

## Solutions to assignment 2

1. The average value of the energy  $\epsilon = \frac{1}{2}mv^2$  is given in the usual way by

$$\langle \epsilon \rangle = \frac{\int d^3v \epsilon e^{-\beta\epsilon}}{\int d^3v e^{-\beta\epsilon}}. \quad (1)$$

Since the integrand depends only on  $v = (v_x^2 + v_y^2 + v_z^2)^{1/2}$ , we can integrate over shells of constant speed:

$$\int d^3v = \int dv_x dv_y dv_z = \int_0^\infty dv 4\pi v^2. \quad (2)$$

Hence,

$$\begin{aligned} \langle \epsilon \rangle &= \frac{\int_0^\infty dv (4\pi v^2) \epsilon e^{-\beta\epsilon}}{\int_0^\infty dv (4\pi v^2) e^{-\beta\epsilon}} \\ &= \frac{m \int_0^\infty dv v^4 e^{-\beta \frac{1}{2}mv^2}}{2 \int_0^\infty dv v^2 e^{-\beta \frac{1}{2}mv^2}}. \end{aligned} \quad (3)$$

Substituting a dimensionless variable

$$x^2 = \frac{m}{k_B T} v^2, \quad dx = \sqrt{\frac{m}{2k_B T}} dv, \quad (4)$$

leads to

$$\begin{aligned} \langle \epsilon \rangle &= \frac{m}{2} \left( \frac{2k_B T}{m} \right) \frac{\int_0^\infty dx x^4 e^{-x^2}}{\int_0^\infty dx x^2 e^{-x^2}} \\ &= k_B T \frac{3\sqrt{\pi}/8}{\sqrt{\pi}/4} = \frac{3}{2} k_B T. \end{aligned} \quad (5)$$

In similar fashion,

$$\begin{aligned} \langle \epsilon^2 \rangle &= \frac{\int_0^\infty dv (4\pi v^2) \epsilon^2 e^{-\beta\epsilon}}{\int_0^\infty dv (4\pi v^2) e^{-\beta\epsilon}} \\ &= \frac{m^2 \int_0^\infty dv v^6 e^{-\beta \frac{1}{2}mv^2}}{4 \int_0^\infty dv v^2 e^{-\beta \frac{1}{2}mv^2}} \end{aligned} \quad (6)$$

rescales to give

$$\begin{aligned} \langle \epsilon^2 \rangle &= \frac{m^2}{4} \left( \frac{2k_B T}{m} \right)^2 \frac{\int_0^\infty dx x^6 e^{-x^2}}{\int_0^\infty dx x^2 e^{-x^2}} \\ &= (k_B T)^2 \frac{15\sqrt{\pi}/16}{\sqrt{\pi}/4} = \frac{15}{4} k_B T. \end{aligned} \quad (7)$$

Thus, we arrive at the required result:

$$\langle \epsilon^2 \rangle - \langle \epsilon \rangle^2 = \left( \frac{15}{4} - \frac{9}{4} \right) (k_B T)^2 = \frac{3}{2} (k_B T)^2. \quad (8)$$

2. The vibrating molecule is described as a system with evenly spaced energy levels  $\epsilon_n = \hbar\omega(n + \frac{1}{2})$ . The average energy is given by

$$U = \frac{\sum_{n=0}^\infty \epsilon_n e^{-\beta\epsilon_n}}{\sum_{n=0}^\infty e^{-\beta\epsilon_n}}. \quad (9)$$

Written explicitly, this becomes

$$\begin{aligned} U &= \frac{\sum_{n=0}^\infty \hbar\omega(n + \frac{1}{2}) e^{-\beta\hbar\omega(n + \frac{1}{2})}}{\sum_{n=0}^\infty e^{-\beta\hbar\omega(n + \frac{1}{2})}} \\ &= \frac{\sum_{n=0}^\infty \hbar\omega n e^{-\beta\hbar\omega(n + \frac{1}{2})}}{\sum_{n=0}^\infty e^{-\beta\hbar\omega(n + \frac{1}{2})}} + \frac{\hbar\omega}{2} \\ &= \hbar\omega \frac{\sum_{n=0}^\infty n e^{-\beta\hbar\omega n}}{\sum_{n=0}^\infty e^{-\beta\hbar\omega n}} + \frac{\hbar\omega}{2}. \end{aligned} \quad (10)$$

Expanding the sum and collecting the geometric series leads to

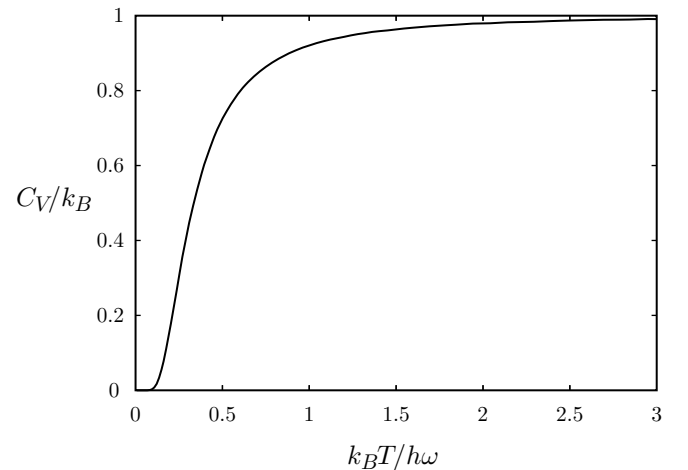
$$\begin{aligned} U &= \hbar\omega \frac{e^{-\beta\hbar\omega} + 2e^{-2\beta\hbar\omega} + 3e^{-3\beta\hbar\omega} + \dots}{1 + e^{-\beta\hbar\omega} + e^{-2\beta\hbar\omega} + \dots} + \frac{\hbar\omega}{2} \\ &= \hbar\omega e^{-\beta\hbar\omega} \frac{1 + 2e^{-\beta\hbar\omega} + 3e^{-2\beta\hbar\omega} + \dots}{1 + e^{-\beta\hbar\omega} + e^{-2\beta\hbar\omega} + \dots} + \frac{\hbar\omega}{2} \\ &= \hbar\omega e^{-\beta\hbar\omega} \frac{(1 + e^{-\beta\hbar\omega} + e^{-2\beta\hbar\omega} + \dots)^2}{1 + e^{-\beta\hbar\omega} + e^{-2\beta\hbar\omega} + \dots} + \frac{\hbar\omega}{2} \\ &= \hbar\omega e^{-\beta\hbar\omega} (1 + e^{-\beta\hbar\omega} + e^{-2\beta\hbar\omega} + \dots) + \frac{\hbar\omega}{2} \\ &= \frac{\hbar\omega e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} + \frac{\hbar\omega}{2} \\ &= \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} + \frac{\hbar\omega}{2}. \end{aligned} \quad (11)$$

The result above is equivalent to  $U = \hbar\omega(\langle n \rangle + \frac{1}{2})$ , where  $\langle n \rangle = f_{BE}(\hbar\omega)$ . In other words, the vibration can be interpreted as a bosonic degree of freedom with energy  $\hbar\omega$  per excitation.

The temperature derivative of the average energy is

$$\begin{aligned} C_V &= \frac{dU}{dT} = - \frac{\hbar\omega(-\hbar\omega/k_B T^2) e^{\hbar\omega/k_B T}}{(e^{\hbar\omega/k_B T} - 1)^2} \\ &= k_B e^{-\beta\hbar\omega} \left( \frac{\beta\hbar\omega}{1 - e^{-\beta\hbar\omega}} \right)^2. \end{aligned} \quad (12)$$

And this is how it looks:



3. Unlike noninteracting classical particles, which are wholly independent, noninteracting fermions and bosons feel a statistical repulsion and attraction that is quantum mechanical in origin. For example, if we put a single bosonic particle in a  $\mathbf{p} = 0$  momentum state, then a second particle added to the system will be favourably disposed to occupy the same state. On the other hand, if we put a single fermionic particle in a  $\mathbf{p} = 0$  momentum state, then a second particle added to the system is simply not allowed to occupy the same state. (Remember, this is how a Fermi surface comes about!)

Since pressure—coming as it does from the particles striking the walls of the container—is a function of the momentum profile of the particles, we see that the pressures must be ordered  $P_{\text{boson}} < P_{\text{classical}} < P_{\text{fermion}}$ .

4. Suppose that  $\alpha$  is very small. Then  $e^\alpha$ , which has a

powerseries expansion  $e^\alpha = 1 + \alpha + \frac{1}{2!}\alpha^2 + \frac{1}{3!}\alpha^3 + \dots$ , can be approximated by  $e^\alpha = 1 + \alpha$ . In that case, the number of bosons in the ground state is

$$N_0 = \frac{1}{e^\alpha - 1} = \frac{1}{\alpha}. \quad (13)$$

When the system is superfluid, the ground state is macroscopically occupied, and thus  $\alpha = 1/N_0 \ll 1$  is correct.

5. From Eq. (8-72) in the textbook, we know that the superfluid has critical temperature

$$T_c = \frac{h^2}{2mk_B} \left( \frac{N}{2\pi(2.315)V} \right)^{2/3}. \quad (14)$$

The density of liquid Ne is 1.208 g/cm<sup>3</sup>, so

$$\frac{N}{V} = \frac{(1.207 \text{ g/cm}^3)(6.022 \times 10^{23} \text{ mol}^{-1})(10^6 \text{ cm}^3/\text{m}^3)}{20.18 \text{ g/mol}} = 3.601 \times 10^{28} \text{ m}^{-3} \quad (15)$$

and

$$T_c = \frac{(6.626 \times 10^{-34} \text{ J} \cdot \text{s})^2}{2(20 \times 1.66 \times 10^{-27} \text{ kg})(1.381 \times 10^{-23} \text{ J/K})} \left( \frac{3.601 \times 10^{28} \text{ m}^{-3}}{2\pi(2.315)} \right)^{2/3} = 0.895 \text{ K}. \quad (16)$$

Thus, the superfluid transition of <sup>20</sup>Ne at  $T_c = 0.895$  K is pre-empted by freezing at 24.5 K.

### Some Useful Integrals

$$\int_0^\infty dx x^n e^{-x^2} = \begin{cases} \sqrt{\pi}/2 & \text{if } n = 0 \\ 1/2 & \text{if } n = 1 \\ \sqrt{\pi}/4 & \text{if } n = 2 \\ 1/2 & \text{if } n = 3 \\ 3\sqrt{\pi}/8 & \text{if } n = 4 \\ 1 & \text{if } n = 5 \\ 15\sqrt{\pi}/16 & \text{if } n = 6 \end{cases} \quad (17)$$