Physics 234: Solutions to Practice Exam Questions

1.

2. A polynomial with roots at -1, 2, and 3 can be factored as $p(x) \sim (x+1)(x-2)(x-3)$. The requirement that $p(0) = 1$ fixes the prefactor:

$$p(x) = \frac{1}{6}(x+1)(x-2)(x-3) = \frac{1}{6}\bigl(x^3 - 4x^2 + x + 6\bigr).$$

3. The absolute error is bounded by $|f^{(5)}(x)|x^5/5! < (3/25)x^5/5! = x^5/1000$ for all $|x| \le 1$. Evaluated at the point $x = 1/2$, it is at most $\Delta_f = 1/32000$. Hence, the relative error is less than $\epsilon_f = \Delta_f/p(1/2) = 2.43 \times 10^{-5}$, and the number of accurate binary digits is at least $\lfloor -\log_2|\epsilon_f| \rfloor = 15$.

4. Suppose $f(x)$ has a root at x_r. Our guess for its location x^* differs from x_r by a small discrepancy $\Delta x = x_r - x^*$. A series expansion in Δx yields

$$0 = f(x_r) = f(x^* + \Delta x) = f(x^*) + f'(x^*)\Delta x + \frac{1}{2!}f''(x^*)(\Delta x)^2 + \cdots$$

If we truncate the expansion at first order, we have $\Delta x^{(1)} = -f(x^*)/f'(x^*)$. If we extend it one order further, we get

$$\begin{aligned} 0 &= f(x^*) + f'(x^*)\Delta x + \frac{1}{2!}f''(x^*)(\Delta x)^2 = f(x^*) + f'(x^*)\Delta x\left[1 + \frac{f''(x^*)}{2f'(x^*)}\Delta x\right] \\ &\approx f(x^*) + f'(x^*)\Delta x^{(2)}\left[1 + \frac{f''(x^*)}{2f'(x^*)}\Delta x^{(1)}\right] = f(x^*) + f'(x^*)\Delta x^{(2)}\left[1 - \frac{f''(x^*)f(x^*)}{2(f'(x^*))^2}\right], \end{aligned}$$

which leads to

$$\Delta x^{(2)} = -\frac{f(x^*)}{f'(x^*)}\left[1 - \frac{f''(x^*)f(x^*)}{2(f'(x^*))^2}\right]^{-1}.$$

By including the order $(\Delta x)^2$ terms, the convergence of the algorithm is upgraded to cubic: $x_{n+1} - x_n = r_{n+1} - r_n \sim r_n^3$.

5. From the data set $S = \{(1, F_1), \ldots, (4, F_4)\}$, we construct a pseudo-inverse solution

$$\begin{pmatrix} -\hat{k}\hat{x}_0 \\ \hat{k} \end{pmatrix} = \bigl(X^T X\bigr)^{-1} X^T \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}, \quad \text{with } X = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$$

Here, X is the appropriate Vandermonde matrix. Straightforward algebra shows that

$$X^T X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}$$

and

$$\bigl(X^T X\bigr)^{-1} = \frac{1}{4 \cdot 30 - (-10)^2} \begin{pmatrix} 30 & -10 \\ -10 & 4 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 15 & -5 \\ -5 & 2 \end{pmatrix}.$$

Evaluating the pseudo-inverse

$$\begin{aligned}
\begin{pmatrix} -\hat{k}\hat{x}_0 \\ \hat{k} \end{pmatrix} &= \frac{1}{10}\begin{pmatrix} 15 & -5 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} F_1 + F_2 + F_3 + F_4 \\ F_1 + 2F_2 + 3F_3 + 4F_4 \end{pmatrix} \\
&= \begin{pmatrix} F_1 - \frac{1}{2}F_2 - \frac{1}{2}F_4 \\ -\frac{3}{10}F_1 - \frac{1}{10}F_2 + \frac{1}{10}F_3 + \frac{3}{10}F_4 \end{pmatrix}
\end{aligned}$$

leads to the desired result:

$$\begin{aligned}
\hat{k} &= \frac{1}{10}(-3F_1 - F_2 + F_3 + 3F_4) \\
\hat{x}_0 &= \frac{5(2F_1 - F_2 - F_4)}{3F_1 + F_2 - F_3 - 3F_4}
\end{aligned}$$

Alternatively, you might start with the sum of the residuals

$$R^2 = \sum_{i=1}^{4} \left[F_i - k(i - x_0)\right]^2$$

and solve for $\partial R^2/\partial k = \partial R^2/\partial x_0 = 0$.

6. We are told that

$$s(x) = \begin{cases} 1 - x + ax^2 + x^3 & 0 \le x \le 1 \\ 3 + bx + cx^2 - x^3 & 1 < x \le 2 \end{cases}$$

is a natural cubic spline on the interval $[0, 2]$. This means that $s_1(x) = 1 - x + ax^2 + x^3$ and $s_2(x) = 3 + bx + cx^2 - x^3$ are C^2 at $x = 1$, where the piecewise functions meet. That is to say, $s_1(1) = s_2(1)$, $s_1'(1) = s_2'(1)$, and $s_1''(1) = s_2''(1)$. Hence, we evaluate

$$\begin{aligned}
1 - x + ax^2 + x^3 &= 3 + bx + cx^2 - x^3 \\
-1 + 2ax + 3x^2 &= b + 2cx - 3x^2 \\
2a + 6x &= 2c - 6x
\end{aligned}$$

at $x = 1$ to arrive at a system of equations for the coefficients a, b, and c:

$$\begin{aligned}
a - b - c &= 1 \\
2a - b - 2c &= -5 \\
a - c &= -6
\end{aligned}$$

Only two of these equations are linearly independent. We find that $b = -7$, $c = a + 6$. But since the spline is natural, we also know that the endpoints satisfy $s_1''(0) = s_2''(2) = 0$. This tells us that $2a = 2c - 12 = 0$. We conclude that the correct values are $a = 0$, $b = -7$, and $c = 6$.

7. The three function evaluations can be related to $f(a)$ and its derivatives by Talyor expansion:

$$\begin{aligned}
f(a) &= f(a) \\
f(a + h) &= f(a) + f'(a)h + \alpha h^2 + \beta h^3 + \cdots \\
f(a + 2h) &= f(a) + 2f'(a)h + 4\alpha h^2 + 8\beta h^3 + \cdots ,
\end{aligned}$$

where $\alpha = f''(a)/2$ and $\beta = f'''(a)/6$. Then, -3 times the first equation plus 4 times the second minus the third gives

$$\underbrace{(-3+4-1)}_{=0} f(a) + \underbrace{(4-2)h}_{=2h} f'(a) + \alpha \underbrace{(4 \cdot 1 - 4)h^2}_{=0} + \beta \underbrace{(4 \cdot 1 - 8)h^3}_{=-4h^3}.$$

We conclude that

$$\begin{aligned} f'(a) &= \frac{-3f(a) + 4f(a+h) - f(a+2h)}{2h} + 4\beta h^2 \\ &= \frac{-3f(a) + 4f(a+h) - f(a+2h)}{2h} + O(h^2). \end{aligned}$$

8. The number of operations is minimized by rewriting the expression in nested form:

$$\begin{aligned} x^8 + 5x^6 - 2x^4 + x^3 + 7x^2 &= (x^6 + 5x^4 - 2x^2 + x + 7)x^2 \\ &= ((x^4 + 5x^2 - 2)x^2 + x + 7)x^2 \\ &= (((y+5) \times y - 2) \times y + x + 7) \times y \quad \text{where } y = x \times x \end{aligned}$$

The corresponding C implementation requires only 4 multiplications and 4 additions/subtractions.

```
double f(double x)
{
   const double x2 = x*x;
   return (((x2 + 5)*x2 - 2)*x2 + x + 7)*x2;
}
```