(X)^{(34)}$ is divisible by 2.*

Proof. We do our computations in $CK(X)$. To simplify notation, we write c_k, e_k, e instead of $c_k^{CK}, e_k^{CK}, e_1^{CK}$ in this proof.

According to Lemma 4.1, the class of the product $c_2 c_3 c_6 c_7 e^{15} \in CK^{33}(X)$ modulo $v\,CK^{34}(X)$ is represented by the element

$$x := (2e_2 + ve_3)(2e_3 + ve_4)(2e_6 + ve_7)(2e_7 + ve_8)e^{15} \in CK^{33}(X).$$

All we need to prove is the inclusion $x \in I(X)$, i.e.,

(4.5) $$x \in 2\,CK^{33}(X) + v\,CK^{34}(X).$$

Let us consider x as an element of $CK(\bar{X})$. Obviously, $x \in I(\bar{X})^4$. As a first step, we claim and check that

(4.6) $$x \equiv 0 \quad (\mathrm{mod}\ I(\bar{X})^5).$$

Indeed, the expression defining x is a polynomial in v. Its free term is $2^4 e_2 e_3 e_6 e_7 e^{15}$, which is 0 modulo $I(\bar{X})^5$ because $e_2 \equiv e^2$ and $e^{16} \equiv 0$ modulo $I(\bar{X})$ by Lemma 4.3.

Concerning the coefficient to v in x, we need to check that it is 0 modulo $I(\bar{X})^4$. Dividing by 2^3, we get to check that

$$(e_3 e_3 e_6 e_7 + e_2 e_4 e_6 e_7 + e_2 e_3 e_7 e_7 + e_2 e_3 e_6 e_8) e^{15}$$

is 0 modulo $I(\bar{X})$. The first summand in the parentheses modulo $I(\bar{X})$ is

$$e_3 e_3 e_6 e_7 \equiv e_6 e_6 e_7 \equiv 0.$$

Each of the remaining three summands contains e_k with k a 2-power. Since $e_k \equiv e^k$ (mod $I(\bar{X})$) for such k, we get 0 modulo $I(\bar{X})$ after the multiplication by e^{15}.

Next we check that the coefficient to v^2 in x divided by 2^2 is 0 modulo $I(\bar{X})$. This coefficient is a sum of 6 products multiplied by e^{15}. The products containing e_k with k a 2-power are treated as before. The remaining product is $e_3 e_3 e_7 e_7$ and is 0 modulo $I(\bar{X})$ because e_7^2 is so.

For the coefficients to v^3, only the argument on e_k with k a 2-power is needed. The coefficients to higher powers of v are 0 by dimension reason.

We proved claim (4.6). Therefore

(4.7) $$x = 2^5 x_0 + 2^4 v x_1 + 2^3 v^2 x_2 + 2^2 v^3 x_3$$

for some $x_k \in CK^{33+k}(\bar{X})$ with $k = 0, \dots, 3$.

Let L/F be a finite field extension such that q completely splits over L. Note that $CK(X_L) = CK(\bar{X})$. The composition of the change of field homomorphism $CK(X) \to CK(X_L)$ with the norm map $N_{L/F} \colon CK(X_L) \to CK(X)$ is multiplication by the degree $[L : F]$. Since the group $K(\bar{X})$ is torsion-free and the subgroup $K(X) \subset K(\bar{X})$ is of finite index, the composition

$$CK(X_L) \xrightarrow{N_{L/F}} CK(X) \longrightarrow CK(X_L)$$

in the other order is also multiplication by $[L : F]$. The integer $\mathrm{ind}(X)$ is the greatest common divisor of degrees of all such L/F. Since $\mathrm{ind}(X)$ divides 2^4 (e.g., by [16]), it follows that

(4.8) $$2^4 CK(\bar{X}) \subset CK(X).$$

Using inclusion (4.8) we can kill x_0 and x_1 in (4.7). To deal with x_2 and x_3, however, we need some more computations.

Recall that for arbitrary n, the total Chow group $\mathrm{CH}(\bar{X})$ has an additive basis given by the products $e_J^{\mathrm{CH}} := \prod_{k \in J} e_k^{\mathrm{CH}}$, $J \subset \{1, \dots, n\}$. In particular, $\mathrm{CH}_0(\bar{X})$ and $\mathrm{CH}_1(\bar{X})$ are both cyclic, generated respectively by $e_{\{1,\dots,n\}}^{\mathrm{CH}}$ and $e_{\{2,\dots,n\}}^{\mathrm{CH}}$. For our case of $n = 8$, it follows that the group $CK^{36}(\bar{X}) = \mathrm{CH}^{36}(\bar{X}) = \mathrm{CH}_0(\bar{X})$ is generated by $e_{\{1,\dots,8\}} := e_1 e_2 \dots e_8$ so that x_3 is a multiple of $e_{\{1,\dots,8\}}$. It also follows that the group $CK^{35}(\bar{X})$ is generated by $v e_{\{1,\dots,8\}}$ and $e_{\{2,\dots,8\}} := e_2 \dots e_8$. Therefore, we may assume that x_2 is a multiple of $e_{\{2,\dots,8\}}$.

Consider the element

$$y := (2e_2 + ve_3)(2e_4 + ve_5)(2e_6 + ve_7)(2e_7 + ve_8)e^{14} \in CK^{33}(X).$$

Computing as for (4.6), one shows that

$$y \equiv 2^2 v^2 e_{\{2,\dots,8\}} \pmod{I(\bar{X})^5}.$$

It follows that $vey \equiv 2^2 v^3 e_{\{1,\dots,8\}}$ and $2y \equiv 2^3 v^2 e_{\{2,\dots,8\}}$ modulo $I(\bar{X})^6$. Therefore, adding to x an appropriate linear combination of $vey \in v\,CK^{34}(X)$ and $2y \in 2\,CK^{33}(X)$ and using (4.8), we can reduce to the case $x = 0$. $\qquad\square$

Proof of Theorem 1.1. As already explained in the introduction, Theorem 1.1 follows from Corollary 3.4 and Proposition 4.4. We only need to add/recall that for any $k \geq 1$, the image of $c_k^{\mathrm{CH}} \in \mathrm{CH}^k(X)$ under the ring homomorphism φ_X is represented by $c_k^K \in K(X)^{(k)}$. And the image of $e^{\mathrm{CH}} \in \mathrm{CH}^1(X)$ is represented by $e^K \in K(X)^{(1)}$. $\qquad\square$

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