

Physics 234: Lab Test

Tuesday, April 6, 2010

Student's Name: _____ UserID: p234u_____

1. Consider the recursive sequence defined by $x_0 = 2$ and $x_{n+1} := 2 + 2/x_n$. Its asymptotic value $X = \lim_{n \rightarrow \infty} x_n$ is equal to the infinite continued fraction

$$X = 2 + \frac{2}{2 + \frac{2}{2 + \frac{2}{2 + \frac{2}{2 + \dots}}}}$$

Report the value of X converged to six decimal places.

2. The function $f(x) = 5x^3 - x - 1$ has a single root on the interval $I_0 = [0, 1]$. The bisection method applied to $f(x)$, starting from the interval I_0 , produces a sequence of intervals

$$(I_n) = ([0, 1], [0.5, 1], [0.5, 0.75], [0.625, 0.75], [0.6875, 0.75], [0.6875, 0.71875], \dots)$$

(decreasing in width by half at each stage) that more and more tightly bound the root of $f(x)$. What is the rightmost edge of I_{20} ?

$$I_{20} = [0.69762802, \input type="text" value="0.69762897"]$$

3. The golden ratio $\phi = (1 + \sqrt{5})/2$, raised to arbitrary powers, appears in the function $F : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by

$$F(n) = \left\lfloor \frac{\phi^n}{\sqrt{5}} + \frac{1}{2} \right\rfloor.$$

(Hint: the *floor* of x , denoted $\lfloor x \rfloor$, indicates that x is rounded down to the nearest integer—just like a C cast of a floating-point type to an integer type.) You should be able to verify the values $F(1) = F(2) = 1$, $F(10) = 55$, $F(20) = 6765$, and $F(30) = 832040$. What is the value of $F(47)$?

4. An 8×5 matrix A is defined element-wise by

$$A_{ij} = \frac{(-1)^{i+j}}{1 + 2i + (i - j)^4},$$

where the row and column indices range over $i = 1, 2, \dots, 8$ and $j = 1, 2, \dots, 5$ (in typical math style, as opposed to the zero-based indexing of C arrays). What is the 2-norm of A ? That is, what is the *square root of the sum of the squares of all its entries*?

$$\|A\|_2 = \left(\sum_{i=1}^8 \sum_{j=1}^5 |A_{ij}|^2 \right)^{1/2}$$

5. An ellipse E centred on $(0, 0)$ has eccentricity e and a semi-minor axis of length 1. We want to compute its moment of inertia about a vector that passes through the origin perpendicular to the plane of the ellipse. If the ellipse has uniform mass density and total mass $M = 1$, then the moment of inertia is given by

$$I(e) = \int_E d^2r \rho(\mathbf{r}) |\mathbf{r}|^2 = \int_{-1}^1 dy \int_{-\sqrt{(1-y^2)/(1-e^2)}}^{+\sqrt{(1-y^2)/(1-e^2)}} dx (x^2 + y^2) = \int_{-1}^1 dy \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} dx \left(\frac{x^2}{1-e^2} + y^2 \right).$$

Evaluate the integral as follows:

- (a) Set an integer counter N and a floating-point accumulator S to zero.
- (b) Choose two random numbers $x, y \in [-1, 1]$.
- (c) If $x^2 + y^2 \leq 1$ then increment N by one and increment S by $x^2/(1 - e^2) + y^2$.
- (d) Repeat from (b) until N numbers in the tens of millions.
- (e) Report the result $I \doteq S/N$.

You should be able to verify the numerical estimates $I(0) \doteq 0.500$, $I(1/4) \doteq 0.517$, and $I(1/2) \doteq 0.583$. What is the value for $I(2/3)$?

0.700