Physics 234: Lab Test

Thursday, April 8, 2010

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1. Consider the recursive sequence defined by $x_0 = 3$ and $x_{n+1} := 3 + 3/x_n$. Its asymptotic value $X = \lim_{n\to\infty} x_n$ is equal to the infinite continued fraction

$$X = 3 + \cfrac{3}{3 + \cfrac{3}{3 + \cfrac{3}{3 + \cfrac{3}{3 + \cdots}}}}$$

Report the value of X converged to six decimal places.

$\boxed{3.79129}$

2. The function $f(x) = 5x^3 + x - 1$ has a single root on the interval $I_0 = [0, 1]$. The bisection method applied to $f(x)$, starting from the interval I_0, produces a sequence of intervals

$$(I_n) = ([0, 1], [0, 0.5], [0.25, 0.5], [0.375, 0.5], [0.4375, 0.5], [0.46875, 0.5], \ldots,)$$

(decreasing in width by half at each stage) that more and more tightly bound the root of $f(x)$. What is the rightmost edge of I_{18}?

$I_{18} = [0.47251129, \boxed{0.47251511}\ ]$

3. The golden ratio $\phi = (1 + \sqrt{5})/2$, raised to arbitrary powers, appears in the function $F : \mathbb{Z} \to \mathbb{Z}$ defined by

$$F(n) = \left\lfloor \frac{\phi^n}{\sqrt{5}} + \frac{1}{2} \right\rfloor.$$

(Hint: the *floor* of x, denoted $\lfloor x \rfloor$, indicates that x is rounded down to the nearest integer—just like a C cast of a floating-point type to an integer type.) You should be able to verify the values $F(1) = F(2) = 1$, $F(10) = 55$, $F(20) = 6765$, and $F(30) = 832040$. What is the value of $F(46)$?

$\boxed{1836311903}$

4. An 8×5 matrix A is defined element-wise by

$$A_{ij} = \frac{(-1)^{i+j}}{1 + 3i + (i - j)^6},$$

where the row and column indices range over $i = 1, 2, \ldots, 8$ and $j = 1, 2, \ldots, 5$ (in typical math style, as opposed to the zero-based indexing of C arrays). What is the 2-norm of A? That is, what is the *square root of the sum of the squares of all its entries*?

$$\|A\|_2 = \left(\sum_{i=1}^{8} \sum_{j=1}^{5} |A_{ij}|^2 \right)^{1/2}$$

$\boxed{0.45615}$

5. An ellipse E centred on $(0,0)$ has eccentricity e and a semi-minor axis of length 1. We want to compute its moment of inertia about a vector that passes through the origin perpendicular to the plane of the ellipse. If the ellipse has uniform mass density and total mass $M = 1$, then the moment of intertia is given by

$$I(e) = \int_E d^2r\, \rho(\mathbf{r})|\mathbf{r}|^2 = \int_{-1}^{1} dy \int_{-\sqrt{(1-y^2)/(1-e^2)}}^{+\sqrt{(1-y^2)/(1-e^2)}} dx\, (x^2+y^2) = \int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} dx \left(\frac{x^2}{1-e^2}+y^2\right).$$

Evaluate the integral as follows:

(a) Set an integer counter N and a floating-point accumulator S to zero.

(b) Choose two random numbers $x, y \in [-1, 1]$.

(c) If $x^2 + y^2 \leq 1$ then increment N by one and increment S by $x^2/(1-e^2) + y^2$.

(d) Repeat from (b) until N numbers in the tens of millions.

(e) Report the result $I \doteq S/N$.

You should be able to verify the numerical estimates $I(0) \doteq 0.500$, $I(1/4) \doteq 0.517$, and $I(1/2) \doteq 0.583$. What is the value for $I(1/3)$?

$\boxed{0.531}$