

The Dynamics of Majoritarian Blotto Games*

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Abstract

We study *Colonel Blotto* games with sequential battles and a majoritarian objective. For a large class of contest success functions, the equilibrium is unique and characterized by an even split: Each battle that is reached before one of the players wins a majority of battles is allocated the same amount of resources from the player's overall budget. As a consequence, a player's chance of winning any particular battle is independent of the battlefield and of the number of victories and losses the player has accumulated in prior battles. This result is in stark contrast to equilibrium behavior in sequential contests that do not involve either fixed budgets or a majoritarian objective. We also consider the equilibrium choice of an overall budget. For many contest success functions, if the sequence of battles is long enough the payoff structure in this extended games resembles an all-pay auction without noise.

Keywords: Blotto games; dynamic battles; multi-battle contest; all-pay auctions; sequential elections.

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1 Introduction

We study a game that accounts for many generic features of dynamic conflicts, the *sequential majoritarian Blotto contest*. Two players sequentially interact in a series of battles. Victory in each battle is decided stochastically as a function of the players' investments of resources in this battle. A majoritarian rule applies: The player who first wins a given minimum number of battles wins the contest. Furthermore, once the contest has started, the overall amount of resources that can be used by a player is given and cannot be augmented or reduced at a later stage, and unused resources have no scrap value outside the contest.

While static games with given resource budgets have been extensively studied in the literature on Blotto games,¹ dynamic contests with fixed budgets are less well understood and results are available for specific cases only.² Our framework, on the other hand, is very general: The best-of- N contest studied here is of arbitrary odd length, individual battles are noisy, and no specific parametric contest success function (e.g. the Tullock function) is assumed—we impose only mild conditions on the relationship between resource investments and the outcomes of individual battles. Moreover, aggregate resource budgets might either be given or chosen prior to the first battle.

At any given stage of this game, players must decide how much of their remaining budgets to invest trying to win the current battle, and how much to save for possible later battles. The dynamics of this interaction are not obvious. Resources tend to be more valuable at later stages of the game—in the extreme, the outcome of the final battle determines who wins the game, and players may want to preserve resources for such critical contingencies. However, the conflict may end early if one contestant wins a sufficiently large number of battles, and preserving resources for later battles is wasteful if the conflict is already decided before these later battles are reached.

¹The generic Colonel Blotto game dates back to Borel (1921). Among major recent contributions are Roberson (2006), Kvasov (2007), Roberson and Kvasov (2012). Kovenock and Roberson (2010) review this literature and provide a systematic classification of Blotto games.

²An early discussion of sequentiality in Blotto games is in Friedman (1958). More recently, Deck and Sheremeta (2012) consider a sequence of all-pay auctions with fixed total resources and an asymmetric objective: One player wins the game if he wins a single auction, while the opponent must win all auctions. Rinott *et al.* (2012) develop a model in which two teams must allocate fixed resources to their members, who face each other in pairwise contests. The winner of the first battle faces the next member of the opposing team, and so on, until one team has lost all players. Sela and Erez (2013) examine a Blotto model in which players maximize the number of battle victories, rather than a majoritarian objective, and each battle is a symmetric Tullock contest. Similar to the equilibrium in flexible budget models, players invest more in early battles than in later ones. Konrad (2018) studies resource carryovers between sequential battles, using a symmetric Tullock contest success function and a maximum of three battles. In a best-of-3 dynamic Blotto game, Anbarcı *et al.* (2018) show that the weight of battles in the objective function influences spending decisions. Ryvkin (2011) models “fatigue” as a negative resource carryover: A high effort investment in a previous battle makes effort less effective in the future battles.

We find that in every subgame perfect equilibrium, players invest constant amounts of resources in all battles, except possibly after contingencies that are not on the equilibrium path. Hence, the intensity of fighting remains the same throughout the contest. This finding contrasts sharply with results that apply to sequential majoritarian contests in which the contestants’ budgets can be adjusted prior to each single battle (Klumpp and Polborn 2006; Konrad and Kovenock 2009). In such games, equilibrium behavior is history dependent. In particular, a contestant who lost in early rounds may have little incentive to spend resources to catch up with the opponent: If the contestant were to catch up successfully, both players would incur substantial fighting costs in the further course of the game. The anticipation of these costs discourages the player who has fallen behind from catching up (“discouragement effect”), which increases the chance of victory for the frontrunner (“momentum effect”). Since both effects make early round victories especially important, fighting is, on expectation, more intense in early battles than in late battles (“front-loading effect”). The exception is when the race is still close late in the contest—in this case, fighting escalates in the final rounds (“escalation effects” or “showdown effect”).³ None of these behaviors emerges if players dynamically and time-consistently allocate a fixed resource budget.

The assumption of fixed resource budgets is realistic in many potential applications described by our framework:

- In military confrontations—especially rapid ones that have the features of a *blitzkrieg*—commanders may not have the opportunity to replenish their troops and equipment after each battle. In these cases, commanders must allocate fixed military resources to a sequence of confrontations.⁴
- In sporting contests, players face physical constraints which can turn these games into multi-battle Blotto contests. This happens when players cannot replenish their physiological resources between rounds, such as the sets of a tennis match. Such constraints should be more important if the intervals for physical recovery between battles are relatively short.⁵

³Gelder (2014) offers an alternative preference-based explanation for why fighting need not become less intense in late stages, even if one player has fallen far behind the other. If the player who loses the contest cares about whether he or she won at least some battles, he may display “last-stand behavior.”

⁴Konrad (2017) discusses such dynamic conflicts, contrasting Alexander the Great’s military campaign in Asia, which lasted over than ten years, and Napoleon’s campaign against Russia, which lasted less than six months.

⁵Empirical studies of sports contests have produced mixed evidence of momentum effects and front-loading (Ferrall and Smith 1999; Malueg and Yates 2010; Gauriot and Page 2014). Our results suggest that one potential factor that affects both is the extent to which athletic resources budgets are fixed during the competition. The role and empirical significance of physical constraints is further studied in sports medicine (e.g., Skillington *et al.* 2017; Gescheit *et al.* 2017).

- Presidential primaries in the United States consist of a series of elections in which candidates compete for delegates at their party’s nominating convention. The overall amount of resources that candidates can mobilize in such elections is limited for several reasons. First, the main resource being spent is often the time a politician allocates to campaigning in particular districts, which by definition is fixed. Second, moneteray campaign funds are subject to regulations and legal restrictions that can cause a candidate’s election budget to be fixed.⁶ Third, a candidate’s resources will also be fixed if competition takes the form of pledges of political favors to voters, and the aggregate amount of such favors is exogenously given.⁷
- Organizations often allocate resources in a lumpy way to their members, who must decide on the day-to-day spending of these resources. For example, business firms might endow their marketing departments with annual budgets to run a series of marketing campaigns, or they may provide research budgets to their R&D departments which these must allocate to a series of R&D battles with rival firms.

In each of these applications, a set of fixed budgets can explain why the intensity of fighting does not show the types of history dependence that arise in settings with adjustable budgets, and instead remains relatively constant throughout the contest.

History independence also characterizes play in certain group contests, even if budgets are not fixed. Fu *et al.* (2015) and Häfner (2017) examine dynamic team contests in which different players fight in different battles on behalf of their teams. As part of a team, players have a greater incentive to catch up than do individual contestants: If they succeed in bringing their team “back on par” with the opposing team, a *different* team member will bear the cost of fighting in the next battle. In our context, catching up also has lower future costs for a player, but for a different reason: The player uses resources that have already been paid for.

After characterizing the equilibrium, we go on to study the role of the length of dynamic majoritarian Blotto games, given by the number of battles. With a large number of battles, the relationship between budget differences and probability of final victory becomes less stochastic if the noise that is present in a single battle “washes out” over a large number of battles. In this case, a player who has a small single-battle advantage

⁶Most importantly, funds not spent during an election campaign have few alternative uses *for the candidate*, as unused funds must be returned to donors or saved for future election runs by the same candidate for the same office. Thus, for funds raised prior to the campaign, spending these funds has no opportunity cost for the candidate (the only decision is on which election to spend them). In addition, candidates who accept public financing in American elections are limited to the amount received from the state. By law this payment cannot be adjusted in response to other candidates’ spending decisions (see Klumpp *et al.* 2015), resulting in a fixed budget for candidates that accept public funding.

⁷Taylor (2010) argues that candidates may promise federal procurement dollars in order to secure political support in individual states. Since the total amount of such “pork” that can be pledged during a campaign is likely finite, it is best described by a fixed overall budget.

(e.g., because of a slightly larger initial budget) may win a long majoritarian Blotto game with probability close to one. We call this phenomenon the amplification effect that results from an increase in the number of battles in a dynamic Blotto contest. We show that this amplification effect arises for certain contest success functions, including the Tullock function, but not for others. When it does, the players' probability of winning a large N -battle Blotto game, as a function of their initial budgets, resembles the probability of winning in the all-pay auction (see Hillman and Riley 1989; Baye *et al.* 1993.) As this holds even if each of the individual battles is governed by a contest success function with considerable noise, the result provides a microeconomic underpinning for the use of the all-pay auction success function in certain applications.⁸

Finally, we examine the players' incentives to invest in their overall budgets in a resource build-up stage that precedes the Blotto game. At that stage, both players simultaneously choose the resources with which they want to compete in the N -battle contest, and each pays a constant marginal cost per unit of resources. We show that the equilibrium budget choice depends on the length of the contest as well as on the players' costs. If the contest is sufficiently long, equilibrium involves randomization over resource budgets. The profile according to which players randomize their budgets as if they were competing in an all-pay auction without noise is an ε -equilibrium of the N -battle Blotto game, with lower ε for larger N . On the other hand, if the number of battles is not too large, pure strategy equilibria exist. In these equilibria, an increase in the number of battles will increase the players' budgets if the players have symmetric resource costs. If players have asymmetric costs, an increase in the number of battles may decrease equilibrium budgets.

We proceed as follows. In Section 2 we develop our sequential majoritarian Blotto framework and state our main equilibrium characterization result, which is proven in Section 3. Section 4 examines the comparative statics of the outcome of the Blotto game, and its asymptotic properties, as the number of battlefields changes. Section 5 uses these results to examine an enhanced game in which each player's budget is endogenous. Section 6 concludes.

⁸Similar noise-filter processes have been observed in elimination tournaments (see Rosen 1986). Harbaugh and Klumpp (2005) compare a two-stage, four-player Tullock elimination tournament with fixed resource budgets to one in which players have separate budgets in each round. Gradstein and Konrad (1999) and Fu and Lu (2012) study elimination contests in which a large set of players is partitioned into subgroups. The subgroups compete internally, and their winners become the participants in the next contest stage. In these models, an increase in the number of elimination rounds effectively removes the noise that is present in each battle, thereby raising the marginal benefit of effort.

2 Sequential Majoritarian Blotto Games

2.1 The contest

We consider a Blotto tournament with players A and B and an odd number N of battlefields, fought in sequence. Battle N is fought first, battle $N - 1$ is fought second, and so on, with the final battle being battle 1. The player who first wins $n = (N + 1)/2$ battles wins the tournament. At this point, the game ends and no further battles are fought.

Players are endowed with initial resources \bar{a} and \bar{b} which they can invest in the battles to influence their chance of success in each battle. A player cannot spend more than his initial resources in total, and in each battle he cannot use more than the difference between his initial resources and the resources already spent. Each player's objective is to allocate his resources to battles in a consecutive way that maximizes his chance of winning a majority of battles. Players observe the outcome of each battle and the opponent's remaining resources before making simultaneous investment decisions for the next battle. (Note that observing the opponent's remaining resources before each battle is equivalent to observing the opponent's investment made into the previous battle.) Unused resources at the end of the game have no value.

The outcome of each battle is governed by a *contest success function* (CSF)

$$p : [0, \infty)^2 \rightarrow [0, 1],$$

where $p(x, y)$ is the probability that A wins the battle if A spends resources x , and B spends resources y , on the given battle. The probability that B wins the battle is $1 - p(x, y)$. We maintain the following assumptions on the contest success function. p is continuous everywhere except possibly at $(0, 0)$. p is twice differentiable on $\mathbb{R}^2 \setminus (0, 0)$ with $p_x \geq 0$, $p_{xx} \leq 0$, $p_y \leq 0$, $p_{yy} \geq 0$, and these inequalities are strict at all $(x, y) \gg (0, 0)$. We do not require p to be symmetric, i.e., we do not assume that $1 - p(x, y) = p(y, x)$.

In addition to the above assumptions, we impose the following condition:

$$-\frac{p_{xx}(x, y)}{p_x(x, y)} > \frac{p_x(x, y)}{1 - p(x, y)} \quad \text{and} \quad -\frac{p_{yy}(x, y)}{p_y(x, y)} > -\frac{p_y(x, y)}{p(x, y)} \quad \text{for all } (x, y) \gg (0, 0). \quad (1)$$

Condition (1) requires the probability of success to be *sufficiently concave* in a player's own effort. The left-hand side of each inequality in (1) is a curvature measure akin to the Arrow-Pratt measure of risk aversion. The right-hand side is a hazard rate. Loosely speaking, it represents the chance that a marginal unit of effort results in victory conditional on not having won the battle with the effort already invested.

Many commonly used CSFs satisfy our assumptions, including all functions of the form

$$p(x, y) = \frac{f(x)}{f(x) + g(y)}, \quad (2)$$

where $f, g \geq 0$ are twice differentiable, strictly increasing, and weakly concave functions. In particular, the popular *Tullock (1980) lottery function*

$$p^{\text{Tullock}}(x, y) = \begin{cases} x/(x + y) & \text{if } x + y > 0, \\ 1/2 & \text{if } x + y = 0 \end{cases} \quad (3)$$

fits in our model, as do most of the variants of the Tullock function that have been explored in the literature.

2.2 Strategies and solution concept

In principle, a strategy for a player prescribes, for every battle and every history of spending decisions and outcomes in previous battles, an investment into the current battle. We restrict our attention to Markovian strategies, which depend on the game's history only through the total number of victories that A and B have accumulated and the resources the players have remaining in any battle.⁹

To formalize such strategies, we define a *state* of the tournament to be a pair (i, j) such that $i, j \geq 0$ and $1 \leq i + j \leq 2n$. This indicates that A needs to win i battles to win the tournament, and B needs to win j battles to win the tournament. If player A wins a battle, i is reduced by one, and if B wins a battle, j is reduced by one. The initial state is (n, n) . States (i, j) with $i = 0$ or $j = 0$ are terminal states. At these states, one player has won the tournament and no further decisions are made.

States with $i, j \geq 1$ are non-terminal states, at which players must decide how much to invest in the current battle. An *investment function* for player A at non-terminal state (i, j) is a function

$$\alpha_{i,j} : [0, \bar{a}] \times [0, \bar{b}] \rightarrow [0, \bar{a}] \quad \text{s.t.} \quad \alpha_{i,j}(a, b) \leq a.$$

This means that $\alpha_{i,j}(a, b)$ is the investment A makes into the battle at state (i, j) if A 's remaining resources are a and B 's remaining resources are b . An investment function for player B at (i, j) is similarly defined as

$$\beta_{i,j} : [0, \bar{a}] \times [0, \bar{b}] \rightarrow [0, \bar{b}] \quad \text{s.t.} \quad \beta_{i,j}(a, b) \leq b.$$

⁹These are the relevant information sets, as the game is of complete information and players maximize the probability of winning a majority of battles given fixed resources. Payoffs do not directly depend on which specific battles a player wins, or on how the player's resources were allocated to specific battles. At the expense of additional notation, one can extend our analysis to non-Markovian strategies (which condition on full histories) and obtain the same results.

A (*pure*) *continuation strategy* at state (i, j) is then a collection of investment functions at (i, j) and every possible non-terminal state that can be reached from state (i, j) :

$$\begin{aligned}\sigma_{i,j}^A &= \{ \alpha_{i',j'}(\cdot) : (1, 1) \leq (i', j') \leq (i, j) \}, \\ \sigma_{i,j}^B &= \{ \beta_{i',j'}(\cdot) : (1, 1) \leq (i', j') \leq (i, j) \}.\end{aligned}$$

A (*pure*) *strategy* is a continuation strategy at the initial state (n, n) , that is, a collection of investment functions for every state of the game.¹⁰

Each relevant subgame of the tournament originates at an information set $(i, j; a, b)$, consisting of a state (i, j) and a pair of remaining budgets (a, b) . Given $(i, j; a, b)$ and pair of continuation strategies $(\sigma_{i,j}^A, \sigma_{i,j}^B)$, one can compute the probability with which each player wins the tournament, starting at $(i, j; a, b)$. Continuation strategy $\sigma_{i,j}^A$ is a *best response* to $\sigma_{i,j}^B$ at state (i, j) if, for all $a \in [0, \bar{a}]$ and $b \in [0, \bar{b}]$, player A 's probability of winning in subgame $(i, j; a, b)$ is maximized if he uses strategy $\sigma_{i,j}^A$, conditional on B using $\sigma_{i,j}^B$. Player B 's best responses are defined similarly.

A pair $(\sigma_{i,j}^A, \sigma_{i,j}^B)$ of mutual best responses is a (*pure strategy*) *continuation Nash equilibrium* at (i, j) . A (*pure strategy*) *subgame perfect equilibrium* is a profile of strategies (σ^A, σ^B) such, for each non-terminal state (i, j) the associated profile of continuation strategies $(\sigma_{i,j}^A, \sigma_{i,j}^B)$ is a continuation Nash equilibrium at (i, j) .

2.3 Equilibrium characterization

We now present our main equilibrium characterization result. The following definition will be central:

Definition 1. For player A , continuation strategy $\sigma_{i,j}^A$ at state (i, j) is an *even-split continuation strategy* if

$$b > 0 \quad \Rightarrow \quad \alpha_{i',j'}(a, b) = \frac{a}{i' + j' - 1} \quad \text{for all } (i', j') \text{ s.t. } (1, 1) \leq (i', j') \leq (i, j).$$

For B , the definition is analogous. An *even-split strategy* is an even-split continuation strategy at the initial state (n, n) .

A player who uses an even-split strategy allocates his budget evenly and unconditionally across battles, as long as his opponent has positive resources remaining. For example, at the first battle A invests \bar{a}/N and saves the remainder $\bar{a} - \bar{a}/N$ for the next round. At the second battle he spends $(\bar{a} - \bar{a}/N)/(N - 1) = \bar{a}/N$ and saves the remainder $\bar{a} - 2\bar{a}/N$

¹⁰As is common in extensive form games, to facilitate backward induction a player's strategy prescribes actions even for information sets that are not reached under this strategy. For example, if a player exhausts his entire budget in one battle he cannot invest a positive amount in the next. A strategy must still specify how much the player *would* invest in that battle for any amount of remaining resources.

for the third round, where he invests $(\bar{a} - 2\bar{a}/N)/(N - 2) = \bar{a}/N$, and so on. The player does not make this investment dependent on how many battles each player has already won, or on the amount the opponent spends in each battle.

If both players use even-split strategies, player A wins each individual battle with probability $p(\bar{a}/N, \bar{b}/N)$, and B wins each individual battle with probability $1 - p(\bar{a}/N, \bar{b}/N)$. Our main result is that this is the unique outcome of the sequential majoritarian Blotto game.

Proposition 1. *A pure strategy subgame perfect equilibrium exists in which both players use even-split strategies. Furthermore, in every subgame perfect equilibrium, both players use only even-split strategies.*

To understand the second part of the result, note that an even-split strategy, as defined above, does not restrict a player's spending pattern once an opponent has run out of resources. This event will not occur in any equilibrium of our model; however, if (out of equilibrium) a subgame were reached in which one player has zero resources remaining, the other player could have multiple best responses. For example, consider the Tullock function in (3), and imagine that player B has zero resources remaining at some state (i, j) , while A has positive resources remaining. A can win the tournament with probability one by allocating a positive amount of resources to any i of the remaining $i + j - 1$ battles. In particular, A does not have to allocate his resources evenly across all remaining battles (although this would also guarantee victory). Thus, for some CSFs the game can have multiple equilibria. However, in all of these equilibria, observed spending along the equilibrium path is the same—namely, players divide their resources evenly across battles.

2.4 Discussion

Proposition 1 implies that equilibrium behavior in sequential majoritarian Blotto games is markedly different from that in sequential majoritarian *non*-Blotto contests (Klump and Polborn 2006; Konrad and Kovenock 2009). When the players reach state (i, j) they plan for the longest possible path to final victory or defeat, which consists of $i + j - 1$ battles, and allocate equal shares of their remaining budgets to each of these battles. Thus, in equilibrium, player A invests \bar{a}/N in every battle and B invests \bar{b}/N in every battle, independent of which battle it is. Players will not revise their plans in response to winning or losing any particular battle, which means that winning or losing a battle does not create momentum. While the player who loses the first battle will now win the tournament with a smaller probability, he wins the next battle with exactly the same probability as before. Moreover, no disproportionate share of resources is concentrated on early battles, nor does effort escalate as battles become more decisive.

These equilibrium dynamics of the sequential majoritarian Blotto game are surprising. Notice that each player faces two conflicting incentives when deciding on how to allocate his resources across battles: On the one hand, because the game is unlikely to last the entire length of N battles, resources saved for the later part of the tournament are wasted with a positive probability. On the other hand, because later battles are more decisive *if they take place*, players may want to hold resources in reserve for these contingencies. What Proposition 1 means, then, is that these two forces cancel each other exactly in equilibrium. This is true for all contest success functions that satisfy our assumptions, and is not a knife-edge result driven by any particular functional form of p .

It is worth noting that this striking result is highly robust to changes in the underlying parameters in several dimensions, yet extremely sensitive to changes in other dimensions. Broadly, Proposition 1 remains true regardless of almost any difference between the players. More precisely, the result holds if any of the following are true.

- The players start with arbitrarily different budgets.
- The players start with arbitrarily different head-starts, in terms of battles already won. (This follows from the sequential rationality embodied in subgame perfect equilibrium.)
- The players have arbitrarily different “battle efficiencies.” To see what this means, suppose the CSF takes the form $p(x, y) = f(x)/(f(x) + g(y))$. Given budget allocation choices x and y , the functions f and g capture how efficiently a given amount of resource is translated into an effective fighting strength. Our result is robust to arbitrary differences in these functions between players, as long as f and g are strictly increasing and weakly concave.

On the other hand, what our result is not robust to is changes to the game such that the battles are no longer structurally identical, or changes that make some paths to victory or loss preferable to a player, relative to other paths. To be specific, one can show that the following changes to the structure of the game would invalidate our main result.¹¹

- The CSFs are dependent on the battle, instead of each battle being decided by the same function p .
- The battles are worth different amounts of points, and the winner of the overall game is the player who accumulates the majority of points, instead of each battle having same point value.

¹¹See Solomon (2018) for a detailed demonstration that each change would invalidate the even-split nature of the equilibrium.

- A prize is awarded to the winner of each individual battle, in addition to the overall prize for winning the majority of battles.

Thus, we are able to clarify the question of exactly how much symmetry is required to drive this result. The answer is: The players can be arbitrarily different, but the battles, and paths to victory, must be symmetric.

3 Proof of the Main Result

In this section we prove Proposition 1. We first establish existence of an even-split equilibrium, and then show that every equilibrium must be in even-split strategies.

3.1 Existence

For $r \in \{1, \dots, N\}$, let the set of possible states when there are exactly r battles remaining be denoted by

$$T(r) = \{(i, j) \geq (1, 1) : i + j - 1 = r\}.$$

The proof is by induction on the number of battles remaining. Take $2 \leq r \leq N$. Suppose that at all $(i, j) \in T(r - 1)$, it is a continuation Nash equilibrium for both players to use even-split continuation strategies. We will show that a pair of even-split strategies is a continuation Nash equilibrium at all $(i, j) \in T(r)$. Because $T(1) = \{(1, 1)\}$, and continuation strategies at state $(1, 1)$ are even-split (there is exactly one battle remaining and it is clearly optimal to invest any remaining resources in this battle), the result follows.

Fix $2 \leq r \leq N$. Suppose that at all $(i, j) \in T(r - 1)$, an even-split continuation equilibrium exists and is played if (i, j) is reached. Now fix a state $(i_0, j_0) \in T(r)$ for battle r . Suppose that B plays an even-split continuation strategy also at (i_0, j_0) . (The argument is the same when the roles of the players reversed, and omitted.) Let $(a_0, b_0) \gg (0, 0)$ denote the players' remaining budgets at (i_0, j_0) . Since B plays an even-split continuation strategy, B spends $\gamma = b_0/r$ in all battles $r, r - 1, r - 2, \dots$, regardless of the states of these battles and regardless of A 's spending in these battles.¹² Since $\gamma > 0$, A 's probability of winning the tournament is continuous in the resources that A allocates to each battle, and since the set of all such allocations is compact, A will have a best response at (i_0, j_0) . Furthermore, if A allocates $x_0 \in [0, a_0]$ to battle r at state (i_0, j_0) , he will have resources $a_0 - x_0$ remaining when he reaches battle $r - 1$. By the induction hypothesis, A will then allocate $(a_0 - x_0)/(r - 1)$ to battles $r - 1, r - 2, \dots$. We will show that $x_0 = a_0/r$ is A 's optimal investment at state (i_0, j_0) .

¹²Note that battles $r - 1, r - 2, \dots$ may not take place. In general, when we say “spend in battle X” we mean “plan to spend in battle X, if battle X takes place.”

Our approach is the following: If $x_0 \neq a_0/r$, player A invests different amounts in battles r and $r - 1$. We will show that A can then improve his chance of winning by equalizing spending in battles r and $r - 1$, while still allocating $(a_0 - x_0)/(r - 1)$ to battles $r - 2, r - 3, \dots$. Since this reallocation is possible whenever $x_0 \neq a_0/r$, we conclude that A 's best response at (i_0, j_0) is to use the even-split continuation strategy.¹³ To establish the profitability of this deviation, we proceed in three steps. In Step 1, we establish that the reallocation strictly increases A 's chance of winning both battle r and $r - 1$, and strictly decreases A 's chance of losing both battles. In Step 2, we show that this must increase A 's overall chance of winning the tournament at state (i_0, j_0) , provided that A did not spend his entire budget a_0 at state (i_0, j_0) ; that is, we assume that $x_0 < a_0$. In Step 3, finally, we show that $x_0 = a_0$ cannot be a best response to B 's even-split strategy.

Throughout, we assume that $i_0 > 1$ and $j_0 > 1$. The cases where $i_0 = 1$ or $j_0 = 1$ are similar and in the Appendix.

Step 1. Consider the two consecutive battles r and $r - 1$. Define

$$\bar{x} \equiv \frac{x_0 + (a_0 - x_0)/(r - 1)}{2}$$

to be the the average amount spent by A in battles r and $r - 1$, and set $h \equiv |\bar{x} - x_0|$. Note that $0 \leq h \leq \bar{x}$, and $h = 0$ iff $\bar{x} = x_0$, or equivalently, $x_0 = a_0/r = (a_0 - x_0)/(r - 1)$.

The probability that A wins both battles is

$$P(h) = p(\bar{x} + h, \gamma)p(\bar{x} - h, \gamma).$$

P is continuous and differentiable at all $h \in [0, \bar{x}]$, with

$$P'(h) = p_x(\bar{x} + h, \gamma)p(\bar{x} - h, \gamma) - p(\bar{x} + h, \gamma)p_x(\bar{x} - h, \gamma).$$

This term is negative if and only if

$$\frac{p_x(\bar{x} + h, \gamma)}{p(\bar{x} + h, \gamma)} < \frac{p_x(\bar{x} - h, \gamma)}{p(\bar{x} - h, \gamma)}. \quad (4)$$

Because $p_x > 0$ and $p_{xx} < 0$ on \mathbb{R}_{++}^2 , inequality (4) holds and $P'(h) < 0$ for all $h \in [0, \bar{x}]$.

Similarly, the probability that A loses both battles is

$$Q(h) = [1 - p(\bar{x} + h, \gamma)][1 - p(\bar{x} - h, \gamma)].$$

¹³Note that, following the reallocation, A no longer plays an even-split continuation strategy when he reaches battle $r - 1$. (He now spends a different amount in battle $r - 1$ and $r - 2$.) This is not a contradiction to our induction hypothesis, as we do not claim that the new continuation strategy is optimal. The resource shift only establishes that the previous strategy (which was to spend one amount in battle r and a different amount in battles $r - 1, r - 2, \dots$) was *not* optimal.

Q is continuous and differentiable at all $h \in [0, \bar{x}]$, with

$$Q'(h) = -p_x(\bar{x} + h, \gamma)[1 - p(\bar{x} - h, \gamma)] + p_x(\bar{x} - h, \gamma)[1 - p(\bar{x} + h, \gamma)].$$

This term is positive if and only if

$$\frac{p_x(\bar{x} + h, \gamma)}{1 - p(\bar{x} + h, \gamma)} < \frac{p_x(\bar{x} - h, \gamma)}{1 - p(\bar{x} - h, \gamma)} \quad (5)$$

Using (1), condition (5) can be shown to hold¹⁴ so that $Q'(h) > 0$ for all $h \in [0, \bar{x}]$.

Step 2. Next, consider what happens after battle r and battle $r - 1$ are over. The tournament will be in one of the following three states: If A won both battles, then $(i_0 - 2, j_0)$; if A won one battle and lost the other, then $(i_0 - 1, j_0 - 1)$; and if A lost both battles, then $(i_0, j_0 - 2)$. Let

$$Z = \left\{ (i_0 - 2, j_0), (i_0 - 1, j_0 - 1), (i_0, j_0 - 2) \right\}$$

denote the set of these states.

Let $v_{i,j}$ denote the probability that A wins the tournament, conditional on reaching state $(i, j) \in Z$. If (i, j) is a terminal state, then $v_{i,j} = 1$ if $i = 0$ and $v_{i,j} = 0$ if $j = 0$. If $(i, j) \in Z$ is not a terminal state, then $(i, j) \in T(r - 2)$, and A wins the tournament if he wins i of the remaining $r - 2$ battles before B wins j battles. Since player A invests $(a_0 - x_0)/(r - 1)$, and B invests γ , into each of these remaining battles, A wins each battle with probability

$$\rho = p((a_0 - x_0)/(r - 1), \gamma).$$

Because $\gamma > 0$, our assumptions in p imply that $\rho < 1$. Assume now that $x_0 < a_0$; this implies that $(a_0 - x_0)/(r - 1) > 0$ and hence $\rho > 0$. Therefore, the probability with which A wins the tournament, conditional on reaching a non-terminal state $(i, j) \in Z$, can be written as follows:¹⁵

$$v_{i,j} = \sum_{k=0}^{j-1} \binom{i-1+k}{k} \rho^i (1-\rho)^k \in (0, 1). \quad (6)$$

Moreover, $v_{i,j} > v_{i+1,j-1}$ for all (i, j) with $(i, j) \geq (0, 1)$.¹⁶

¹⁴Observe that $\frac{\partial}{\partial x} \frac{p_x}{1-p} = \frac{p_{xx}[1-p] + (p_x)^2}{[1-p]^2} < 0 \Leftrightarrow -\frac{p_{xx}}{p_x} > \frac{p_x}{1-p} \Leftrightarrow (1)$.

¹⁵To understand (6), note that when A wins the tournament, the number of remaining battles B will have won is some integer $0 \leq k \leq j - 1$. Each term in the sum in (6) is the probability of a sequence of i wins and k losses for player A , for given k .

¹⁶This is intuitive, as winning a battle should always be preferred to losing a battle. Nevertheless, a formal proof of the inequality is in the Appendix.

The states in the set Z can hence be ordered according to A 's probability of winning the tournament as follows:

$$1 \geq v_{i_0-2,j_0} > v_{i_0-1,j_0-1} > v_{i_0,j_0-2} \geq 0. \quad (7)$$

Now go back to state (i_0, j_0) . The probability with which A wins the tournament, if he spends $\bar{x} + h$ and $\bar{x} - h$ in battles r and $r - 1$, is

$$z(h) = P(h)v_{i_0-2,j_0} + [1 - P(h) - Q(h)]v_{i_0-1,j_0-1} + Q(h)v_{i_0,j_0-2}. \quad (8)$$

Differentiating (8) with respect to h

$$z'(h) = P'(h)[v_{i_0-2,j_0} - v_{i_0-1,j_0-1}] + Q'(h)[v_{i_0,j_0-2} - v_{i_0-1,j_0-1}].$$

Because $P'(h) < 0$ and $Q'(h) > 0$ for $h \in [0, \bar{x}]$, and using (7), we have $z'(h) < 0$ for all $h \in [0, \bar{x}]$. It follows that A can increase his chance of winning at (i_0, j_0) by reducing h —that is, by reallocating some resources across the two consecutive battles r and $r - 1$, from the battle in which he spends more to the one in which he spends less.

Step 3. It remains to be shown that $x_0 < a_0$. Suppose $x_0 = a_0$. Then after battle r is over, player A spends zero on every remaining battle. In particular, he spends zero on battles $r - 2, r - 3, \dots$, and wins each of these battles with probability $\rho = p(0, \gamma) < 1$. If $p(0, \gamma) > 0$ or if $i_0 = 2$, Step 2 can be applied without modification to show that A should shift some resources from battle r to battle $r - 1$ to improve his overall chance of winning the tournament at state (i_0, j_0) . However, if $p(0, \gamma) = 0$ and $i_0 > 2$, then $v_{i,j} = 0 \forall (i, j) \in Z$ and hence $z(h) = z'(h) = 0 \forall h$. That is, player A wins the tournament with probability zero, and shifting resources from battle r to $r - 1$ does not improve this probability. However, consider an alternative reallocation, by which player A spends any positive share of a_0 on *every* remaining battle. Our assumptions on p imply that A must win each remaining battle with a strictly positive probability, resulting in a strictly positive probability that A wins the tournament. It follows that $x_0 = a_0$ is not a best response to B 's even-split strategy at (i_0, j_0) .

3.2 Uniqueness

While multiple subgame perfect equilibria may exist, all subgame perfect equilibria are in even-split strategies. To establish this result, we make use of the fact that Blotto games are constant-sum games and, therefore, have the following property:¹⁷

¹⁷Lemma 2 is a well-known property of constant-sum games. However, to our knowledge all published proofs establish the property for finite constant-sum games only. For completeness, a proof of the result for games with arbitrary strategy spaces (along the same lines as the standard proof) is in the Appendix.

Lemma 2. Consider a two-player extensive form game with mixed strategy sets S and T . Suppose player A 's expected payoff in strategy profile $(s, t) \in S \times T$ is $\pi(s, t)$ and player B 's expected payoff is $c - \pi(s, t)$, for some c . If the strategy profiles (s_1, t_1) and (s_2, t_2) are Nash equilibria of the game, so are (s_1, t_2) and (s_2, t_1) .

Since subgame perfect equilibria are strategy profiles that induce Nash equilibria in every subgame of an extensive form game, Lemma 2 immediately extends to subgame perfect equilibria.

Let $(\sigma_{\text{even}}^A, \sigma_{\text{even}}^B)$ be a pair of even-split subgame perfect equilibrium strategies of the sequential majoritarian Blotto game (which exists, as shown above). Suppose that (σ^A, σ^B) is another pure strategy subgame perfect equilibrium. By Lemma 2, the profile $(\sigma^A, \sigma_{\text{even}}^B)$ is then also a subgame perfect equilibrium. Recall that in Step 2 in Section 3.1 we showed that $z'(h)$ is *strictly* negative. This implies that, if player B plays an even-split strategy, then *every* subgame-perfect best reply by player A is an even-split strategy.¹⁸ Hence, σ^A must be an even-split strategy, and by reversing the roles of the players one can similarly show that σ^B must be an even-split strategy.

Next, suppose there exists a subgame perfect equilibrium in mixed strategies, i.e., at least one player randomizes over at least two pure strategies. Suppose this is player A , and call his mixed strategy ξ^A . By Lemma 2, the profile $(\xi^A, \sigma_{\text{even}}^B)$ is also a subgame perfect equilibrium, which implies that every pure strategy in the support of ξ^A is a subgame perfect best reply to σ_{even}^B . Again using Step 2 in Section 3.1, this means that every pure strategy in the support of ξ^A must be an even-split strategy. Therefore, any randomization in equilibrium must involve only even-split strategies. \square

4 Contest Length and the Probability of Victory

This section examines how the outcome of the sequential majoritarian Blotto game depends on the the number of battlefields.

4.1 Individual-battle advantage

Suppose we take a sequential N -battle game and increase N . If we do not change the initial resource budgets, the players will be forced to spread their fixed resources over an increasingly large number of battlefields. Proposition 1 establishes that players react to this change by scaling back their investments proportionally in all battles. We will show that this scaling-back typically helps the player who is more likely to win an *individual* battle of the contest. We call this player the “advantaged player.” In some important cases the advantaged player is the player with the larger resources, but this does not always have to be so.

¹⁸A subgame-perfect best reply is a strategy that is a best reply at all subgames.

Note that in every equilibrium of the N -battle Blotto game, player A wins each individual battle with probability $p(\bar{a}/N, \bar{b}/N)$, and B wins each battle with probability $1 - p(\bar{a}/N, \bar{b}/N)$. To help motivate our notion of single-battle advantage, consider the following contest success function:

$$p(x, y) = \begin{cases} \frac{x + c}{x + c + y + d} & \text{if } x + c + y + d > 0, \\ 1/2 & \text{if } x + c + y + d = 0 \end{cases} \quad (c, d \geq 0). \quad (9)$$

If $c = d = 0$, the function in (9) boils down to the Tullock lottery function (3). Note that the Tullock function is homogeneous of degree zero, so that each player's probability of winning a single battle depends on the players relative (instead of absolute) efforts in that battle. In this case, the length of the tournament has no effect on the distribution of outcomes in any individual battle—in equilibrium of the N -battle tournament, player A wins each battle with probability $p(\bar{a}/N, \bar{b}/N) = p(\bar{a}, \bar{b})$. Thus, player A has an advantage if $p(\bar{a}, \bar{b}) > 1/2$, and player B has an advantage if $p(\bar{a}, \bar{b}) < 1/2$. The identity of the advantaged player depends on the players' resource budgets but not on the length of the contest. Moreover, if p is symmetric (i.e., $p(x, y) = 1 - p(y, x)$), the advantaged player is the player who has the larger initial resource budget.

On the other hand, if $c > 0$ or $d > 0$ or both, $p(\bar{a}/N, \bar{b}/N)$ depends on N and converges to $c/(c + d)$ as N increases. As long as N is sufficiently large, player A has an advantage if $c > d$, while B has an advantage if $c < d$. In this case, the identity of the advantaged player does not depend on \bar{a} or \bar{b} . Moreover, if p is symmetric (i.e., $c = d$), neither player has an advantage in the limit, regardless of the relative size of their initial resource budgets.

4.2 An amplification result

Recall that player A wins the N -battle game if and only if A wins $n = (N + 1)/2$ battles before player B wins n battles. Thus, player A 's win probability can be expressed in the same way as (6), replacing i and j with n and replacing ρ with $p(\bar{a}/N, \bar{b}/N)$:

$$\pi_N(\bar{a}, \bar{b}) \equiv \sum_{k=0}^{n-1} \binom{n+k-1}{k} p(\bar{a}/N, \bar{b}/N)^n (1 - p(\bar{a}/N, \bar{b}/N))^k. \quad (10)$$

Player B 's probability of winning is then $1 - \pi_N(\bar{a}, \bar{b})$.

The following result provides conditions under which a large number of battlefields amplifies a player's existing individual-battle advantage.

Proposition 3. Consider a sequential majoritarian Blotto game with N battlefields and initial resource endowments $\bar{a} > 0$ and $\bar{b} > 0$. Define $\bar{p} = \lim_{N \rightarrow \infty} p(\bar{a}/N, \bar{b}/N)$.

- (a) If $\bar{p} > 1/2$ ($\bar{p} < 1/2$), player A (B) has a per-battle advantage for sufficiently large N . As $N \rightarrow \infty$, the probability that player A (B) wins the tournament converges to 1.

If p is homogeneous of degree zero and symmetric, statement (a) can be strengthened to the following:

- (b) If $\bar{a} > \bar{b}$ ($\bar{a} < \bar{b}$), player A (B) has a per-battle advantage for all N . The probability that player A (B) wins the tournament is strictly increasing in N , and converges to 1 as $N \rightarrow \infty$.
- (c) If $\bar{a} = \bar{b}$, then neither player has an advantage, and both players win the tournament with probability $1/2$, for all N .

Proposition 3 implies that splitting a shorter tournament into a larger number of battles tends to increase the win probability of the advantaged player and tends to decrease the win probability of the disadvantaged player. This, in turn, implies that the advantaged player prefers to lengthen the contest, and the disadvantaged player prefers to shorten it.

To gain an intuition for this result, it will be helpful to transform a player's win probability into the following form (which is also used in the proof of Proposition 3 in the Appendix):

Lemma 4. The expression in (10) is equivalent to

$$\pi_N(\bar{a}, \bar{b}) = \sum_{k=n}^N \binom{N}{k} p(\bar{a}/N, \bar{b}/N)^k (1 - p(\bar{a}/N, \bar{b}/N))^{N-k}. \quad (11)$$

Note that (11) is simply the probability that A wins more than half out of a total N independent battles, assuming that all N battles are fought out (instead of ending the game once the first player has won a majority of battles). The same logic as in the Condorcet Jury Theorem now applies: If winning the contest depends only on winning a majority of battles, and the outcomes of the battles are independent, then even a slightly higher probability of victory in every individual battle compounds over the course of a long tournament. In the limit, a long enough contest selects the advantaged player with certainty.

However, Proposition 3 is more than just an application of the Condorcet Jury Theorem, as the win probabilities in the individual battles are not exogenous in our context. Instead they are chosen by the players to maximize their chances of winning

the N -battle tournament. Thus, Proposition 3 relies on the even-split nature of the equilibrium in the N -battle tournament, which we established in Propositions 1.

4.3 Further examples

We conclude the section with a several examples that illustrate further aspects of the dependence of players' win probabilities on the length of the contest, in cases that are *not* covered by Proposition 3.

Example 1. Consider the contest success function in (9), with $c = d = 1$: $p(x, y) = (x + 1)/(x + y + 2)$. Suppose $\bar{a} = 3$ and $\bar{b} = 1$.¹⁹ In the even-split equilibrium of the N -battle tournament, player A wins each battle with probability $p(\bar{a}/N, \bar{b}/N) = (3/N + 1)/[4/N + 2] > 1/2$, which means player A has an individual-battle advantage for all N . Note, however, that $p(\bar{a}/N, \bar{b}/N)$ is decreasing in N and converges to $1/2$. An increase in the number of battles is now accompanied by a decrease in the advantaged player's probability of winning each battle. The left panel in Figure 1 shows that the second effect dominates the first: As $N \rightarrow \infty$, player A 's probability of winning the entire tournament decreases and converges to $1/2$.

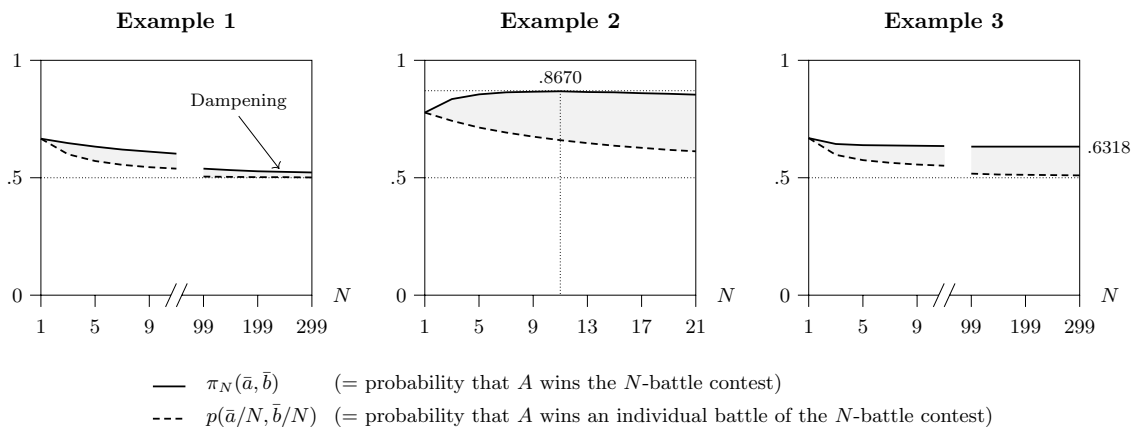


Figure 1: Win probability as a function of contest length.

Example 2. Consider the same contest success function as in Example 1, but suppose that $\bar{a} = 20$ and $\bar{b} = 5$. In contrast to the previous example, an increase in the number of battles first amplifies A 's per-battle advantage. The center panel in Figure 1 shows that $\pi_N(\bar{a}/N, \bar{b}/N)$ reaches a maximum of 0.8670 at $N = 11$ and begins to decrease thereafter.

¹⁹The contest success function is not covered by Proposition 3 (a) because $\bar{p} = 1/2$, and it is not covered by Proposition 3 (b)–(c) because p is not homogeneous of degree 0.

Example 3. Consider the contest success function

$$p(x, y) = \frac{1}{2} \left[1 + \frac{(x - y)(x^2 + y^2)^{1/4}}{x + y} \right]$$

and suppose that $\bar{a} = 0.25$ and $\bar{b} = 0.05$. Since p is symmetric but $\bar{a} > \bar{b}$, player A wins each battle with probability $p(\bar{a}/N, \bar{b}/N) > 1/2$, so player A has an individual-battle advantage for all N .²⁰ As was the case in Example 1, $p(\bar{a}/N, \bar{b}/N)$ is decreasing in N and converges to $1/2$ as $N \rightarrow \infty$. The right panel in Figure 1 shows that, unlike in the previous example, player A 's probability of winning the tournament converges to 0.6318.

5 Endogenous Budget Choice

We now study an extended model in which players simultaneously choose the size of their budgets at an initial fundraising stage, observe these choices, and then play the N -battle Blotto game. We call such games *sequential Blotto game with endogenous budgets*. The budget choice adds an additional layer of competition to our model. While the second stage—the sequential Blotto tournament—is still a constant-sum game, the overall model is now variable-sum.

We assume that both players attach value 1 to a tournament win and 0 to a loss. The cost function of choosing budgets \bar{a} and \bar{b} is linear and given by $C_A(\bar{a}) = c_A \bar{a}$ and $C_B(\bar{b}) = c_B \bar{b}$, respectively, with $c_A, c_B > 0$. We focus on the case where p is symmetric and homogeneous of degree zero. Note that this structure is equivalent to one in which players have the symmetric linear cost functions but asymmetric valuations of winning.

After the initial fundraising stage is completed, our main result implies that both players will split their budgets evenly across the N stages of the Blotto contest. The probability that player A wins the contest is then given by (10). At the fundraising stage, therefore, player A maximizes the following payoff function with respect to \bar{a} :

$$u_N^A(\bar{a}, \bar{b}) = \pi_N(\bar{a}, \bar{b}) - C_A(\bar{a}) = \sum_{k=n}^N \binom{N}{k} p(\bar{a}, \bar{b})^k (1 - p(\bar{a}, \bar{b}))^{N-k} - c_A \bar{a}. \quad (12)$$

(We used the fact that p is homogeneous of degree zero, i.e., $p(\bar{a}/N, \bar{b}/N) = p(\bar{a}, \bar{b})$.) Similarly, player B maximizes

$$u_N^B(\bar{a}, \bar{b}) = (1 - \pi_N(\bar{a}, \bar{b})) - C_B(\bar{b}) = \left[1 - \sum_{k=n}^N \binom{N}{k} p(\bar{a}, \bar{b})^k (1 - p(\bar{a}, \bar{b}))^{N-k} \right] - c_B \bar{b}. \quad (13)$$

²⁰One can verify that this CSF satisfies our assumptions on p , including the concavity condition (1), in the relevant ranges $x \in [0, 0.25]$ and $y \in [0, 0.05]$. Thus, an even-split equilibrium exists.

Thus, the sequential Blotto game with endogenous budgets can be interpreted as a one-shot, variable-sum contest with linear costs and contest success function $\pi_N(\cdot)$. A Nash equilibrium of this one-shot game describes the initial budget choices in subgame perfect equilibrium of our extended Blotto game.

5.1 Pure strategy equilibrium

Let us first consider the possibility of a pure strategy equilibrium. Note that player $i \in \{A, B\}$ will never choose a budget greater than $1/c_i$, as the value of competing in the second stage is at most 1. If player B chooses budget $\bar{b} \in (0, 1/c_B]$, player A 's payoff function u^A is continuous in \bar{a} , and a value $\bar{a} \in [0, 1/c_A]$ exists that maximizes (12). This value satisfies the first-order condition

$$\frac{\partial u_N^A}{\partial \bar{a}} = 0 \iff p_x(\bar{a}, \bar{b}) \left[n \binom{2n-1}{n} p(\bar{a}, \bar{b})^{n-1} (1 - p(\bar{a}, \bar{b}))^{n-1} \right] = c_A. \quad (14)$$

Similarly, given $\bar{a} \in [0, 1/c_A]$, player B 's optimal budget satisfies the first-order condition

$$\frac{\partial u_N^B}{\partial \bar{b}} = 0 \iff -p_y(\bar{a}, \bar{b}) \left[n \binom{2n-1}{n} p(\bar{a}, \bar{b})^{n-1} (1 - p(\bar{a}, \bar{b}))^{n-1} \right] = c_B. \quad (15)$$

Dividing (14) by (15) yields the following necessary condition for an interior equilibrium in pure strategies:

$$\frac{-p_x(\bar{a}, \bar{b})}{p_y(\bar{a}, \bar{b})} = \frac{c_A}{c_B}. \quad (16)$$

Since p is homogeneous of degree zero, by Euler's Theorem we further have

$$\bar{a} p_x(\bar{a}, \bar{b}) + \bar{b} p_y(\bar{a}, \bar{b}) = 0, \quad (17)$$

and combining (16) and (17) shows that, in any interior pure strategy equilibrium, the players' *relative* budgets satisfy

$$\frac{\bar{a}}{\bar{b}} = \frac{c_B}{c_A}. \quad (18)$$

To say more about the *absolute* levels of \bar{a} and \bar{b} , we use (18) to write

$$p(\bar{a}, \bar{b}) = p\left(\bar{a}, \bar{a} \frac{c_A}{c_B}\right) = p(c_B, c_A) \equiv \bar{p}. \quad (19)$$

Substituting (19) into (14), we get

$$p_x\left(\bar{a}, \bar{a} \frac{c_B}{c_A}\right) = c_A \left[n \binom{2n-1}{n} \bar{p}^{n-1} (1 - \bar{p})^{n-1} \right]^{-1}. \quad (20)$$

If n increases to $n+1$, the term $n\binom{2n-1}{n}$ in (20) changes by a factor $2(2n+1)/n = 4 + 2/n$, which is strictly greater than 4 for all n , and converges to 4 as $n \rightarrow \infty$. The term $\bar{p}^{n-1}(1-\bar{p})^{n-1}$ changes by a factor $\bar{p}(1-\bar{p}) \in (0, 1/4]$. Consider the following two cases:

- If $c_A = c_B$ then $\bar{p} = 1/2$ and $\bar{p}(1-\bar{p}) = 1/4$. In this case, the term in square brackets increases in n , which implies that the right-hand side of (20) decreases.
- If $c_A \neq c_B$ then $\bar{p} \neq 1/2$ and $\bar{p}(1-\bar{p}) < 1/4$. In this case, the term in square brackets in (20) decreases in n if n is sufficiently large, which implies that the right-hand side of (20) increases.

As for the left-hand side, since p is homogeneous of degree zero, p_x is homogeneous of degree -1 . It follows that the value of \bar{a} that solves (20) must be increasing in n (and hence N) if $c_A = c_B$, but will be asymptotically decreasing if $c_A \neq c_B$. Since \bar{b} is proportional to \bar{a} by (18), the same property applies to player B 's budget choice. This leads to the following result:

Proposition 5. *Fix c_A, c_B , and p symmetric and homogeneous of degree 0. Consider two sequential Blotto games with endogenous budgets, involving N and N' battles, respectively, with $N' > N$. If interior pure strategy equilibria exist for both games, the following is true:*

1. *If players are equally cost effective (i.e., $c_A = c_B$), the equilibrium budgets \bar{a} and \bar{b} are larger in the N' -battle contest than in the N -battle contest.*
2. *If players are not equally cost effective (i.e., $c_A \neq c_B$), and N is sufficiently large, the equilibrium budgets \bar{a} and \bar{b} are larger in the N -battle contest than in the N' -battle contest.*

This result highlights the important role of cost asymmetries in our extended model. Even a slight cost asymmetry could lead to the opposite comparative statics of the pure strategy equilibrium with respect to N , compared to the case of symmetric costs.

5.2 Mixed strategy equilibrium

Pure strategy equilibria will not exist once N is sufficiently large. To see this, note that (for homogeneous and symmetric p) Proposition 3 implies that player A 's probability of winning the tournament converges pointwise to

$$\lim_{N \rightarrow \infty} \pi_N(\bar{a}, \bar{b}) = p^{\text{all-pay}}(\bar{a}, \bar{b}) = \begin{cases} 1 & \text{if } \bar{a} > \bar{b}, \\ 1/2 & \text{if } \bar{a} = \bar{b}, \\ 0 & \text{if } \bar{a} < \bar{b}. \end{cases} \quad (21)$$

This is the contest success function of a one-shot, all-pay auction without noise.

Baye *et al.* (1996; Theorem 1, p. 293) show that the one-shot contest with success function $p^{\text{all-pay}}$, symmetric prizes $V = 1$, and constant marginal costs c_A and c_B , has a unique Nash equilibrium. In this equilibrium, the “strong” player (player A , say, with $c_A < c_B$) randomizes \bar{a} uniformly over the interval $[0, 1/c_B]$. The “weak” player (player B with $c_B < c_A$) sets $\bar{b} = 0$ with probability $1 - c_A/c_B$, and with the remaining probability c_A/c_B randomizes \bar{b} uniformly over $[0, 1/c_B]$. If the players are symmetric (i.e., $c_A = c_B = c$), this profile reduces to both players drawing their budgets uniformly from $[0, 1/c]$.

In the all-pay auction, a player wins with certainty if he spends an infinitesimal amount more than his opponent. As an approximation of a N -battle Blotto game with endogenous budgets and large N , this implies that any player whose budget is slightly larger than that of the opponent wins with probability almost one. It is easy to see that this implies that at least one player would like to deviate from any pure strategy profile (\bar{a}, \bar{b}) . Instead, any subgame perfect equilibrium of the large- N Blotto game must involve randomization at the fundraising stage.

The question is whether the uniform equilibrium that arises in the all-pay auction limit game approximates the first-stage equilibrium behavior in our extended Blotto game, if N is large but finite. There are, to our knowledge, no results that imply that this is the case, and we will not pursue this issue here.²¹ We can, however, show that uniform randomization is an ε -equilibrium of large N -battle games. (ε -equilibria are strategy profiles in which no player can improve his payoff by more than ε when switching to another strategy.)

Proposition 6. *Let p be a symmetric and h.o.d. 0 contest success function. Without loss of generality, assume that $c_A \leq c_B$. For every $\varepsilon > 0$, there exists N^* such that the following is true for all $N > N^*$: In the N -battle Blotto game with endogenous budgets, an ε -equilibrium exists in which player A randomizes his budget using to cumulative distribution function $G_A(\bar{a}) = c_B \bar{a}$, and player B randomizes his budget using to the cumulative distribution function $G_B(\bar{b}) = 1 - c_A/c_B + c_A \bar{b}$.*

Proposition 6 shows that the all-pay auction without noise can be regarded as a suitable approximation of majoritarian Blotto games with endogenous budgets, provided the contest success function that governs each individual battle of the Blotto game is symmetric and homogeneous of degree 0.

²¹The question whether equilibria of discontinuous games are similar to equilibria of close-by continuous games has long been of interest to game theorists. Dasgupta and Maskin (1986, p. 38) make a statement which suggests that, for the type of discontinuity present in the all-pay auction, this is the case; however, a formal result is not proven. Börgers (1991) examines the approximation of a continuous game by a sequence of discontinuous games (whereas we have a sequence of continuous games converging to a discontinuous game). Bagh (2010) studies the approximation of a discontinuous game by a sequence of continuous games, but to verify his conditions one needs to know the equilibria of the continuous games (whereas we know the equilibrium of the discontinuous game).

Lastly, what happens if p is not homogeneous of degree zero? Play in the Blotto game with endogenous budgets can be very different from that described in Proposition 6. To illustrate this, consider the CSF $p(x, y) = (x + 1)/(x + y + 2)$. One can show that the probability that player A wins in equilibrium of an N -battle Blotto game converges to $\lim_{N \rightarrow \infty} \pi_N(\bar{a}, \bar{b}) = 1/2$. Thus, in the limit a player's budget has no influence on his probability of winning. Moreover, convergence of $\pi_N(\cdot)$ is uniform, which implies that a pair of *zero* budgets is an ε -equilibrium of every sufficiently long Blotto game.

6 Conclusion

We examined a two-player, multi-battle competition characterized by three main features: The battles are fought sequentially; the player who first wins a majority of battles wins the contest; and both players are endowed with fixed budgets that must be allocated across the battles. We called such games sequential majoritarian Blotto games. The tradeoffs the players face when deciding how much of their resources to invest in each battle are not trivial. Yet, we showed that a very simple strategy—namely, to allocate the same proportion of one's initial resources to every battle—uniquely describes equilibrium behavior in this game. Therefore, the intensity of fighting as well as the probability with which a given player wins each individual battle remains constant from the beginning to the end of the game. These dynamics contrast sharply with those arising in equilibrium of sequential majoritarian non-Blotto games. Our findings are robust to the introduction of differences across players, but sensitive to the introduction of differences across battles.

We also derived sufficient conditions under which a Condorcet Jury Theorem-type “amplification effect” emerges in games with a large number of battles, i.e., the player who is more likely to win any individual battle wins the N -battle game with probability almost one if N is large. For homogeneous and symmetric contest success functions, this is the player with the larger initial resource budget. In an extended model in which players invest in their resource budgets prior to the Blotto contest, the amplification effect implies that the games' payoff functions approach those of the all-pay auction without noise, as N becomes large. The mixed strategy equilibrium of the all-pay auction then approximates (in the sense of ε -approximation) the players' budget choice in this extended game. For contests of shorter length, pure strategy equilibria may exist. When they do, budgets may respond differently to an increase in the contest length, depending on whether the players' costs are symmetric or not.

Overall, the dynamics of effort allocation in sequential majoritarian Blotto contests contrast sharply with those arising in equilibrium of sequential majoritarian non-Blotto games. The question of whether the resources that a player can mobilize can be chosen and augmented during the contest, or whether players have a fixed budget from which

they must draw resources over time, is essential. This distinction can explain why the intensity of fighting is strongly path dependent in some conflicts but not in others.

Appendix

Remaining steps in the proof of Proposition 1

Here we complete the proof of the existence of an even-split equilibrium in the sequential majoritarian Blotto game. For the most part, the argument was developed in Section 3.1. What remains to be done is the following:

- A.** Show that $v_{i,j} > v_{i+1,j-1}$ for all $(i,j) \geq (0,1)$, where $v_{i,j}$ is defined in (6).
- B.** Repeat the same steps as in the text for the following cases: (i) $i_0 = 1$ and $j_0 > 1$; (ii) $i_0 > 1$ and $j_0 = 1$.

Part A. Since $\rho \in (0,1)$, (6) implies that

$$v_{i,j} \in (0,1) \quad \forall (i,j) \geq (1,1), \quad v_{0,j} = 1 \quad \forall j \geq 1, \quad v_{i,0} = 0 \quad \forall i \geq 1. \quad (22)$$

The proof is by induction. Fix $d \geq 1$ and suppose that

$$v_{i,j} > v_{i+1,j-1} \quad \forall (i,j) \geq (0,1) \text{ s.t. } i+j = d. \quad (23)$$

Take $(i,j) \geq (0,1)$ with $i+j = d+1$. If $i \geq 1$ and $j \geq 2$, write

$$v_{i,j} = \rho v_{i-1,j} + (1-\rho)v_{i,j-1} \quad \text{and} \quad v_{i+1,j-1} = \rho v_{i,j-1} + (1-\rho)v_{i+1,j-2}.$$

By (23) we have

$$v_{i,j} - v_{i+1,j-1} = \rho(v_{i-1,j} - v_{i,j-1}) + (1-\rho)(v_{i,j-1} - v_{i+1,j-2}) > 0.$$

If $i = 0$ then $j \geq 2$ and (22) implies

$$v_{0,j} = 1, \quad v_{1,j-1} \in (0,1) \quad \Rightarrow \quad v_{0,j} - v_{1,j-1} > 0.$$

Similarly, if $j = 1$ then $i \geq 1$ (22) implies

$$v_{i,1} \in (0,1), \quad v_{i+1,0} = 0 \quad \Rightarrow \quad v_{i,1} - v_{i+1,0} > 0.$$

Thus, for all $(i,j) \geq (0,1)$ with $i+j = d+1$, we have $v_{i,j} > v_{i+1,j-1}$. Now note that (23) is clearly true for $d = 1$ ($v_{0,1} = 1 > 0 = v_{1,0}$), and the result follows.

Part B. First consider case (i). Since $i_0 = 1$, once player A wins battle r he has won the tournament. Therefore, the term $P(h)$ (i.e., the probability that A wins battle r and battle $r - 1$) is not defined. The term $Q(h)$ (i.e., the probability that A loses battle r and battle $r - 1$) is defined as before and strictly increasing in h . After battle r and (if necessary) battle $r - 1$ are over, the tournament will be in some state

$$(i, j) \in Z = \left\{ (0, j_0), (0, j_0 - 1), (1, j_0 - 2) \right\}.$$

For $\gamma > 0$ and $x_0 \in [0, a_0]$, the probabilities $v_{i,j}$ that player A wins the tournament at $(i, j) \in Z$ can therefore be ranked as follows:

$$1 = v_{0,j_0} = v_{0,j_0-1} > v_{1,j_0-2} \geq 0.$$

Going back to state (i_0, j_0) , player A wins the tournament with probability

$$z(h) = 1 - Q(h) + Q(h)v_{1,j_0-2}.$$

Note that $z'(h) = Q'(h)[v_{1,j_0-2} - 1] < 0$, and it follows that A can increase his chance of winning the tournament by reducing h .

Next, consider case (ii). Since $j_0 = 1$, once player A loses battle r he has lost the tournament. Therefore, the term $Q(h)$ is not defined; the term $P(h)$ is defined as before and strictly decreasing in h . After battle r and (if necessary) battle $r - 1$ are over, the tournament will be in some state

$$(i, j) \in Z = \left\{ (i_0 - 2, 1), (i_0 - 1, 0), (i_0, 0) \right\}.$$

For $\gamma > 0$ and $x_0 \in [0, a_0]$, the probabilities $v_{i,j}$ that player A wins the tournament at $(i, j) \in Z$ can therefore be ranked as follows:

$$1 \geq v_{i_0-2,1} > v_{i_0-1,0} = v_{i_0,0} = 0.$$

Going back to state (i_0, j_0) , player A wins the tournament with probability

$$z(h) = P(h)v_{i_0-2,1}.$$

Note that $z'(h) = P'(h)v_{i_0-2,1} < 0$, and it follows that A can increase his chance of winning the tournament by reducing h . \square

Proof of Lemma 2

In Nash equilibrium of a constant-sum game, player A maximizes π by choice of $s \in S$, and player B minimizes π by choice of $t \in T$. Let (s_1, t_1) and (s_2, t_2) be two Nash

equilibria. Because (s_1, t_1) is an equilibrium,

$$\pi(s_1, t_1) \geq \pi(s_2, t_1) \quad \text{and} \quad \pi(s_1, t_1) \leq \pi(s_1, t_2) \quad (24)$$

(otherwise player A would deviate from s_1 to s_2 , or player B would deviate from t_1 to t_2). Similarly, because (s_2, t_2) is an equilibrium,

$$\pi(s_2, t_2) \geq \pi(s_1, t_2) \quad \text{and} \quad \pi(s_2, t_2) \leq \pi(s_2, t_1). \quad (25)$$

Combining the four weak inequalities in (24)–(25), we get $\pi(s_1, t_1) \geq \pi(s_2, t_1) \geq \pi(s_2, t_2) \geq \pi(s_1, t_2) \geq \pi(s_1, t_1)$, and it follows that

$$\pi(s_1, t_1) = \pi(s_2, t_1) = \pi(s_2, t_2) = \pi(s_1, t_2) = \pi(s_1, t_1). \quad (26)$$

By definition, $\pi(s_1, t_1)$ is the maximum payoff A can obtain if B uses strategy t_1 . By the first equality in (26), if A uses strategy s_2 against t_1 he obtains exactly this maximum payoff; hence s_2 is a best reply to t_1 . Similarly, $\pi(s_2, t_2)$ is the minimum payoff that A can obtain if he uses strategy s_2 . By the second equality in (26), if B uses strategy t_1 against s_2 then A obtains exactly this minimum payoff; hence t_1 is a best reply to s_2 . It follows that the profile (s_2, t_1) must also be a Nash equilibrium. An analogous argument, using the third and fourth equality in (26), establishes that (s_1, t_2) is a Nash equilibrium. \square

Proof of Lemma 4

Consider a vector $\nu = (\nu_1, \nu_2, \dots, \nu_N)$ of i.i.d. draws $\nu_i \in \{0, 1\}$, where $Pr[\nu_i = 1] = \rho$. Given $\nu \in \{0, 1\}^N$, define

$$W_i(\nu) = \sum_{i'=1}^i \nu_{i'}, \quad m(\nu) = \begin{cases} \min\{i : W_i = n\} & \text{if } W_N \geq n, \\ -1 & \text{otherwise.} \end{cases}$$

Observe that $W_N(\nu) \geq n$ if and only if $m(\nu) \geq n$. Thus,

$$\begin{aligned} Pr[W_N(\nu) \geq n] &= \sum_{W_N=n}^N \binom{N}{W_N} \rho^{W_N} (1-\rho)^{N-W_N} \\ &= \sum_{m=n}^N \binom{m-1}{m-n} \rho^n (1-\rho)^{m-n} = Pr[m(\nu) \geq n]. \end{aligned} \quad (27)$$

To understand the second sum in (27), take $m \geq n$. By definition of $m(\nu)$, $\rho(\nu) = m$ if and only if $|\{i \leq m : \nu_i = 1\}| = n$, $|\{i \leq m : \nu_i = 0\}| = m - n$, and $\nu_m = 1$. Each term in the sum is the probability that ν has these properties for given m , and we iterate

over $m \in \{n, \dots, N\}$. Now substitute k for W_N in the first sum in (27), substitute k for $m - n$ in the second sum, and substitute $p(\bar{a}/N, \bar{b}/N)$ for ρ in both sums, to get

$$\begin{aligned} \sum_{k=n}^N \binom{N}{k} p(\bar{a}/N, \bar{b}/N)^k (1 - p(\bar{a}/N, \bar{b}/N))^{N-k} \\ = \sum_{k=0}^{n-1} \binom{n+k-1}{k} p(\bar{a}/N, \bar{b}/N)^n (1 - p(\bar{a}/N, \bar{b}/N))^k. \end{aligned}$$

These are the expressions in (11) and (10), respectively. \square

Proof of Proposition 3

In Proposition 1 we have shown that, in every subgame perfect equilibrium of the sequential best-of- N Blotto game, along the equilibrium path players A and B invest $\bar{a}/N > 0$ and $\bar{b}/N > 0$ into each battle, respectively. Thus, player A wins each individual battle with probability $\rho_N = p(\bar{a}/N, \bar{b}/N)$. Using Lemma 4, player A 's probability of winning the tournament can be written as

$$\pi_N(\bar{a}, \bar{b}) = \sum_{k=n}^N \binom{N}{k} (\rho_N)^k (1 - \rho_N)^{N-k}. \quad (28)$$

To show part (a) of the result, suppose $\bar{p} > 1/2$ (if $\bar{p} < 1/2$ the argument is analogous). There exists $\delta > 0$ and N^* such that $\rho_N > 1/2 + \delta$ for all $N > N^*$. Since, for given N , (28) increases in ρ_N , a lower bound for $\pi(\bar{a}, \bar{b}, N)$ can be obtained by substituting $\rho_N = 1/2 + \delta$ into (28):

$$\underline{\pi} = \sum_{k=n}^N \binom{N}{k} (1/2 + \delta)^k (1/2 - \delta)^{N-k},$$

so that $\pi(\bar{a}, \bar{b}, N) \geq \underline{\pi}$ for all $N > N^*$. Since $\delta > 0$, the Condorcet Jury Theorem (see, e.g., Boland 1998) now implies that $\underline{\pi} \rightarrow 1$, and therefore $\pi(\bar{a}, \bar{b}, N) \rightarrow 1$, as $N \rightarrow \infty$.

To show parts (b) and (c) of the result, suppose p is homogeneous of degree zero and symmetric. Then

$$\rho_N = p(\bar{a}/N, \bar{b}/N) = p(\bar{a}, \bar{b}) \begin{cases} \geq \\ \leq \end{cases} 1/2 \quad \text{iff} \quad \bar{a} \begin{cases} \geq \\ \leq \end{cases} \bar{b}$$

for all N . Thus, the Condorcet Jury Theorem can be applied directly to show that $\bar{a} > \bar{b}$ implies $\pi(\bar{a}, \bar{b}, N)$ is strictly increasing in N , and $\pi(\bar{a}, \bar{b}, N) \rightarrow 1$ as $N \rightarrow \infty$. (The results for $\bar{a} < \bar{b}$ and $\bar{a} = \bar{b}$ are analogous.) \square

Proof of Proposition 6

As we explained in the text, in anticipation of the even-split equilibrium in the Blotto subgame, we can view the budget choice stage as a one-shot contest in which A and B invest efforts x and y ,²² pay linear costs $C_A(x) = c_A x$ and $C_B(y) = c_B y$, and A wins with probability

$$\pi_N(x, y) \equiv \sum_{k=n}^N \binom{N}{k} p(x, y)^k (1 - p(x, y))^{N-k},$$

where p is symmetric, homogeneous of degree 0, increasing in x and decreasing in y . Note that these properties of p transfer to π_N . Efforts larger than $1/c_i$ are not individually rational; hence the strategy space can be restricted to $[0, 1/c_A] \times [0, 1/c_B]$.

Consider the all-pay auction game with constant marginal effort costs c_A and c_B and contest success function

$$p^{\text{all-pay}}(x, y) = \begin{cases} 1 & \text{if } x > y, \\ 1/2 & \text{if } x = y, \\ 0 & \text{if } x < y \end{cases}$$

Note that $|\pi_N(x, y) - p^{\text{all-pay}}(x, y)| \leq 1/2$ for all x, y . Let $G_A(x) = c_B x$ and $G_B(y) = 1 - c_A/c_B + c_A y$ be the effort distributions in the Nash equilibrium of this all-pay auction. We will show that for every $\varepsilon > 0$ there exists $\bar{N}(\varepsilon)$ such that $N > \bar{N}(\varepsilon)$ implies the following: For every $x \in [0, 1/c_A]$, $E[\pi_N(x, y)|x, y \sim G_B]$ is within ε of $E[p^{\text{all-pay}}(x, y)|x, y \sim G_B]$; and for every $y \in [0, 1/c_B]$, $E[1 - \pi_N(x, y)|y, x \sim G_A]$ is within ε of $E[1 - p^{\text{all-pay}}(x, y)|y, x \sim G_A]$. The result is then immediate.

Fix $\lambda \in (0, \infty)$ and denote by

$$q_N(\lambda) \equiv \pi_N(x, \lambda x) = \pi_N(1, \lambda)$$

player A 's success probability in the N -battle contest if B spends λ times the amount A spends (assuming A spends a positive amount). Monotonicity of p implies that $q_N(\lambda)$ decreases strictly in λ ; and symmetry implies $q_N(\lambda) = 1 - q_N(1/\lambda)$ and $q_N(1) = 1/2$.

For $\varepsilon \in (0, 1)$, let $\lambda_N(\varepsilon) > 1$ be the unique λ that solves

$$q_N(\lambda) = \frac{\varepsilon}{2}.$$

This means: In order to hold A to a success probability of $\varepsilon/2$, player B must spend $\lambda_N(\varepsilon)$ times the amount A spends. Equivalently, by the symmetry and homogeneity properties of π_N , player A wins with probability $1 - \varepsilon/2$ if B spends $1/\lambda_N(\varepsilon)$ times the

²²Throughout the proof we replace \bar{a} with x and \bar{b} with y for better readability.

amount A spends. $\lambda_N(\varepsilon)$ is well defined for sufficiently large N . Moreover, as $N \rightarrow \infty$ Proposition 3 implies that $\pi_N(x, y) \rightarrow p^{\text{all-pay}}(x, y)$ pointwise. For given ε , this implies that $\lambda_N(\varepsilon) \rightarrow 1$ as $N \rightarrow \infty$.

Since we restrict attention to $(x, y) \in [0, 1/c_A] \times [0, 1/c_B]$, as long as $(x, y) \neq (0, 0)$ we have

$$\left| \pi_N(x, y) - p^{\text{all-pay}}(x, y) \right| \geq \frac{\varepsilon}{2} \text{ iff } y \in \left[\frac{1}{\lambda_N(\varepsilon)}x, \min \left\{ \frac{1}{c_B}, \lambda_N(\varepsilon)x \right\} \right]. \quad (29)$$

The length of the interval $[x/\lambda(\varepsilon), \min\{1/c_B, \lambda(\varepsilon)x\}]$ is at most $[1 - 1/\lambda_N(\varepsilon)^2]/c_B$; see Figure 2.

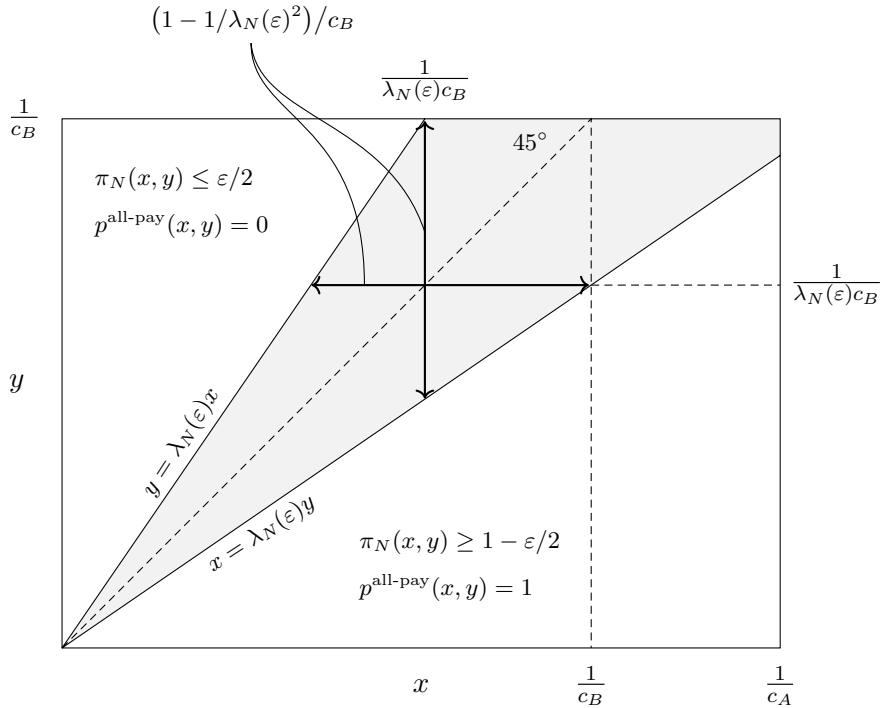


Figure 2: Partitioning of the strategy space.

If player B randomizes effort y according to the cumulative distribution function $G_B(y) = 1 - c_A/c_B + c_A y$, player A 's expected probability of winning the N -battle Blotto game with effort $x \in [0, 1/c_A]$ can be written as

$$E[\pi_N(x, y)|x, y \sim G_B] = E[p^{\text{all-pay}}(x, y)|x, y \sim G_B] \pm D_N^A(\varepsilon),$$

where $D_N^A(\varepsilon)$ is a function satisfying

$$\begin{aligned} 0 < D_N^A(\varepsilon) &\leq \left(1 - \frac{c_A}{c_B}\right) \frac{\varepsilon}{2} + \frac{c_A}{c_B} \left(\left(1 - \frac{1}{\lambda_N(\varepsilon)^2}\right) \frac{1}{2} + \frac{1}{\lambda_N(\varepsilon)^2} \frac{\varepsilon}{2} \right) \\ &= \frac{\varepsilon}{2} + \frac{c_A}{c_B} \left(1 - \frac{1}{\lambda_N(\varepsilon)^2}\right) \left(\frac{1}{2} - \frac{\varepsilon}{2}\right). \end{aligned}$$

A sufficient condition for $D_N^A(\varepsilon) \leq \varepsilon$ is

$$\lambda_N(\varepsilon) \leq \left(1 - \frac{\varepsilon}{1 - \varepsilon} \frac{c_B}{c_A}\right)^{-1/2} \equiv \bar{\lambda}^A(\varepsilon).$$

Note that $\bar{\lambda}^A(\varepsilon) > 1$ for all $0 < \varepsilon < 2c_A/[c_A + c_B]$, and $\lambda_N(\varepsilon) \rightarrow 1$ as $N \rightarrow \infty$. It follows that for every $\varepsilon > 0$ there exists $\bar{N}^A(\varepsilon)$ such that, for $N > \bar{N}^A(\varepsilon)$, $E[\pi_N(x, y)|x, y \sim G_B]$ differs from $E[p^{\text{all-pay}}(x, y)|x, y \sim G_B]$ by no more than ε .

Next, repeat the exercise for player B , who wins with probability $1 - \pi_N$ in the Blotto game and with probability $1 - p^{\text{all-pay}}$ in the all-pay auction, respectively. Assume that player A randomizes x uniformly on $[0, 1/c_B] \subseteq [0, 1/c_A]$. Given $x \in [0, 1/c_B]$ we have

$$\left| (1 - \pi_N(x, y)) - (1 - p^{\text{all-pay}}(x, y)) \right| \geq \frac{\varepsilon}{2} \text{ iff } x \in \left[\frac{1}{\lambda_N(\varepsilon)} y, \min \left\{ \frac{1}{c_B}, \lambda_N(\varepsilon) y \right\} \right]. \quad (30)$$

The length of the interval on the right side of (30) is at most $[1 - 1/\lambda_N(\varepsilon)^2]/c_B$, and player B 's expected probability of winning the N -battle Blotto game with effort $y \in [0, 1/c_B]$ can be written as

$$E[1 - \pi_N(x, y)|y, x \sim G_A] = E[1 - p^{\text{all-pay}}(x, y)|y, x \sim G_A] \pm D_N^B(\varepsilon),$$

where D_N^B is a function satisfying

$$0 < D_N^B(\varepsilon) \leq \left(1 - \frac{1}{\lambda_N(\varepsilon)^2}\right) \frac{1}{2} + \frac{1}{\lambda_N(\varepsilon)^2} \frac{\varepsilon}{2}.$$

A sufficient condition for $D_N^B(\varepsilon) \leq \varepsilon$ is

$$\lambda_N(\varepsilon) \leq \left(\frac{1 - \varepsilon}{1 - 2\varepsilon}\right)^{1/2} \equiv \bar{\lambda}^B(\varepsilon).$$

Note that $\bar{\lambda}^B(\varepsilon) > 1$ for all $0 < \varepsilon < 1$, and $\lambda_N(\varepsilon) \rightarrow 1$ as $N \rightarrow \infty$. It follows that for every $\varepsilon > 0$ there exists $\bar{N}^B(\varepsilon)$ such that, for $N > \bar{N}^B(\varepsilon)$, $E[1 - \pi_N(x, y)|y, x \sim G_A]$ differs from $E[1 - p^{\text{all-pay}}(x, y)|y, x \sim G_A]$ by no more than ε .

Now set $\bar{N}(\varepsilon) \equiv \max\{\bar{N}^A(\varepsilon), \bar{N}^B(\varepsilon)\}$ and the result follows. \square

References

- [1] Anbarcı, Nejat, Kutay Cingiz, and Mehmet S. Ismail (2018). “Multi-Battle n -Player Dynamic Contests.” Maastricht University Graduate School of Business and Economics, Discussion Paper RM/18/003.
- [2] Bagh, Adib (2010). “Variational Convergence: Approximation and Existence of Equilibria in Discontinuous Games.” *Journal of Economic Theory* 145, 1244–1268.
- [3] Baye, Michael R., Dan Kovenock, and Casper G. de Vries (1996). “The All-Pay Auction With Complete Information.” *Economic Theory* 8 (2), 291–305.
- [4] Boland, Philip J. (1998). “Majority Systems and the Condorcet Jury Theorem.” *Journal of the Royal Statistical Society Series D (The Statistician)* 38 (3), 181–189.
- [5] Borel, Émile (1921). “La theorie du jeu les equations integrales a noyau symetrique.” *Comptes Rendus del Academie* 173, 1304–1308. English translation by Savage, L. (1953), “The Theory of Play and Integral Equations With Skew Symmetric Kernels,” *Econometrica* 21, 97–100.
- [6] Börgers, Tilman (1991). “Upper Hemicontinuity of the Correspondence of Subgame-Perfect Equilibrium Outcomes.” *Journal of Mathematical Economics* 20, 89–106.
- [7] Dasgupta, Partha and Eric Maskin (1986). “The Existence of Equilibrium in Discontinuous Economic Games, II: Applications.” *Review of Economic Studies* 53 (1), 27–41.
- [8] Deck, Cary A. and Roman M. Sheremeta (2012). “Fight or Flight? Defending Against Sequential Attacks in the Game of Siege.” *Journal of Conflict Resolution* 56 (6), 1069–1088.
- [9] Ferrall, Christopher and Anthony A. Smith, Jr. (1999). “A Sequential Game Model of Sports Championship Series: Theory and Estimation.” *Review of Economics and Statistics* 81 (4), 704–719.
- [10] Friedman, Lawrence (1958). “Game-Theory Models in the Allocation of Advertising Expenditures.” *Operations Research* 6 (5), 699–709.
- [11] Fu, Qiang and Jingfeng Lu (2012). “The Optimal Multi-Stage Contest.” *Economic Theory* 51 (2), 351–382.
- [12] Fu, Qiang, Jingfeng Lu, and Yue Pan (2015). “Team Contests With Multiple Pairwise Battles.” *American Economic Review* 105 (7), 2120–2140.

- [13] Gelder, Alan (2014), “From Custer to Thermopylae: Last Stand Behavior in Multi-Stage Contests.” *Games and Economic Behavior* 87, 442–466.
- [14] Gescheit, Danielle T., Rob Duffield, Missa Skein, Neil Brydon, Stuart J. Cormal, and Machar Reid (2017). “Effects of Consecutive Days of Match Play on Technical Performance in Tennis.” *Journal of Sports Sciences* 35 (20), 1988–1994.
- [15] Gradstein, Mark and Kai A. Konrad (1999). “Orchestrating Rent Seeking Contests.” *Economic Journal* 109 (458), 536–545.
- [16] Gauriot, Romain and Lionel Page (2014). “Does Success Breed Success? A Quasi-Experiment on Strategic Momentum in Dynamic Contests.” Queensland University of Technology Business School, Discussion Paper 28.
- [17] Häfner, Samuel (2017). “A Tug-of-War Team Contest.” *Games and Economic Behavior* 104, 372–391.
- [18] Harbaugh, Rick and Tilman Klumpp (2005). “Early Round Upsets and Championship Blowouts.” *Economic Inquiry* 43, 316–329.
- [19] Harris, Christopher and John Vickers (1987). “Racing With Uncertainty.” *Review of Economic Studies* 54 (1), 1–21.
- [20] Hillman, Arye L. and John G. Riley (1989). “Politically Contestable Rents and Transfers.” *Economics & Politics* 1 (1), 17–39.
- [21] Klumpp, Tilman, Hugo M. Mialon, and Michael A. Williams (2015). “Leveling the Playing Field? The Role of Public Campaign Funding in Elections.” *American Law and Economics Review* 17, 361–408.
- [22] Klumpp, Tilman and Mattias K. Polborn (2006). “Primaries and the New Hampshire Effect.” *Journal of Public Economics* 90, 1073–1114.
- [23] Klumpp, Tilman and Kai A. Konrad (2018). “Sequential Majoritarian Blotto Games.” Working paper.
- [24] Konrad, Kai A. (2009). *Strategy and Dynamics in Contests*. Oxford University Press.
- [25] Konrad, Kai A. (2018). “Budget and Effort Choice in Sequential Colonel Blotto Campaigns.” Unpublished manuscript.
- [26] Konrad, Kai A. and Dan Kovenock (2009). “Multi-Battle Contests.” *Games and Economic Behavior* 66, 256–274.
- [27] Kovenock, Dan and Brian Roberson (2010). “Conflicts With Multiple Battlefields.” CESifo working paper: Empirical and Theoretical Methods, No. 3165.

- [28] Kvasov, Dmitriy (2007). “Contests With Limited Resources.” *Journal of Economic Theory* 127, 738–748.
- [29] Malueg, David A. and Andrew J. Yates (2010). “Testing Contest Theory: Evidence from Best-of-Three Tennis Matches.” *Review of Economics and Statistics* 92 (3), 689–692.
- [30] Rinott, Yosef, Marco Scarsini, and Yaming Yu (2012). “A Colonel Blotto Gladiator Game.” *Mathematics of Operations Research* 37 (4), 574–590.
- [31] Roberson, Brian (2006). “The Colonel Blotto Game.” *Economic Theory* 29, 1–24.
- [32] Roberson, Brian and Dmitriy Kvasov (2012). “The Non-Constant-Sum Colonel Blotto Game.” *Economic Theory* 51, 397–433.
- [33] Rosen, Sherwin (1986). “Prizes and Incentives in Elimination Tournaments.” *American Economic Review* 76 (4), 701–715.
- [34] Ryvkin, Dmitry (2011). “Fatigue in Dynamic Tournaments.” *Journal of Economics and Management Strategy* 20 (4), 1011–1041.
- [35] Sela, Aner and Eyal Erez (2013). “Dynamic Contests With Resource Constraints.” *Social Choice and Welfare* 41 (4), 863–882.
- [36] Skillington, S. Andrew, Robert H. Brophy, Rick W. Wright, and Matthew V. Smith (2017), “Effect of Pitching Consecutive Days in Youth Fast-Pitch Softball Tournaments on Objective Shoulder Strength and Subjective Shoulder Symptoms.” *American Journal of Sports Medicine* 45 (6), 1413–1419.
- [37] Solomon, Adam (2018). “Strategy and Fundraising in Sequential Majoritarian Elections.” Working paper. July 24, 2018, <https://ssrn.com/abstract=3095105>.
- [38] Taylor, Andrew J. (2010). “Does Presidential Primary and Caucus Order Affect Policy?” *Political Research Quarterly* 63 (2), 398–409.
- [39] Tullock, Gordon (1980). “Efficient Rent-Seeking.” In: J. M. Buchanan, R. D. Tollison, and G. Tullock (eds.) *Toward a Theory of the Rent-Seeking Society*. College Station, Texas.