Physics 308: Statistical, Molecular, and Solid State Physics

In-class Midterm Exam

Thursday, February 14, 2008, 09:30–10:50

Student's Name: ________________________

Multiple Choice Questions (24 points)

Answer by circling one of (a), (b), (c), etc. Please be sure it is clear which one you have selected.

1. In general, thermalization is a process that involves the sharing of

   (a) energy and momentum

   (b) energy only

   (c) momentum only

   between particles.

2. An ensemble average involves repeated measurements over times

   (a) much longer than

   (b) much shorter than

   the characteristic thermalization time scale of the system.

3. A classical system is found in states A, B, and C with probabilities $P(A)$, $P(B)$, and $P(C)$. These satisfy $P(B) = 2P(A)$ and $P(C) = 2P(B)$. Some property of the system, call it x, takes the values x_A, x_B, and x_C in the corresponding states. What is the average value of x?

   (a) $\langle x \rangle = \frac{1}{3}x_A + \frac{2}{3}x_B + \frac{4}{3}x_C$

   (b) $\langle x \rangle = \frac{1}{\sqrt{7}}x_A + \sqrt{\frac{2}{7}}x_B + \frac{2}{\sqrt{7}}x_C$

   (c) $\langle x \rangle = \frac{1}{7}x_A + \frac{2}{7}x_B + \frac{4}{7}x_C$

   (d) $\langle x \rangle = \frac{1}{5}x_A + \frac{2}{5}x_B + \frac{2}{5}x_C$

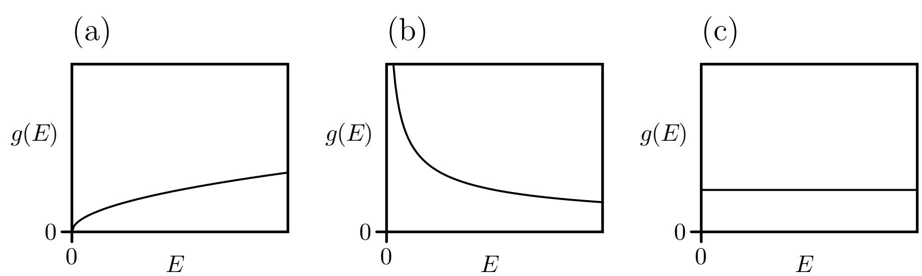
4. What does integration over the *phase space* mean for classical point particles living on a surface?

   (a) $\int d^3r\, d^3v$

   (b) $\int dx\, dy\, dv_x\, dv_y$

   (c) $\int dE\, E^{1/2}$

5. If the particles in question 4 have only kinetic energy $\frac{1}{2}m(v_x^2 + v_y^2)$, how does $g(E)$, the number of states with energy E, scale with E?



6. A system with two energy levels $\epsilon_1 = 0$ and $\epsilon_2 = \Delta$ is in thermal equilibrium at temperature $T = \Delta/(k_B \log 2)$. What is the ratio n_2/n_1, where n_i is the occupation of level i?

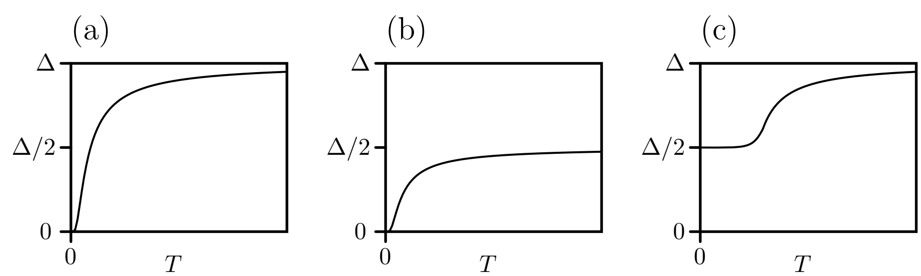
   (a) $\frac{1}{2}$

   (b) $\log 2$

   (c) 1

   (d) 2

7. Which picture correctly illustrates the energy of the two-level system in question 6 as a function of T?



8. Which is a possible set of units for Boltzmann's constant, k_B?

   (a) K/J (Kelvin/Joule)

   (b) eV/K (electron Volt/Kelvin)

   (c) K/mol (Kelvin/mole)

   (d) J/mol (Joule/mole)

9. An ideal gas consists of . . .

   (a) completely noninteracting point particles.

   (b) particles so strongly interacting that thermal equilibrium is assured.

   (c) particles with interactions just strong enough to establish thermal equilibrium but otherwise negligible.

   (d) particles with harmonic internal degrees of freedom.

10. Which integrals would be most useful in calculating $\langle v^2 \rangle$ from the Maxwell-Boltzmann speed distribution?

(a) $\int_0^\infty dx\, x^2 e^{-x^2} = \frac{\sqrt{\pi}}{4}$ and $\int_0^\infty dx\, e^{-x^2} = \frac{\sqrt{\pi}}{2}$

(b) $\int_0^\infty dx\, x^3 e^{-x^2} = \frac{1}{2}$ and $\int_0^\infty dx\, x e^{-x^2} = \frac{1}{2}$

(c) $\int_0^\infty dx\, x^4 e^{-x^2} = \frac{3\sqrt{\pi}}{8}$ and $\int_0^\infty dx\, x^2 e^{-x^2} = \frac{\sqrt{\pi}}{4}$

(d) $\int_0^\infty dx\, x^2 e^{-x} = 2$ and $\int_0^\infty dx\, e^{-x} = 1$

(e) $\int_0^\infty dx\, x^3 e^{-x} = 6$ and $\int_0^\infty dx\, x e^{-x} = 1$

11. What, according to the equipartition theorem, is the *specific heat per molecule* of an ideal diatomic gas?

(a) $\frac{5}{2} k_B T$

(b) $\frac{7}{2} k_B T$

(c) $\frac{7}{2} k_B$

(d) $\frac{9}{2} k_B$

12. Which of the following is not a degree of freedom realized in a diatomic molecule?

(a) translational
(b) torsional
(c) rotational
(d) vibrational

13. The Maxwell-Boltzann distribution is peaked at a most probable speed, $v_{\rm mp}$. Which of the following statements is correct?

(a) $v_{\rm mp} < \langle v \rangle < \langle v^2 \rangle^{1/2} < \langle v^3 \rangle^{1/3} < \cdots$

(b) $v_{\rm mp} = \langle v \rangle = \langle v^2 \rangle^{1/2} = \langle v^3 \rangle^{1/3} = \cdots$

(c) $v_{\rm mp} > \langle v \rangle > \langle v^2 \rangle^{1/2} > \langle v^3 \rangle^{1/3} > \cdots$

14. The partition function, $Z = \sum_x e^{-\beta E_x}$, is a sum of the Boltzmann weight over all possible configurations x of the system. Which is a correct expression for the average energy of the system?

(a) $-\frac{1}{Z}\frac{dZ}{d\beta}$

(b) $\frac{d \log Z}{d\beta}$

(c) $-\frac{1}{\beta}\frac{dZ}{d\beta}$

(d) $\frac{d \log Z}{dZ}$

15. For nonrelativistic, massive particles, the kinetic energy operator is

(a) $\frac{\hat{\mathbf{p}}^2}{2m} = -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2$

(b) $\frac{\hat{\mathbf{p}}^2}{2m} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$

(c) $\frac{\hat{\mathbf{p}}^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial z^2}$

(d) $\hat{\mathbf{p}}c = \frac{\hbar c}{i}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

16. In order to interpret a wavefunction $\psi(x)$ as a probability amplitude, how must it be normalized?

(a) $\int dx\, \psi(x) = 0$

(b) $\int dx\, \psi(x) = 1$

(c) $\int dx\, |\psi(x)| = 1$

(d) $\int dx\, |\psi(x)|^2 = 1$

17. If, under particle interchange, $\psi(x_2, x_1) = -\psi(x_1, x_2)$, ...

(a) the wavefunction is fermionic and $\psi(x, x) = 0$.

(b) the wavefunction is bosonic and $\psi(x, x) = 0$.

(c) the wavefunction is fermionic and $\psi(x, x) = -1$.

(d) the wavefunction is bosonic and $\psi(x, x) = 1$.

18. ^{3}He is a composite object consisting of an odd number of fermions (protons, neutrons, and electrons) bound together. Which statement is most correct?

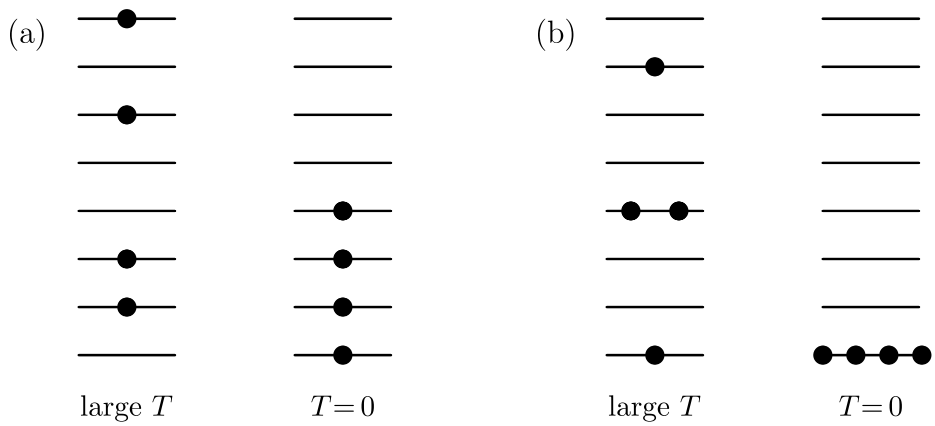
(a) ^{3}He atoms are fermions and thus cannot form a superfluid.

(b) ^{3}He atoms are bosons, but their superfluid transition is pre-empted by freezing.

(c) ^{3}He atoms are bosons that become superfluid below a critical temperature T_c.

(d) ^{3}He atoms are fermions, but bound pairs of ^{3}He atoms are bosons and thus can form a superfluid at low temperatures.

19. The following diagrams show energy levels and their occupancies for particles with different quantum statistics. Which is consistent with a superfluid at zero temperature?



20. For blackbody radiation, the energy density of photons with frequency between ω and $\omega + d\omega$ is

$$u_T(\omega)d\omega = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3 d\omega}{e^{\hbar\omega/k_B T} - 1}.$$

If we assume that a star of radius R emits as a blackbody at temperature T, its total power output will be given by its energy density times its surface area. How does the power output scale?

(a) R^2T^3

(b) R^3T^4

(c) R^2T^4

(d) R^3T^5

21. At zero temperature, the ground state of a gas of electrons is characterized by its Fermi energy E_F. How does E_F depend on the number density N/V of the gas?

(a) $\left(\frac{N}{V}\right)^{1/3}$

(b) $\left(\frac{N}{V}\right)^{2/3}$

(c) $\left(\frac{N}{V}\right)^{5/2}$

(d) $\left(\frac{N}{V}\right)^{5/3}$

22. Which of the following is NOT true for a gas of noninteracting electrons at low temperatures?

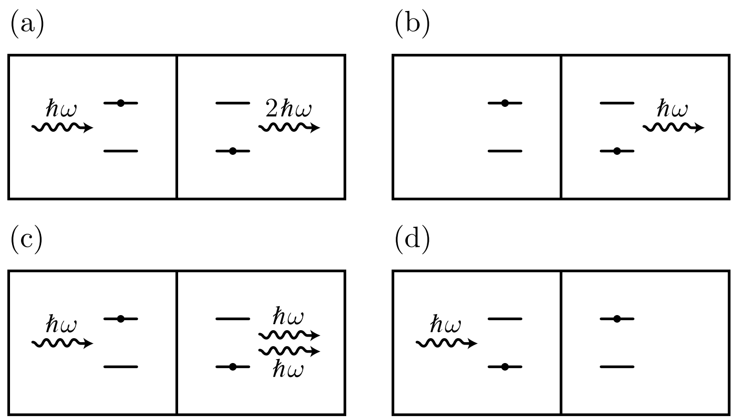
(a) Thermal excitations occur primarily within an energy window $\sim k_B T$ of the Fermi energy.

(b) The average energy of the gas is enhanced by a factor $E_F/k_B T$ over its classical value.

(c) The heat capacity of the gas is enhanced by a factor $E_F/k_B T$ over its classical value.

(d) The Fermi energy increases with the addition of more electrons to the system.

23. Consider a two-level system with states separated by $\hbar\omega$ in energy. Which one of the following before and after tableaux depict the process called *stimulated emission*? (Horizontal lines represent levels and filled circles indicate which levels are occupied; the squiggly lines are photons.)



24. The number and energy densities of a bosonic excitation are

$$\frac{N}{V} = \frac{1}{h^3} \int d^3p \, \frac{1}{e^{\beta E(\mathbf{p})} - 1} \quad \text{and} \quad \frac{U}{V} = \frac{1}{h^3} \int d^3p \, \frac{E(\mathbf{p})}{e^{\beta E(\mathbf{p})} - 1},$$

where $E(\mathbf{p})$ relates the energy to the momentum $\mathbf{p}$. If $E(\mathbf{p}) \sim |\mathbf{p}|^\alpha$ ($\alpha > 0$), what is the temperature dependence?

(a) $\frac{N}{V} \sim 1, \quad \frac{U}{V} \sim T$

(b) $\frac{N}{V} \sim T^{1+\alpha}, \quad \frac{U}{V} \sim T^{2+\alpha}$

(c) $\frac{N}{V} \sim T^{1/\alpha}, \quad \frac{U}{V} \sim T^{1+1/\alpha}$

(d) $\frac{N}{V} \sim T^{3/\alpha}, \quad \frac{U}{V} \sim T^{1+3/\alpha}$

Long Answer Question (10 points)

Answer ONE of the following questions (numbered 25 and 26). If you do attempt both, indicate which one should be graded. Please write clearly in the space provided and show the details of your calculations.

25. A gas of noninteracting, classical particles living on a line is subject to the potential

$$V(x) = \begin{cases} \infty & x < 0 \\ V_0 x/a & x \geq 0 \end{cases}$$

where V_0 has units of energy and a units of length.

(a) Show that the probability of finding a particle between x and $x + dx$ is

$$P(x)dx = \frac{dx}{a} \frac{V_0}{k_B T} \exp\left(-\frac{x}{a} \frac{V_0}{k_B T} \right).$$

(b) Compute the average position, $\langle x \rangle$. (Feel free to borrow an integral evaluation from question 10.)

(c) What is $\langle x \rangle$ in the limit $T \to 0$? Why?

(d) Compute the average energy $\langle E\rangle = \frac{1}{2}m\langle v^2\rangle + V_0\langle x\rangle/a$.

(e) Explain why $\langle E \rangle \neq 2 \times \frac{1}{2} k_B T = k_B T$.

26. The Hamiltonian of a particle confined to a one-dimensional box of length L is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x),$$

where the potential is

$$V(x) = \begin{cases} 0 & \text{if } 0 < x < L \\ \infty & \text{otherwise} \end{cases}$$

(a) Show explicitly that $\psi_n(x) = \sin(n\pi x/L)$ is a solution to the time-independent Schrödinger equation for $n = 1, 2, 3, \ldots$

(b) What is the energy eigenvalue E_n corresponding to ψ_n? (Hint: it should have the form $E_n = E_1 n^\alpha$, where α is an integer)

(c) What is the ground state energy of three electrons (spin 1/2) in the box?

(d) What is the ground state energy of three ^{4}He atoms (spin 0) in the box?

(e) What is the ground state energy of N electrons in the box? To answer this, you may or may not want to use one of the following relations:

$$\sum_{n=1}^{N} n = \frac{1}{2}N^2 + \frac{1}{2}N \qquad \sum_{n=1}^{N} n^2 = \frac{1}{3}N^3 + \frac{1}{2}N^2 + N$$

$$\sum_{n=1}^{N} n^3 = \frac{1}{4}N^4 + \frac{1}{2}N^3 + \frac{1}{4}N^2 \qquad \sum_{n=1}^{N} n^4 = \frac{1}{5}N^5 + \frac{1}{2}N^4 + \frac{1}{3}N^3 - \frac{1}{30}N$$