Physics 234: Exercise 5

1. Newton's method is an iterative scheme for finding the roots of $f(x)$ that starts from an initial guess x_0 and generates a sequence of improved estimates according to

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}.$$

Unless something goes wrong, the sequence converges to a limit $L = \lim_{n\to\infty} x_n$ that satisfies $f(L) = 0$. The rate of convergence can be determined by looking at how fast the residuals $r_n = L - x_n$ vanish. Equivalently, one can look at how fast the number of accurate binary digits

$$b_n \sim -\log_2 \left| \frac{r_n}{L} \right|$$

grows. Show that, so long as the root at $x = L$ is simple, the convergence is characterized by $r_{n+1} = r_n^2$ and $b_n \sim 2^n$.

2. Repeat this analysis for the secant method. Remember that it starts from *two* initial guesses and iterates according to

$$x_{n+1} := x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})},$$

with $f'(x_n)$ replaced by a finite difference constructed between x_{n-1} and x_n. Show that (again, for simple roots only) the convergence is characterized by $r_{n+1} = r_n r_{n-1}$ and $b_n \sim \phi^n$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio.

3. How do the results of questions 1 and 2 change when the root is multiple.

4. Each Newton iteration is equivalent to extrapolating the local tangent line back to the x-axis. The rule is derived by expanding around the current best guess to linear order in the discrepancy Δx between the guess and the true location of the root.

   (a) Show that by expanding to *second* order, one can derive the following:

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)} \left[1 - \frac{f''(x_n) f(x_n)}{2(f'(x_n))^2} \right]^{-1}.$$

   (b) You've already shown that the Newton method converges quadratically. What can you say about the rate of convergence for the new rule above?

5. Consider the finite difference approximation

$$f'(a) = \frac{-3f(a) + 4f(a+h) - f(a+2h)}{2h} + O(h^p).$$

What is the value of p?

6. Construct a 5-point finite-difference approximation to $f''(x_2)$ on the ordered set of points $\{x_0, x_1, \ldots, x_4\}$.