

Solutions to assignment 3

1. Surface waves on liquid ${}^4\text{He}$ have a dispersion relation $\epsilon(k) = \hbar(\sigma/\rho)^{1/2}k^{3/2}$. Here, the wavevector $\mathbf{k} = (k_x, k_y)$ is restricted to two dimensions, so its sum has the form

$$\sum_{\mathbf{k}} \rightarrow (\text{Area}) \int \frac{d^2k}{(2\pi)^2}.$$

Since the excitations are bosonic, their average energy density (i.e., their energy per unit area) is

$$\begin{aligned} u &= \int \frac{d^2k}{(2\pi)^2} \epsilon(k) f_{\text{BE}}(\epsilon(k)) \\ &= \int \frac{(2\pi k) dk}{(2\pi)^2} \frac{\epsilon(k)}{e^{\beta\epsilon(k)} + 1} \\ &= \frac{\hbar}{2\pi} \sqrt{\frac{\sigma}{\rho}} \int_0^\infty dk \frac{k^{5/2}}{e^{\beta\hbar(\sigma/\rho)^{1/2}k^{3/2}} + 1}. \end{aligned}$$

Rescaling variables with $x = [\hbar(\sigma/\rho)^{1/2}/k_B T]^{2/3} k$ gives

$$\begin{aligned} u &= \frac{\hbar}{2\pi} \sqrt{\frac{\sigma}{\rho}} \left(\frac{k_B T \rho^{1/2}}{\hbar \sigma^{1/2}} \right)^{(2/3) \times (7/2)} \int_0^\infty dx \frac{x^{5/2}}{e^{x^{3/2}} + 1} \\ &= \frac{\hbar}{2\pi} \sqrt{\frac{\sigma}{\rho}} \left(\frac{k_B T \rho^{1/2}}{\hbar \sigma^{1/2}} \right)^{7/3} \int_0^\infty dx \frac{x^{5/2}}{e^{x^{3/2}} + 1} \\ &\doteq \frac{1}{2\pi \hbar^{4/3}} \left(\frac{\rho}{\sigma} \right)^{2/3} (k_B T)^{7/3} \times 0.678. \end{aligned}$$

2. The power per unit area R arriving at Earth is given by the Stefan-Boltzmann law: $R = \sigma T^4$, where σ is Stefan's constant. For a 5% decrease in the sun's temperature,

$$\frac{R(T) - R(0.95T)}{R(T)} = 1 - (0.95)^4 = 0.186.$$

There is an 18.6% decrease in power.

The Stefan-Boltzmann law can be understood as follows. We learned that the energy density of blackbody radiation goes as (see Eq. 8-80)

$$u(E) dE = \frac{8\pi E^3 dE}{c^3 h^3 (e^{E/k_B T} - 1)}.$$

This means that the total energy radiated is given by the integral

$$R = \int_0^\infty dE u(E) \sim \int_0^\infty dE \frac{E^3}{e^{E/k_B T} - 1}.$$

By power counting (one factor of dE and three factors of E), we immediately see that $R \sim T^4$.

3. We start by rewriting Eq. 8-93 as

$$\frac{n(E)}{V} = \frac{\pi}{2} \left[\frac{8mc^2}{(hc)^2} \right]^{3/2} \frac{E^{1/2}}{e^{(E-E_F)/k_B T} + 1}.$$

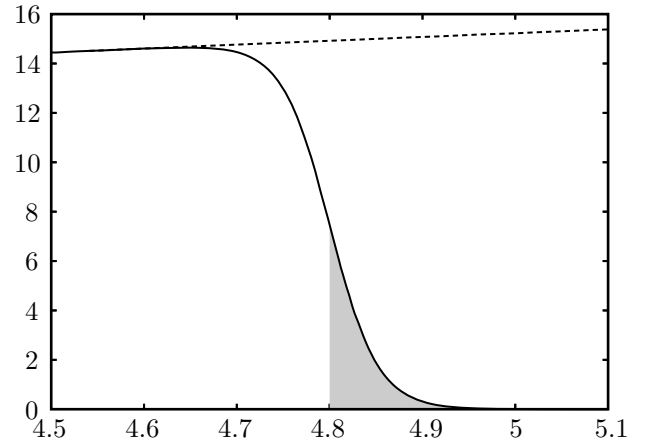
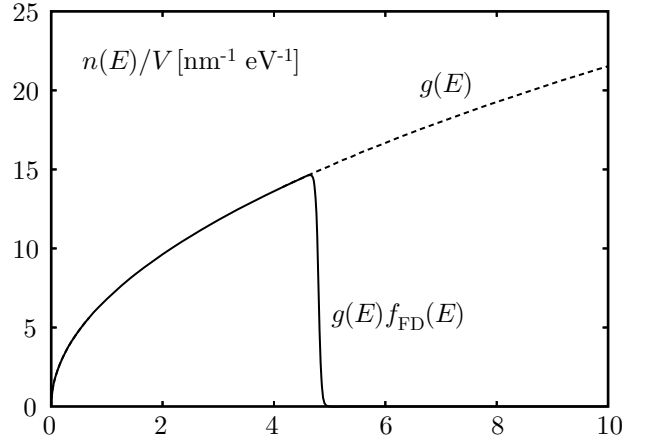
Inside the back cover of the text, we find the constants

$$\begin{aligned} mc^2 &= 0.511 \text{ MeV}, \\ hc &= 1.2398 \times 10^{-6} \text{ eV m}, \\ k_B &= 8.6173 \times 10^{-5} \text{ eV K}^{-1}. \end{aligned}$$

At 300 K (room temperature), $k_B T = 0.02585 \text{ eV}$; hence

$$\begin{aligned} \frac{n(E)}{V} &= \frac{\pi}{2} \left[\frac{8(5.11 \times 10^5 \text{ eV})}{(1.2398 \times 10^{-6} \text{ eV m})^2} \right]^{3/2} \\ &\quad \times \frac{E^{1/2}}{e^{(E/\text{eV} - 4.8)/0.02585} + 1} \\ &= (6.81 \text{ eV}^{-1} \text{ nm}^{-1}) \frac{(E/\text{eV})^{1/2}}{e^{(E/\text{eV} - 4.8)/0.02585} + 1}. \end{aligned}$$

A plot of this function look like this:



The second panel shows a magnified view in the energy range $|E - E_F| < 0.3$. The shaded area represents the electrons that have been thermally promoted above the Fermi level.

Now let's tabulate some values.

| E (eV) | $n(E)/V$ (eV ⁻¹ nm ⁻³) |
|----------------|---|
| 4.5 | 14.5 |
| 4.6 | 14.6 |
| 4.7 | 14.5 |
| 4.8 (= E_F) | 7.46 |
| 4.85 | 1.89 |
| 4.9 | 0.306 |
| 4.95 | 0.0456 |
| 5 | 0.00665 |
| 5.05 | 0.000966 |
| 5.1 | 0.000140 |

With these in hand, we can employ Simpson's rule to integrate the weight above E_F :

$$\frac{0.05}{2} [7.46 + 2(1.89 + 0.306 + 0.0456 + 0.00665 + 0.000966 + 0.000140)] = 0.025 \times 11.96 \approx 0.3.$$

That is, roughly 0.3 electrons/nm³ below E_F have been excited to levels above E_F .

4. A long, thin wire can be thought of as a one-dimensional system. According to Eq. 10-30, the Fermi energy is

$$E_F = \frac{h^2}{32m} \left(\frac{N}{L} \right)^2 = \frac{(hc)^2}{32mc^2} \left(\frac{N}{L} \right)^2.$$

We can assume that the linear number density is related to its density per unit volume in the obvious way. Using the value for Mg found in Table 10-3, we get $N/L =$

$(8.61 \times 10^{28} \text{ m}^{-3})^{1/3} = 4.41 \times 10^9 \text{ m}^{-1}$. This gives,

$$E_F = \frac{(1.2398 \times 10^{-6} \text{ eV m})^2}{32(5.11 \times 10^5 \text{ eV})} (4.41 \times 10^9 \text{ m}^{-1})^2 = 1.83 \text{ eV}.$$

5. The energy degeneracy is $g(E) = AE^{1/2}$. Hence, the total number of fermions is

$$N = \int_0^{E_F} AE^{1/2} dE = \frac{2}{3} AE_F^{3/2}.$$

The number that are within $k_B T$ of the Fermi level is

$$\begin{aligned} N_F(T) &= \int_{E_F - k_B T}^{E_F} AE^{1/2} dE \\ &= \frac{2}{3} A [E_F^{3/2} - (E_F - k_B T)^{3/2}] \\ &= \frac{2}{3} AE_F^{3/2} \left[1 - \left(1 - \frac{k_B T}{E_F} \right)^{3/2} \right]. \end{aligned}$$

Since $k_B T \ll E_F$, we can write

$$\begin{aligned} N_F(T) &\approx \frac{2}{3} AE_F^{3/2} \left[1 - \left(1 - \frac{3 k_B T}{2 E_F} \right) \right] \\ &= \frac{2}{3} AE_F^{3/2} \left(\frac{3 k_B T}{2 E_F} \right) \\ &= N \left(\frac{3 k_B T}{2 E_F} \right) \ll N. \end{aligned}$$

For Cu, $E_F = 7.04 \text{ eV}$; at 300 K, the fraction within $k_B T$ is

$$\frac{N_F(T)}{N} = \frac{3(0.02585 \text{ eV})}{2(7.04 \text{ eV})} = 0.0055.$$