Price-Quality Competition in a Mixed Duopoly*

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Abstract

We examine competition between a private and a public provider in markets for merit goods such as education, healthcare, housing, recreation, or culture. The private firm provides a high-price/high-quality variety of the good and serves richer individuals, while the public firm provides a low-price/low-quality variety and serves poorer individuals. We first characterize the private competitor's best response to changes in the public firm's price and quality. This enables us to examine the distributional effects of policies that affect the price or quality of the public firm's product. We then numerically characterize the public firm's optimal provision policy, taking the private response into consideration. Our results have implications for the financing of publicly provided goods, and for whether additional resources, if available, should be spent on reducing the price or enhancing the quality of these goods.

Keywords: Mixed duopoly; quality differentiation; public provision of private goods; crowding-out/crowding-in; funding of public services; distribution.

JEL codes: D21; D43; H11; H42; H44; I00; L38

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1 Introduction

Many goods and services are provided by private firms as well as public entities. Education, healthcare, health insurance, housing, housing-related financial services, as well as recreational and cultural amenities, are important examples. The public provision of such merit goods is an indirect way of redistributing incomes,[1] and major public investments are often made with the stated goal of enhancing access to these goods.

This paper investigates the interaction between a public and a private provider in a market for such goods. Our model has two important features. First, the two firms compete in terms of both price and quality. Second, consumers are heterogeneous in their incomes. We focus on market outcomes in which the private firm provides a high-price/high-quality variety of the good and serves higher-income individuals, while the public firm provides a low-price/low-quality variety and serves lower-income individuals.[2] Within this framework, we examine the effects of a number of policies that change the price and quality of the publicly provided good. These policies can be interpreted as reflecting shifts in the policy maker's objective, constraints, or both. We then address three key questions: (a) How do these policies affect the price and quality of the privately provided variety? (b) How do they affect the welfare of individuals with different incomes? (c) What does an optimal public policy look like?

To give a concrete example, consider competition between a public and a private school. Imagine a policy initiative aimed at making education more accessible to low-income students by reducing public school tuition. If public education is to be provided at the same quality, the affordability initiative requires an increase in the public school's funding level. Absent such a subsidy, the only way for the public school to meet its new mandate is to cut costs by lowering quality. We will demonstrate that these two scenarios—a fully funded affordability initiative and an unfunded one—can lead to opposite responses by the private school. Under certain conditions on consumer preferences and the income distribution, the private school responds with a higher quality in the former scenario but a lower quality in the latter. Intuitively, in the first case the public school becomes more attractive to all students. The resulting increase in competitive pressure forces the private school to become more attractive itself, which it does in part by increasing its quality.

[1]For example, the public provision of education and healthcare accounts for a significant share of income redistribution in developed countries; see Poterba 1996.

[2]While not universal, this pattern is common in settings where the objective of public provision is to offer affordable access to certain goods. For example, public universities charge lower tuition rates than elite private universities, but have larger classes and employ less famous professors. Public healthcare providers charge lower treatment fees than private clinics, but have longer wait times and less luxurious facilities. Municipal golf courses charge lower membership or green fees than private country clubs, but are less well maintained and more crowded.

In the second case, the public school becomes a less attractive option for high-income students, who care relatively more about the quality of education and relatively less about its price—and these are precisely the students who enroll at the private school. Hence, the unfunded affordability program allows the private school to reduce quality, and thereby cut costs, without losing students to the public competitor.

After characterizing the private firm's best response, we investigate the distributional and welfare effects of these policy changes. Public policy affects the welfare of individuals across the income spectrum through two different channels: Lower-income individuals are directly affected by changes in the price and quality of the publicly provided variety, while higher-income individuals are affected indirectly via the private firm's strategic response. In the education example above, low-income individuals benefit under both policies, while individuals with sufficiently high incomes benefit in the first scenario (the fully funded affordability program) but are hurt in the second (the unfunded affordability program). It is then possible that *all* individuals—including those with very high incomes—are better off under an affordability program that is paid for by a progressive income tax, compared to an unfunded program that is implicitly paid for by a reduction in the public provider's quality. In particular, high-income individuals may prefer such a tax-financed alternative because it induces the private provider to keep its quality high.

Finally, we characterize an overall market equilibrium in a numerical instance of our model, assuming that the public firm is a Stackelberg leader and maximizes a weighted sum of consumer welfare and producer surplus. We first show that, if the price and quality of the public variety are sufficiently small, the private firm will indeed locate above the public firm in the price-quality spectrum—which is the sorting we assumed in our model. We then show that the public firm provides its variety at lower prices and lower quality levels, the stronger its redistributive concern (i.e., the smaller the substitution elasticity in the consumer welfare portion of its objective). Interestingly, including a concern for private profits induces the public firm to lower its price and quality even further.

The remainder of the paper is organized as follows. In Section 2, we relate our work to existing research on both price-quality competition and mixed markets. In Section 3 we present our formal model. Section 4–5 contain our theoretical results concerning the private firm's best response to changes in public policy: In Section 4 we develop the general comparative statics results necessary for our analysis, and in Section 5 we apply these results to a number of specific policy experiments. We then illustrate the distributional implications of these policies in Section 6. In Section 7, we examine an overall equilibrium in which the public firm acts as a Stackelberg leader and the private firm is free to locate anywhere on the price-quality spectrum. Section 8 concludes with a synthesis and discussion of our findings. Proofs that are too long to be included in the main text are in the Appendix.

2 Relation to the Literature

This paper is related to, and contributes to, two strands of previous literature. The first is in the area of industrial organization and concerns competition between profit-maximizing private firms on the price dimension as well as quality dimension. The second is in the area of public economics and concerns competition between public and private firms in so-called mixed markets. We now locate our work within each of these areas.

Literature on price-quality competition. A number of authors have examined how firms choose the quality of their products as a strategic variable (Shaked and Sutton 1982; Choi and Shin 1992; Motta 1993; Ferreira and Thisse 1996; Wauthy 1996; Lehmann-Grube 1997; Wang and Yang 2001; Garella and Petrakis 2008). One of the main findings of this literature is that endogenous quality differentiation divides consumers into market segments, and thereby relaxes the competitive pressure faced by each firm. With very few exceptions (to be discussed below), the papers in this literature model consumer heterogeneity as differences in a "taste for quality" parameter. The advantage of this approach is that it yields very tractable models of firm competition in quality. We depart from the assumption of taste heterogeneity and examine, instead, a framework with homogenous preferences but income inequality. This is technically more challenging, but provides the foundation for the analysis of distributional effects of public policies: Governments provide goods such as education and healthcare not because they are concerned about the welfare of consumers who have little taste for education or healthcare, but because they are concerned about the welfare of consumers who value these goods but cannot afford them (or can afford only an insufficient amount). To analyze policies that redistribute resources to those consumers, a model with income differences is necessary.

To our knowledge, only two early papers—Gabszewicz and Thisse (1979) and Shaked and Sutton (1982)—examine quality differentiation in models where consumers differ in incomes but not tastes. In the former, quality is exogenous and firms only choose prices, while in the latter quality is chosen by the firms, but is costless.[3] The main objective of both papers is to prove existence of an oligopolistic equilibrium in their respective settings; however, neither is concerned with characterizing the distributional implications of market outcomes. Our goal, on the other hand, is to investigate these

[3] A firm may prefer to produce a low-quality good, even if production of a high-quality good had the same cost, to separate itself from the high-quality competitor. This will allow the low-quality firm to retain some rents that would be competed away if all firms set the same quality.

distributional implications. We obtain a rich set of results characterizing the private firm's best response, and hence the welfare changes experienced by the private firm's customers. In order to prove these results, we develop a "duality method" that reformulates the private firm's problem, from one of choosing an optimal price-quantity pair into one of choosing a marginal income level above which consumers purchase from the private firm (see Section 4.1). Once reformulated in this way, many of the technical difficulties that prevented a fuller characterization of outcomes in Shaked and Sutton (1982) disappear. Thus, our paper also makes a methodological contribution, which we imagine could prove useful in further studies.

Literature on mixed markets. Our paper is also related to an extensive literature on mixed markets, which examines the interaction between private firms and public enterprises (De Fraja and Delbono 1990; Cremer *et al.* 1991; Barros 1995; Anderson *et al.* 1997; Matsumura 1998; Pal 1998; Bárcena-Ruiz 2007; Ishida and Matsushima 2009; Matsumura and Ogawa 2014; Lasram and Laussel (2019)). In this literature, the firms' products are typically modeled as either perfect or imperfect substitutes with a fixed demand function, which implies that quality is not an endogenous choice made by the firms.[4] In contrast, our model allows both the private firm and the public firm to set the quality of their products. More importantly, the mixed markets literature has in large part been motivated by the liberalization of industries that used to be comprised of state-owned monopolies, such as the telecommunications, railroad, or utilities industries. The goal in this context is to establish equal competition between former public monopolies and new private entrants, by eliminating subsidies and other advantages enjoyed by the public firms. This is not the case in the markets we consider. To the contrary, public providers of education, healthcare, and the like often enjoy significant financial subsidies that are not available to their private rivals, and providing such advantages is an explicit part of redistributive government policy.

3 Model and Preliminary Results

We now introduce our formal model of price-quality competition in a mixed duopoly. After setting up the basic market structure, we describe the choice problems faced by consumers, the private firm, and the public firm. A discussion of several of the modeling choices we make is provided at the end of this section.

[4]There are some exceptions. Ishibashi and Matsumura (2006) studies a patent race to investigate the role of quality-improving R&D competition between a public and a private firm; similarly, Matsumura and Sunada (2013) consider demand-increasing advertising competition in a mixed oligopoly.

3.1 The market

We study a market with two goods, X and Y. Good Y is a good such as education, healthcare, or recreation, while good X is a numeraire good ("everything else"). Every individual demands zero or one units of Y. The price of a unit of good X is one; thus, if an individual with income m purchases one unit of Y at price p she consumes $m-p$ units of X.

The quality of good Y is denoted by $\theta \geq 0$. An individual who consumes x units of good X and one unit of good Y of quality θ obtains utility $u(x,\theta)$. We assume that the utility function u is strictly quasi-concave, twice continuously differentiable, and satisfies the following properties for $(x,\theta) \gg (0,0)$: $u_x > 0$, $u_\theta > 0$, $u_{xx} \leq 0$, $u_{\theta\theta} \leq 0$, $u_{x\theta} > 0$. The last property ensures that higher-income consumers have a higher willingness to pay for quality.

Good Y is sold by two suppliers, a private firm and a public firm. The private firm's price and quality are denoted p_r and θ_r, and the public firm's price and quality are denoted p_b and θ_b. For now, we assume that $(p_r,\theta_r) \gg (p_b,\theta_b) \gg (0,0)$, and we refer to (p_r,θ_r) as the "premium variety" and to (p_b,θ_b) as the "basic variety." The prices and qualities of both suppliers will be made endogenous later. In addition to these varieties, consumers can always choose an outside option, $(0,0)$, which we interpret as not consuming good Y. The cost of producing one unit of Y at quality θ is $c(\theta)=\theta$ for both providers.[5]

Individuals differ in their incomes. We assume that incomes are distributed according to an atomless distribution F with convex support, density f, and strictly increasing hazard rate $\lambda(m) = f(m)/[1-F(m)]$. An individual cannot spend more than her income. Thus, high-income individuals, i.e., those with $m \geq p_r$, can buy either the basic or the premium variety of good Y. Middle-income individuals, i.e., those with $m \in [p_b, p_r)$, can only buy the basic variety. Individuals with incomes $m < p_b$ are unable to buy good Y.

3.2 The individual's choice

Each individual decides whether or not to purchase one unit of good Y and, if she purchases one unit, from which of the two providers. Depending on (p_b,θ_b) and (p_r,θ_r), one or both providers can have a zero market share. For example, if the private firm's price-quality combination is very unattractive compared to the price-quality combination offered by its public competitor, all individuals prefer to buy the basic variety (if they

[5]This functional form is without loss of generality. As long as $c'(\theta) > 0$ and $c''(\theta) \geq 0$, one can always adjust the utility function to translate the model into one where $c(\theta) = \theta$.

buy good Y at all). For the purpose of this paper, this is not an interesting case.[6] We want to focus on outcomes in which some individuals buy the basic variety, others buy the premium variety, and yet others do not buy good Y at all. In such situations, the two firms are engaged in meaningful competition; at the same time there is scope for government policy aimed at increasing the number of consumers who purchase good Y ("expanding access"). The following result states that, in this case, the market is segmented by income:

Lemma 1. *Suppose that a positive mass of individuals purchase the basic variety, a positive mass purchase the premium variety, and a positive mass do not purchase good Y. There exist thresholds $0 < \underline{m} < \overline{m}$ such that individuals with incomes $m < \underline{m}$ are the ones who do not consume Y, individuals with incomes $m \in [\underline{m}, \overline{m})$ are the ones who buy the basic variety, and individuals with incomes $m \geq \overline{m}$ are the one who buy the premium variety.*[7]

Note that this market segmentation result mirrors similar results in Gabszewicz and Thisse (1979) and Shaked and Sutton (1982), derived for the specific utility function $u(x, \theta) = x\theta$.

3.3 The private firm's problem

The private firm chooses quality θ_r and price p_r to maximize its profit,

$$\pi(\theta_r, p_r) \;=\; \big(1 - F(\overline{m})\big)\big(p_r - \theta_r\big). \tag{1}$$

The term $1 - F(\overline{m})$ is the number of units sold by the private firm, and the term $p_r - \theta_r$ is the per-unit profit margin. The two first-order conditions with respect to p_r and θ_r can be written as follows:

$$\frac{\partial \overline{m}}{\partial p_r}\big(p_r - \theta_r\big) \;=\; \frac{1}{\lambda(\overline{m})} \;=\; -\frac{\partial \overline{m}}{\partial \theta_r}\big(p_r - \theta_r\big). \tag{2}$$

It follows that, at the private firm's optimal price-quality combination,

$$\frac{\partial \overline{m}}{\partial p_r} \;=\; -\frac{\partial \overline{m}}{\partial \theta_r}. \tag{3}$$

[6]It also cannot be a long-run equilibrium of the market, as the private firm would either exit the market or would adjust its price and quality so as to gain a positive measure of consumers, and the same is true for a public firm that has no customers.

[7]The marginal individual of income $\underline{m}$ could also choose not to purchase good Y, and the marginal individual of income $\overline{m}$ could also purchase the basic variety. These changes would not affect the results.

If this condition did not hold, the private firm could change quality and price by identical amounts and gain customers (decrease $\overline{m}$) while leaving its profit margin unchanged. This would increase the firm's profit, which is impossible at the profit maximum.

Note that the income threshold $\overline{m}$, at which a consumer is indifferent between purchasing the basic or premium variety, is defined by the indifference condition

$$u(\overline{m} - p_b, \theta_b) \;=\; u(\overline{m} - p_r, \theta_r). \tag{4}$$

Differentiating the right-hand side of (4) with respect to p_r and θ_r, and treating $\overline{m}$ as a function of these variables, we have

$$\frac{\partial \overline{m}}{\partial p_r} \;=\; \frac{u_x(\overline{m} - p_r, \theta_r)}{u_x(\overline{m} - p_r, \theta_r) - u_x(\overline{m} - p_b, \theta_b)} \;>\; 0,$$

$$\frac{\partial \overline{m}}{\partial \theta_r} \;=\; \frac{u_\theta(\overline{m} - p_r, \theta_r)}{u_x(\overline{m} - p_b, \theta_b) - u_x(\overline{m} - p_r, \theta_r)} \;<\; 0,$$

and substituting these expressions back in (3) we obtain

$$u_x(\overline{m} - p_r, \theta_r) \;=\; u_\theta(\overline{m} - p_r, \theta_r). \tag{5}$$

That is, the marginal utility of the quantity of X equals the marginal utility of the quality of Y, for the individual located at the income threshold $\overline{m}$. This is also easy to interpret: If, for example, $u_\theta(\overline{m} - p_r, \theta_r)$ was larger than $u_x(\overline{m} - p_r, \theta_r)$, the private firm could raise its quality by some amount and its price by a larger amount, thereby increasing its profit margin while leaving the number of customers unchanged.

Condition (5) also implies that $u_x(m - p_r, \theta_r) < u_\theta(m - p_r, \theta_r)$ for all $m > \overline{m}$. Thus, all buyers of the premium variety (except the marginal buyer) would benefit from a higher quality at a higher price. This observation will play an important role in our analysis.

3.4 The public firm's problem

The public firm is an agent of the government that is charged with implementing the government's social welfare objective and given a budget to do so. Many of the questions we are interested in can be answered without specifying the precise welfare objective the government pursues. Instead, we can think of the public firm receiving a mandate to, say, make its good more affordable. Such a mandate may reflect an increased concern for redistribution in the government's social welfare function. However, in order to examine how the firm implements this mandate with a given budget, and how the corresponding changes in the public firm's price and quality affect the provision of the private variety, a specification of an underlying social welfare function is not needed. (This is the approach we take in Section 4–6 of this paper.)

On the other hand, to pin down actual equilibrium values of p_b, θ_b, p_r, and θ_r (which we do in Section 7), we have to specify an objective function for the public firm. Abstracting away from any potential principal-agent problems, we assume that this objective coincides with the government's social welfare objective, and is to maximize a weighted sum of consumer welfare and private firm profits. Specifically, let

$$U(m) \;=\; \begin{cases} u(m,0) & \text{if } m < \underline{m}, \\ u(m-p_b,\theta_b) & \text{if } \underline{m} \le m < \overline{m}, \\ u(m-p_r,\theta_r) & \text{if } m \ge \overline{m} \end{cases}$$

denote the indirect utility of an individual with income m. We then define the public firm's objective function as follows:

$$W \;=\; \underbrace{\left(\int_0^\infty U(m)^{1-\psi} dF(m)\right)^{\frac{1}{1-\psi}}}_{\substack{\text{Consumer}\\ \text{welfare}}} \;+\; \underbrace{\phi\pi(\theta_r,p_r)}_{\substack{\text{Private firm}\\ \text{profit}}}, \tag{6}$$

where $\psi \ge 0$ and $\phi \ge 0$. The first component in (6) is a constant-elasticity-of-substitution welfare function, where the value of the substitution elasticity $1/\psi$ reflects the firm's (or government's) redistributive concern. Note that $\psi = 0$ corresponds to a utilitarian consumer welfare function, which morphs into a Cobb-Douglas welfare function as $\psi \to 1$, and a Rawlsian welfare function as $\psi \to \infty$. The second component in (6) reflects a concern for the private firm's profit, where the weight $\phi \ge 0$ represents the degree to which the public firm (or government) cares about this profit relative to consumer welfare.[8]

The public firm chooses p_b and θ_b to maximize W, subject to the budget constraint

$$\big(F(\overline{m}) - F(\underline{m})\big)\big(\theta_b - p_b\big) \;\le\; B, \tag{7}$$

where $B \ge 0$ is a fixed subsidy the public firm receives. If the constraint binds we have $\theta_b \ge p_b$, that is, the public firm sets price (weakly) below cost. Note that (7) is a constraint on the firm's operating loss, with the parameter B reflecting, for example, a

[8] Adding the public firm's own profit into its objective function does not change the analysis, as the public firm faces a budget constraint that will generally be binding, and hence it always earns a constant profit. Furthermore, accounting for the excess burden of taxation used to finance the public firm does not change the analysis qualitatively.

block grant. Some public providers may instead face a constraint on its per-customer loss, in which case its budget constraint becomes $\theta_b - p_b \leq b$, where $b \geq 0$ is the per-customer subsidy. Our comparative results do not depend on the precise form of the budget constraint. However, when we derive the optimal public provision policy in Section 7 we use the budget constraint (7).

Finally, when choosing p_b and θ_b, the public firm anticipates the optimal choice of p_r and θ_r by the private competitor. In other words, we assume a Stackelberg model of competition, with the public firm being the leader and the private firm being the follower.

3.5 Remarks

Several of our modeling choices deserve a brief discussion. First, we assume that the private firm supplies the premium variety and the public firm the basic variety. This sorting is commonly observed in markets in which public entities participate to pursue social welfare goals, and the primary focus of our theoretical analysis is on the comparative statics of equilibria that have this property. In Section 7 we examine conditions under which this sorting arises in equilibrium. In particular, we show that, with a sufficiently strong redistributive concern in its objective function (i.e., a sufficiently large parameter ψ), the public firm chooses a price-quality pair for which the private best response is, in fact, a location "above" the public firm (i.e., a higher price and quality).

Second, we assume that the private and the public firm have the same cost function. This symmetry is a simplification, and we could assume instead that the cost of producing one unit of a given quality is strictly larger for the public firm than for the private firm, reflecting greater production efficiency under the profit maximizing incentive.[9] Our results in Sections 4–6 would be unaffected by this change, as they do not depend on the public firm's cost function. The results in Section 7 would not change qualitatively.

Third, we assume that the private firm maximizes profits. In reality, of course, many private firms are not-for-profit. For example, most private universities in the United States are not-for-profit institutions (and the few that are for-profit tend to operate at the lower end of the quality spectrum). Not-for-profit firms pursue certain objectives subject to the constraints that revenue must cover cost and that profits, should they arise, cannot be distributed to shareholders. Nevertheless, the behavior of such firms can resemble that of for-profit firms. For example, not-for-profit firms may maximize the difference between revenues and costs in order to pay for capital investments, build up their endowments, or achieve similar goals that are instrumental in the pursuit of

[9] For example, in a survey of empirical studies that compare the relative performance of private and public firms, Megginson and Netter (2001) conclude that private firms tend to operate more efficiently, and are more profitable, than their public counterparts.

the firm's long-term objective. Since our results do not depend on whether the private firm's profit is distributed to shareholders or retained within the organization, the model captures the behavior of such not-for-profit firms as well.

Fourth, we assume that the public firm moves first and the private firm second. It is beyond the scope of this paper to derive the public firm's Stackelberg leadership as the equilibrium of an endogenous timing game.[10] What our choice reflects is the idea that the public firm adjusts its price and quality in response to changes in policy objectives and fiscal constraints, which occur relatively infrequently. On the other hand, the private firm adjusts its price and quality in response to changes in its competitive environment, which includes the public firm, and such adjustments can happen comparatively quickly.

Fifth, we assume that there is exactly one private and one public firm, and that each firm supplies exactly one variety. Since consumers are heterogeneous in their incomes, and hence in their willingness to pay for quality, new niche firms could enter profitably.[11] Alternatively, existing firms could increase their profits if they offered additional varieties. We must sidestep these issues in order to keep our model tractable.

Lastly, we point out that our model treats quality as an explicit choice variable. In some applications, quality may instead be an outcome that is indirectly determined through other choices made by firms. For example, consider school competition in the presence of peer effects, i.e., the quality of a school depends on the average ability of its students. Schools set tuition, financial aid, etc., while student composition (and hence school quality) is determined in equilibrium via the sorting of students to schools.[12] Again in the interest of tractability, we do not consider these channels, and instead assume that each firm chooses both its price and its quality directly.

4 The Private Firm's Best Response

In this section we characterize the profit maximizing price-quality choice of the private firm and examine how this choice adjusts to changes in the public variety. The section is organized as follows. First, in Section 4.1, we transform the private firm's problem into a dual problem. This dual problem closely resembles a conventional monopolistic pricing problem in which the firm chooses a location on a demand curve. Changes in the public variety alter this demand curve, so that studying the private firm's response

[10] This topic has received some attention in the mixed oligopoly literature; see, e.g., Pal (1998), Ino and Matsumura (2010), Amir and De Feo (2014).

[11] This depends, however, on the particular assumptions made on the timing of price and quality choices; see, e.g., Shaked and Sutton (1982).

[12] Papers that examine such indirect quality choices include Epple and Romano (1998), Epple *et al.* (2006), and De Fraja and Landeras (2006).

to a policy change becomes a problem of examining a monopolist re-optimizing after a demand change. We then use this dual formulation to characterize the private firm's best response to two kinds of changes. The first type is a direct change in the public firm's price or quality and is analyzed in Section 4.2. The second type is a change that affects the distribution of consumers' disposable incomes (such as a change in the income tax rate) and is analyzed in Section 4.3.

4.1 Dual formulation

Fix a given public variety (p_b, θ_b). Also fix a marginal individual $\overline{m}$ and suppose the private firm serves all individuals with incomes $\overline{m}$ and above. The (p_r, θ_r)-pair that maximizes the firm's profit, conditional on selling to individuals with incomes $m \geq \overline{m}$, satisfies the following two conditions.

First, individual $\overline{m}$ is indifferent between purchasing the public and private variety. This is condition (4), which we rewrite as

$$I(p_r, \theta_r | \overline{m}) \;\equiv\; u(\overline{m} - p_r, \theta_r) - u(\overline{m} - p_b, \theta_b) \;=\; 0.$$

The locus $I(\cdot|\overline{m}) = 0$ is the indifference curve consisting of all (p_r, θ_r)-pairs for which individuals $\overline{m}$'s utility is constant and equal to $u(\overline{m} - p_b, \theta_b)$. Second, for individual $\overline{m}$ the marginal utilities of X and the quality of Y are equal. This is condition (5), which we rewrite as

$$H(p_r, \theta_r | \overline{m}) \;\equiv\; \frac{u_x(\overline{m} - p_r, \theta_r)}{u_\theta(\overline{m} - p_r, \theta_r)} \;=\; 1.$$

The locus $H(\cdot|\overline{m}) = 1$ is individual $\overline{m}$'s "iso-MRS curve." In Figure 1, the left graph plots both curves for a fixed public variety $A = (p_b, \theta_b)$ and a marginal individual with income $\overline{m}_1$. Note that our assumptions on u imply an upward sloping and convex indifference curve and a downward sloping iso-MRS curve. The intersection of these curves lies at the point B, meaning that the point B maximizes the private firm's profit conditional on a marginal buyer with income $\overline{m}_1$.

Now construct a new intersection using a different marginal individual, with income $\overline{m}_2 > \overline{m}_1$, say. The new iso-MRS curve, $H(\cdot|\overline{m}_2) = 1$, is a copy of the previous curve shifted to the right by exactly $\overline{m}_2 - \overline{m}_1$. The new indifference curve, $I(\cdot|\overline{m}_2) = 0$, is a "stretched out" copy of the previous curve, with the horizontal distance between the two curves always being less than $\overline{m}_2 - \overline{m}_1$.[13] Thus, the intersection of the new indifference

[13] Suppose the curve stayed in place. Then individual $\overline{m}_2$ would strictly prefer the points on that curve to A (see Lemma 1). Therefore, to make him indifferent, p_r must increase given θ_r. On the other hand, if the curve moved to the right by exactly $\overline{m}_2 - \overline{m}_1$, the utility level associated with the new curve would be exactly the same as that associated with the previous one, which is $u(\overline{m}_1 - p_b, \theta_b)$. Since this is less than

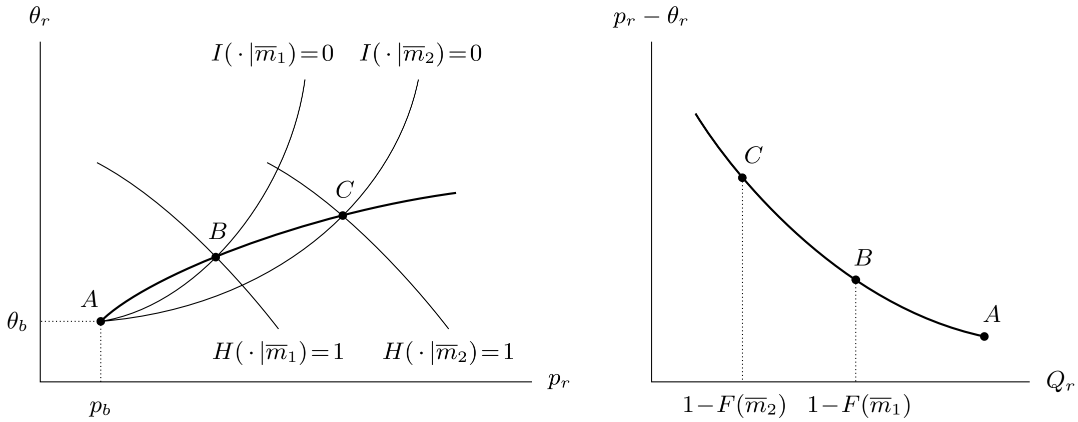

Figure 1: The private firm's price-quality locus (left) and resulting demand curve (right).

curve and the new iso-MRS curve, C, lies to the right of, and above, the previous intersection, B.

Repeating these steps for other values of $\overline{m}$, one can trace out an upward sloping curve containing all (p_r, θ_r)-pairs that can be constructed in this fashion. In the diagram, this is the thick curve passing through B and C. We shall call this curve the firm's price-quality locus. Each point on the locus is a profit-maximizing private variety for some value of $\overline{m}$. Because the slope of each indifference curve is exactly one at that intersection, and the price-quality locus crosses the indifference curves from above, it must have a slope strictly between zero and one. Thus, as one moves along the locus from A toward C, the variables $\overline{m}$, p_r, and θ_r all increase, as does the difference $p_r - \theta_r$.

In a final step, we can translate the price-quality locus into a "demand curve" that plots the quantity of the private firm, $Q_r = 1 - F(\overline{m})$ against its net price, or profit margin, $p_r - \theta_r$. This demand curve is depicted in the right diagram of Figure 1. Similar to a monopolist choosing a price-quantity pair from a given demand curve, we can think of the private firm as choosing a combination of a quantity and profit margin from the curve shown in the right diagram of Figure 1. The total profit the private firm earns is the area of rectangle between the origin and the chosen point on the curve, and the firm selects the point at which this area is maximized.

$u(\overline{m}_2 - p_b, \theta_b)$, individual $\overline{m}_2$ would strictly prefer A to points on the new curve. Therefore, to make him indifferent, the increase in p_r must be less than $\overline{m}_2 - \overline{m}_1$, given θ_r. It follows that the new indifference curve lies to the right of the old one, with a horizontal distance between the two curves that is always less than $\overline{m}_2 - \overline{m}_1$.

4.2 Effects of changes in p_b and θ_b

Suppose the public firm changes its variety from (p_b, θ_b) to $(\hat{p}_b, \hat{\theta}_b)$. Denote by

$$\Delta(m) \equiv u(m - \hat{p}_b, \hat{\theta}_b) - u(m - p_b, \theta_b) \tag{8}$$

the resulting utility change for an income-m consumer when purchasing the public variety. Pick a point on the private firm's price-quality locus, S, say. Let this point be associated with marginal individual $\overline{m}$ and assume that this individual is made worse off by the change (i.e., $\Delta(\overline{m}) < 0$). Note that the change in the public variety does not affect the iso-MRS curve $H(\cdot|\overline{m}) = 1$, as this curve does not depend on p_b or θ_b. However, it does affect the indifference curve $I(\cdot|\overline{m}) = 0$: Since $\Delta(\overline{m}) < 0$, the new indifference curve is associated with a lower utility than the original curve, and is thus located below and to the right of the original curve. Thus, the policy change moves point S along the iso-MRS curve in a south-easterly direction to S', as depicted in the left diagram in Figure 2. By the same logic, had we assumed that $\Delta(\overline{m}) > 0$ point S' would be located to the north-west of S.

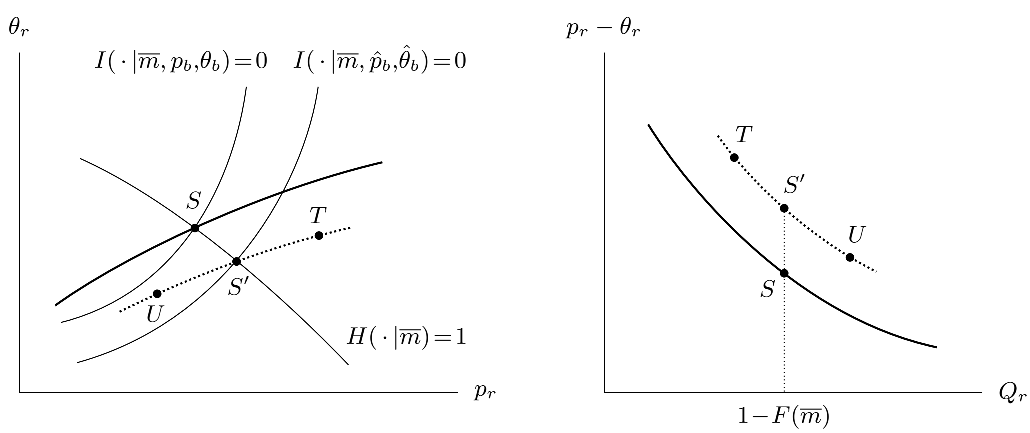

Figure 2: Local effects of changes in (p_b, θ_b) on the private firm's price-quality locus (left) and the demand curve (right).

If the private firm did not change its output, the above argument implies that the firms should increase p_r and decrease θ_r if $\Delta(\overline{m}) < 0$; and that it should decrease p_r and increase θ_r if $\Delta(\overline{m}) > 0$. This is intuitive: If $\Delta(\overline{m}) < 0$, the public variety becomes a less attractive substitute for the marginal individual at the public-private threshold, thus reducing the competitive pressure faced by the private firm. It can hence increase its price and reduce its quality without losing customers. (In fact, if it did not increase price or reduced quality the firm would attract new customers with incomes just below the original cutoff $\overline{m}$.) The opposite holds if $\Delta(\overline{m}) > 0$.

Of course, following a change in the public firm's price and quality the private firm will generally adjust its output, by selecting a different marginal consumer $\overline{m}$. In Figure 2, this corresponds to a move along the new price-quality locus starting at S', either to north-east (increasing $\overline{m}$/decreasing Q_r) or to the south-west (decreasing $\overline{m}$/increasing Q_r). If the private firm reduces its output, e.g., by moving from S' to T, its price at the new optimum T must be higher than at the original optimum S, as T is unambiguously to the right of S. However, the effect on quality is ambiguous—it depends on whether the downward move from S to S' is outweighed by the upward move from S' to T. Conversely, if the private firm increases its output by moving from S' to U, its quality at the new optimum U is unambiguously lower than at the original optimum S, while the price effect is ambiguous.

Similar arguments can be made if $\Delta(\overline{m}) > 0$. Thus, we obtain a total of four cases, depending on (i) whether the change in the public variety makes the private firm's marginal consumer better or worse off, and (ii) whether the private firm expands or contracts in response. In each case, we can determine the direction of the total change in either the private firm's price or its quality, but not both. The four cases are summarized in the following result:

Lemma 2. *Let $\overline{m}$ be the income threshold at which a consumer is indifferent between the private and the public variety. Now consider a change in the public variety from (p_b, θ_b) to $(\hat{p}_b, \hat{\theta}_b)$ and define $\Delta(m)$ as in (8).*

(a) If $\Delta(\overline{m}) < 0$ the following holds: If the private firm reduces its output, p_r increases while the effect on θ_r is uncertain. If the private firm expands its output, θ_r decreases while the effect on p_r is uncertain.

(b) If $\Delta(\overline{m}) > 0$ the following holds: If the private firm reduces its output, θ_r increases while the effect on p_r is uncertain. If the private firm expands its output, p_r decreases while the effect on θ_r is uncertain.

Lemma 2 places some constraints on the adjustments of the private firm's price, quality, and quantity, following a change in the public variety. To say more, however, we need to examine the firm's quantity response in more detail.

Consider again the case $\Delta(\overline{m}) < 0$. As discussed already, if the private firm were to maintain its quantity it would increase its price and reduce its quality, resulting in a higher profit margin and thus an upward shift of the firm's demand curve at the current quantity $Q_r = 1 - F(\overline{m})$. This is illustrated in the right diagram in Figure 2. However, whether the firm responds to this change with an increase or decrease in output depends not on whether demand has increased or decreased, but on whether demand has become more or less elastic. If it is more elastic after the change—as would be the case, for example, if the demand curve simply shifted upward with no change to its slope—the firm will expand; Lemma 2 then implies that θ_r must be lower at the firm's new optimum. Similarly, if elasticity decreases the firm will reduce its output and, therefore, increase p_r.

It is difficult to determine the effect of changes in p_b and θ_b on the elasticity of the private firm's demand in general. For the special case of homothetic preferences (i.e., those that can be represented by a utility function that is homogeneous of degree 1), we can state the following:

Lemma 3. *Suppose the consumer's preferences are homothetic. Let $\overline{m}$ be the income threshold at which a consumer is indifferent between the private and the public variety. Now consider a change in the public variety from (p_b, θ_b) to $(\hat{p}_b, \hat{\theta}_b)$ and define $\Delta(m)$ as in (8), using a utility function that is homogeneous of degree 1 (which exists in the homothetic case).*

(a) If $\Delta'(\overline{m}) > \lambda(\overline{m})\Delta(\overline{m})$, the private firm's demand becomes more elastic at $1 - F(\overline{m})$ after the policy change, resulting in a higher quantity supplied by the private firm.

(b) If $\Delta'(\overline{m}) < \lambda(\overline{m})\Delta(\overline{m})$, the private firm's demand becomes less elastic at $1 - F(\overline{m})$ after the policy change, resulting in a lower quantity supplied by the private firm.

Combining the conditional price and quality adjustments listed Lemma 2 with the quantity adjustments listed in Lemma 3, we get the following result:

Proposition 4. *Suppose the consumer's preferences are homothetic. Let $\overline{m}$ be the income threshold at which a consumer is indifferent between the private and the public variety. Now consider a change in the public variety from (p_b, θ_b) to $(\hat{p}_b, \hat{\theta}_b)$ and define $\Delta(m)$ as in (8), using a utility function that is homogeneous of degree 1. The following table characterizes the private firm's best response to the change:*

	$\Delta'(\overline{m}) > \lambda(\overline{m})\Delta(\overline{m})$:	$\Delta'(\overline{m}) < \lambda(\overline{m})\Delta(\overline{m})$:
$\Delta(\overline{m}) < 0$:	*(a)* $Q_r \nearrow$, $\theta_r \searrow$	*(b)* $Q_r \searrow$, $p_r \nearrow$
$\Delta(\overline{m}) > 0$:	*(c)* $Q_r \nearrow$, $p_r \searrow$	*(d)* $Q_r \searrow$, $\theta_r \nearrow$

In Sections 5.1–5.3, we will apply this result to examine the effects of various policies aimed at making the publicly provided good more affordable, or at increasing its quality.

4.3 Effects of changes in the income distribution

In the preceding Section 4.2, we examined how the private firm adjusts its price, quality, and output in response to changes in the public firm's price and quality, holding constant the income distribution. We can also examine how the private firm responds to changes in the income distribution, holding constant the public firm's price and quality. We will compare income distributions that can be ranked as follows:

Definition 1. Income distribution $\hat{F}$ is *less dispersed* than income distribution F if there exists a non-decreasing function $\sigma : [0, \infty) \to \mathbb{R}$ such that $\hat{F}(m) = F(m + \sigma(m))$ for all m in the support of $\hat{F}$.

For example, one income distribution is less dispersed than another if it results from applying a strictly increasing contraction to individual incomes.[14] In the left side of Figure 3, for example, the incomes on the vertical axis are less dispersed than the incomes on the horizontal axis; the corresponding contraction is the solid curve that crosses the 45°-line from above.

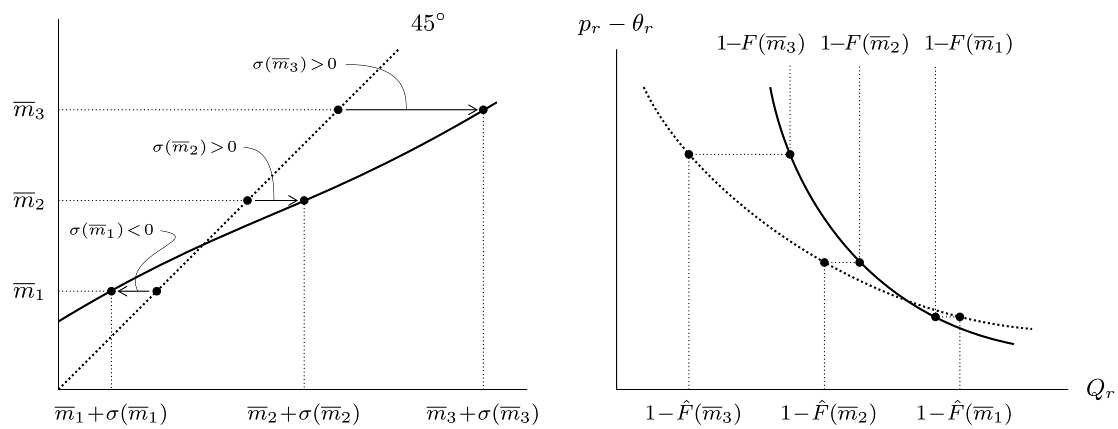

Figure 3: Dispersion of the income distribution (left) and its effect on the private firm's the demand curve (right).

Everything else equal, a change of the income distribution does not alter the private firm's price-quality locus, because this locus is independent of F. However, the translation

[14]More precisely, if $\hat{F}$ is less dispersed than F according to Definition 1, the mapping from $m + \sigma(m)$ to m is a strictly increasing and *non-expansive* mapping on $\mathbb{R}_+$. If in addition $\sigma'(m) > 0$ for all m, the mapping is a contraction.

of the price-quality locus into the demand curve is affected. Recall that for each profit margin $p_r - \theta_r$ and associated threshold income $\overline{m}$, the firm sells a quantity of $1 - F(\overline{m})$ units. If F is replaced by a less dispersed distribution $\hat{F}$, the firm's demand curve in Figure 1 flattens. This is illustrated on the right side of Figure 3, where the steeper (solid) curve is the demand curve arising from the more dispersed income distribution, and the flatter (dotted) curve is the demand curve arising from the less dispersed income distribution.

The following result provides a condition under which this flattening of the demand curve leads the private firm to reduce both its price and its quality:

Proposition 5. *Let $\overline{m}$ be the income threshold at which a consumer is indifferent between the private and the public variety. Now consider a change in the income distribution from F to $\hat{F}$, where $\hat{F}$ is less dispersed than F. Further, suppose that $\hat{F}(\overline{m}) > F(\overline{m})$. Then, as a result of the change in the income distribution, the private firm reduces both its price p_r and its quality θ_r.*

Note that any income tax schedule with non-negative marginal tax rates induces a mapping from gross to net incomes that is a contraction, and hence results in a distribution of net incomes that is less dispersed than the distribution of gross incomes. Knowledge of how the private firm adjusts its price and quality to such a change is therefore important when we evaluate the effects of using income taxation to finance subsidies paid to the public firm. In Section 5.4 below, we will apply Proposition 5 to examine the effects of using an income tax to finance the public firm's operating loss. In particular, we will show that the condition $\hat{F}(\overline{m}) > F(\overline{m})$ in Proposition 5 is equivalent to the condition that the individual at the public-private income threshold $\overline{m}$ faces a strictly positive tax liability (instead of having to pay no taxes or being a recipient of transfers).

5 Policy Applications

The results developed in Section 4 permit insights into the effects of several policies that affect the way the public firm operates. These changes may stem from two sources. The first is a shift in the policy maker's objectives. For example, Megginson and Netter (2001) note that "...government objectives can change from one administration to the next," and such changes are relayed to public agencies in the form of directives and new mandates. The second source is a change in the policy maker's constraints. For example, a reduction in general tax revenue during a recession may force the policy maker to cut the subsidies to some public firms.

Whatever their source, in our model these changes will result in adjustments to the price and/or quality of the public variety. For instance, a public university that is faced with a mandate to enroll more low-income students, but does not receive additional funding, will need to lower its tuition and simultaneously reduce quality to offset the reduced subsidy per student (e.g., by increasing class size or reducing expenditures on facilities). Similarly, a public healthcare provider whose subsidy is cut, but that is required by law to provide a certain level of care, will have no choice but to increase the price for its services.[15] We now use the results developed in the previous section to examine the private firm's response to such policies.

5.1 Price variations while maintaining quality

Many government agencies have an expressly stated objective of providing affordable access to the goods they provide. If this objective becomes more important relative to other goals, the policy maker may direct the public firm to reduce its price. Suppose that the public firm is fully compensated for any revenue lost, so that its quality can stay the same as before. We call this policy an *externally funded affordability program*.[16] In our price-quality diagram, such a policy corresponds to a horizontal move of the point (p_b, θ_b) to the left.

Similarly, we can envision a scenario in which the price of the public variety increases while its quality stays constant. This scenario reflects a policy maker who is forced to cut a subsidy paid to the public firm (e.g., as a result of lower government revenue during a recession) and at the same time wants to maintain, or must maintain, a certain quality standard (e.g., a standard required by law). In our price-quality diagram, this corresponds to a horizontal move of the point (p_b, θ_b) to the right.

To see how the private firm reacts to these changes, consider the first scenario (the price reduction). Because $u_x > 0$, the price reduction makes all individuals better off if they purchase the public variety; thus, we have $\Delta(m) > 0$ for all m. Furthermore, because $u_{xx} \leq 0$, the utility gain is (weakly) smaller for individuals with higher incomes; thus, $\Delta'(m) \leq 0$ for all m. Combining these two observations puts us in case (d) of Proposition 4, and the private firm's response is a decrease in output Q_r together with an increase in quality θ_r. In the second scenario, the arguments are exactly reversed, and

[15] Note that these adjustments arise because the public firm's budget constraint (7) must be satisfied. A similar adjustment occurs in Ishida and Matsushima (2009), where a change in the public sector wage necessitates adjustments in the firm's price and quantity.

[16] The reason for choosing this term is that the subsidy reflects an inflow of funds into the industry under consideration; hence, the price reduction is financed "outside of the model."

case (a) of Proposition 4 applies: The private firm responds with an increase of Q_r and a decrease of θ_r. These market adjustments are summarized in the following result:

Proposition 6. *(Externally funded affordability program/cutting subsidies under a fixed quality standard) Assume that the consumers' preferences are homothetic. Consider a government policy that decreases/increases the public firm's price but does not change the public firm's quality. As a result of the policy, the public firm's quantity increases/decreases, while the private firm's quantity decreases/increases and its quality increases/decreases.*

Note that the two firms' quantities move in opposite directions. For example, the externally funded affordability program results in an increase in the public supply of good Y that is accompanied by a decrease in the private supply, thereby generating a crowding-out effect.

5.2 Quality variations while maintaining price

Governments may care about not only the affordability but also the quality of the goods they provide. Thus, we now consider a policy that increases the quality of the public variety, which necessarily raises the public firm's cost per unit. Suppose that the public firm is fully compensated for this cost increase, so that its price can stay the same as before. In our price-quality diagram, this *externally funded quality improvement* corresponds to an upward move of the point (p_b, θ_b).

By reversing this scenario, we can describe a situation in which the public firm's quality decreases while its price remains constant. Such a situation may arise when the policy maker reduces a subsidy paid to the public firm (e.g., as a result of lower government revenue during a recession) while, at the same time, raising the price of the publicly provided good is not an option (e.g., it may be politically infeasible because of a previous election promise to not increase tuition of a public university, or to not raise public health insurance premiums). In this case, the public firm will be forced to reduce its quality in order to save costs, resulting in a downward move of the point (p_b, θ_b) in price-quality space.

Consider the first scenario (the quality increase). Since $u_\theta > 0$, the quality increase makes all individuals better off if they purchase the public variety. Thus, we still have $\Delta(m) > 0$ for all m, as was true with the externally funded price reduction examined in Section 5.1. At the same time, since $u_{x\theta} > 0$, the quality increase benefits richer individuals relatively more. Therefore, and unlike in the price reduction scenario, we have $\Delta'(m) > 0$ for all m. Without further assumptions either case (c) or (d) in Proposition 4 could apply. If the policy change is a quality decrease of the public varity the arguments are reversed, putting us either in case (a) or (b).

If we assume a Cobb-Douglas utility function and a uniform income distribution, we can show that the relevant cases in Proposition 4 are (d) and (a), respectively. This leads to the following result:

Proposition 7. *(Externally funded quality improvements/cutting subsidies under a fixed affordability standard) Assume that the consumers' preferences can be represented by a Cobb-Douglas utility function and that incomes are uniformly distributed. Consider a government policy that increases/decreases the public firm's quality but does not change the public firm's price. As a result of the policy, the public firm's quantity increases/decreases, while the private firm's quantity decreases/increases and its quality increases/decreases.*

Thus, the private firm's responses are similar to those analyzed in Section 5.1: An externally funded quality improvement policy reduces the private firm's output and increases its quality, while a subsidy cut under a fixed affordability standard has the reverse effect. Furthermore, since the two firms' quantities move in opposite directions—a larger public output is associated with a smaller private output, and vice versa—a crowding-out effect emerges again.

5.3 Simultaneous variations in price and quality

Next, consider a scenario in which the public firm must either operate on a cost-recovery basis (i.e., $B = 0$), or receives a per-customer subsidy $b \geq 0$ for its operation. Suppose the government mandates that the publicly provided good become more affordable, but does not provide an additional subsidy to compensate the public firm for its lost revenue. This means that the public provider must cut costs by the same amount as it cuts revenue. This cost reduction can only be achieved by lowering quality by an amount equal to the price cut, resulting in a diagonal move of the point (p_b, θ_b) downward and to the left. The same scenario in reverse describes a situation in which the government mandates that the public firm increase its quality, but provides no additional subsidy to offset the increased cost of public provision. In this case, the public firm must increase its price, so that we have a diagonal move of the point (p_b, θ_b) upward and to the right

Unlike in the previous two cases, where only one of the two variables p_b and θ_b changed, the effect of a simultaneous price and quality change on the public firm's customers will not be uniform. To see this, consider a reduction in the price and quality of the public variety: Low-income individuals, whose marginal utility of income is higher than their marginal utility of quality, benefit from the change; the opposite holds for high-income individuals, who are hurt by the change. The individual who matters for our results is the marginal individual with income $\overline{m}$, and this individual is made worse off by the change if he purchases the public variety. We know this because of the private firm's profit maximizing behavior: Recall from condition (5) that the private firm sets price and quality so that the marginal utility of both goods is equal for an income-$\overline{m}$ consumer; that is, $u_x(\overline{m} - p_r, \theta_r) = u_\theta(\overline{m} - p_r, \theta_r)$. Since $u_{xx} \leq 0$, $u_{\theta\theta} \leq 0$, $u_{x\theta} > 0$, and $(p_b, \theta_b) \ll (p_r, \theta_r)$, we must have $u_x(\overline{m} - p_b, \theta_b) < u_\theta(\overline{m} - p_b, \theta_b)$. Thus, a decrease of p_b and θ_b by the same amount must decrease $u(\overline{m} - p_b, \theta_b)$, so that $\Delta(\overline{m}) < 0$.

At the same time, the utility loss is even larger for consumers with income $m > \overline{m}$, who care relatively more about quality and relatively less about price. This implies that $\Delta'(\overline{m}) < 0$, and without further assumptions either case (a) or (b) of Proposition 4 could apply. Similarly, if the policy change is a simultaneous increase in the price and quality of the public varity, the arguments are reversed, putting us either in case (c) or (d).

As in the previous section, if we assume a Cobb-Douglas utility function and a uniform income distribution, we can say more. In particular, the relevant cases in Proposition 4 are (a) and (d), respectively. This leads to the following result:

Proposition 8. *(Unfunded affordability programs/unfunded quality improvements) Assume that the consumers' preferences can be represented by a Cobb-Douglas utility function and that incomes are uniformly distributed. Consider a government policy that decreases/increases the public firm's price and quality by the same amount. As a result of this policy, the private firm's quantity increases/decreases and its quality decreases/increases.*

Note that the private firm's response to an unfunded affordability program is of the opposite direction as its response to an externally funded affordability program, which we examined in Section 5.1. However, we can no longer determine whether the public firm's output increases or decreases. If it increases, the two firms adjust their quantities in the same directions, so that the unfunded affordability program generates a crowding-in effect. An example of such a co-movement of Q_b and Q_r is given in Section 6. On the other hand, the private firm's response to an unfunded quality improvement is now of the same direction as its response to a fully funded quality improvement, described in Section 5.2.

5.4 Funding the public firm through income taxation

In our final policy experiment, we replace an external subsidy to the public firm by an internally funded one, that is, a subsidy financed by taxing the consumers *in the market*. Such a scenario might arise in situations where a government that historically relied on resource revenue to fund its operations is confronted with a negative resource price shock. If this government wants to maintain the quality of the goods it provides without raising the price it charges for these goods, it must find other sources of revenue, and income taxation is one possibility.[17] Unlike the previous policies we examined, this policy does not change the public firm's price or quality but instead changes the distribution of disposable income.

Consider the case where the public firm incurs an operating loss—that is, $p_b < \theta_b$—and hence requires a government subsidy. Further, consider an income tax schedule $T : [0, \infty) \to \mathbb{R}$, where $T(m)$ is the tax paid by income-m individuals (if $T(m) < 0$, income-m individuals receive a transfer). T is continuous and weakly increasing, and satisfies $T'(m) < 1$ where differentiable (i.e., marginal tax rates are always below 100%). After an individual with gross income m has paid his tax $T(m)$, he allocates his net income $m - T(m)$ to goods X and Y. We say that the tax schedule T *finances the public firm* if there exist $\underline{m}$, $\overline{m}$, p_r, θ_r such that

$$u(\overline{m} - T(\overline{m}) - p_b, \theta_b) = u(\overline{m} - T(\overline{m}) - p_r, \theta_r), \tag{9}$$

$$u(\underline{m} - T(\underline{m}) - p_b, \theta_b) = u(\underline{m} - T(\underline{m}), 0), \tag{10}$$

$$\frac{1}{\lambda(\overline{m})(p_r - \theta_r)} = \frac{u_x(\overline{m} - T(\overline{m}) - p_r, \theta_r)}{u_x(\overline{m} - T(\overline{m}) - p_r, \theta_r) - u_x(\overline{m} - T(\overline{m}) - p_b, \theta_b)}, \tag{11}$$

$$\frac{1}{\lambda(\overline{m})(p_r - \theta_r)} = \frac{u_\theta(\overline{m} - T(\overline{m}) - p_r, \theta_r)}{u_x(\overline{m} - T(\overline{m}) - p_b, \theta_b) - u_x(\overline{m} - T(\overline{m}) - p_r, \theta_r)}, \tag{12}$$

$$(\theta_b - p_b)\big(F(\overline{m}) - F(\underline{m})\big) = \int_0^\infty T(m)dF(m). \tag{13}$$

(9)–(10) are the indifference conditions that define the thresholds $\underline{m}$ and $\overline{m}$, in terms of the marginal individual's after-tax incomes. (11)–(12) are the private firm's profit-maximizing conditions, also taking into account the altered income distribution after taxation. (13) states that the tax revenue must cover the public firm's operating loss.

Since $T'(m) < 1$, the proportion of individuals in the population with net income less than or equal to $m - T(m)$ is the same as the proportion of individual with gross incomes less than or equal to m. Thus, it can be expressed as $\hat{F}(m - T(m)) = F(m)$, where F is the gross income distribution and $\hat{F}$ is the net income distribution. Moreover, since T is non-decreasing, $\hat{F}$ is a less dispersed income distribution than F according to Definition 1. This property allow us (via Proposition 5) to compare the private firm's price and quality under the tax scenario to an alternative scenario in which the subsidy is paid for externally:

[17] An example of such a jurisdiction is the the province of Alberta, Canada, which reacted to a decrease in oil prices in 2014 by raising personal income taxes.

Proposition 9. *Fix (p_b, θ_b) with $p_b < \theta_b$, so that the public firm makes an operating loss. Consider two scenarios to pay for this operating loss:*

(S) *An external subsidy. — For this case, let (p_r^S, θ_r^S) denote the private firm's price and quality, and $\overline{m}^S$ the income of the individual who is indifferent between the public and private variety.*

(T) *A tax schedule T that finances the public firm. — For this case, let (p_r^T, θ_r^T) denote the private firm's price and quality.*

If individuals with income $\overline{m}^S$ pay a positive tax in the tax scenario (i.e., if $T(\overline{m}^S) > 0$), then $p_r^T < p_r^S$ and $\theta_r^T < \theta_r^S$.

To stick with the aforementioned example of a resource price shock, Proposition 9 states the following: If the government institutes a tax schedule that raises exactly the revenue required to keep the public firm's price and quality the same as before, the private competitor reacts to this policy by reducing both its price and its quality.[18]

6 Distributional Implications

We now illustrate the distributional effects of the policy changes we discussed so far. We computationally explore how individuals across the income spectrum are affected when the price and quality of the publicly provided variety change and the private firm responds optimally.

We assume that incomes are uniformly distributed on $[0, 1]$. We further assume Cobb-Douglas preferences represented by $u(x, \theta) = x\theta$, which is the utility function used in Shaked and Sutton (1982). In this case, the income thresholds $\underline{m}$ and $\overline{m}$ are given by

$$\underline{m} = p_b \quad \text{and} \quad \overline{m} = \frac{p_r\theta_r - p_b\theta_b}{\theta_r - \theta_b}.$$

We consider a baseline policy scenario in which the public variety is set at $(p_b, \theta_b) = (0.15, 0.15)$. This baseline scenario is shown in the first row of Table 1. The private firm responds by setting its variety to $(p_r, \theta_r) = (0.428, 0.293)$, and the marginal individual who is indifferent between the private and public variety has income $\overline{m} = 0.721$. This means that the private firm sells to the individuals in the top 27.9% of the income distribution, the public firm sells to the middle 57.1%, and the bottom 15% cannot afford

[18]Note that Proposition 9 does not say anything about the private firm's response to a policy that replaces an *unfunded* government program with a tax-funded one. Such a policy would simultaneously affect the public firm's price or quality *and* the distribution of disposable incomes. We do not have an analytical result for this case; however, in Section 6 we explore this question computationally.

the good. Because the public firm's price equals its quality it does not require a subsidy (i.e., $B = 0$).

Policy	p_b	θ_b	p_r	θ_r	$\underline{m}$	$\overline{m}$	Q_b	Q_r	B	τ
Baseline scenario	.150	.150	.428	.293	.150	.721	.571	.279	0	0
(a) Affordability mandate										
1. No funding	.075	.075	.434	.206	.075	.640	.565	.360	0	0
2. External funding	.075	.105	.447	.323	.075	.770	.695	.230	.052	0
3. Tax funding	.075	.150	.398	.308	.075	.797	.722	.203	.054	.127
(b) Quality mandate										
1. No funding	.225	.225	.434	.357	.225	.790	.565	.210	0	0
2. External funding	.150	.225	.460	.400	.150	.860	.710	.140	.053	0
3. Tax funding	.150	.225	.412	.380	.150	.910	.760	.090	.057	.158

Table 1: Numerical examples for $u(x, \theta) = x\theta$ and $m \sim U[0, 1]$.

6.1 Improving affordability

Consider now a policy initiative intended to reduce the size of the unserved population, by lowering the price of the public variety to $p_b = 0.075$. We examine three ways of achieving this goal: (1) An unfunded price reduction, which requires a reduction in quality to $\theta_b = 0.075$; (2) a fully funded price reduction without a change in quality, financed through an externally provided subsidy $B > 0$; and (3) a price reduction that leaves quality unchanged and requires a subsidy B funded through an income tax rate τ, with income up to 0.075 exempt.[19] Panel (a) in Table 1 contains these three scenarios.

Relative to the baseline scenario the private firm's quality decreases and its quantity increases in case 1 (see Proposition 8). The opposite is true in case 2 (Proposition 6). In case 3, we apply a tax rate of $\tau = 12.7\%$ to all incomes over 0.075. Given the resulting after-tax income distribution and public variety $(p_b, \theta_b) = (0.075, 0.15)$, the private firm raises quality, cuts price, and reduces quantity relative to the baseline scenario. In the resulting market outcome the public firm incurs an operating loss of $Q_b(\theta_b - p_b) = 0.054 = B$. Relative to case 2, which has the same (p_b, θ_b) but without the tax, the private firm sets a lower price and quality (Proposition 9).

[19]That is, the tax paid by an income-m individual is $T(m) = \max\{0, \tau(m - 0.075)\}$ and total tax revenue raised is $R = \int_{0.075}^{1} \tau(m - 0.075)dm = 0.4278\tau$.

Panel (a) of Figure 4 plots the indirect utility of individuals in the baseline scenario as well as the three affordability scenarios. In our example, an externally funded price reduction makes all individuals better off, compared to the baseline scenario.[20] An unfunded price reduction makes individuals who previously could not (or just barely) afford good Y better off, but hurts all others. Interestingly, compared to this unfunded price reduction, one that is funded by an income tax is preferred by all individuals, including those with high enough incomes to purchase the private variety. Even though they are taxed to pay for lowering the price of a variety they do not consume, high-income consumers benefit from the quality response of the private firm. There is no guarantee that these two affordability scenarios can always be Pareto-ranked. However, our example demonstrates that in a mixed-duopoly model with endogenous quality choice, raising taxes to pay for a price reduction of the public variety can have appealing welfare properties, compared to the alternative of reducing quality to pay for the price reduction.

6.2 Improving quality

Next, consider a policy initiative aimed at increasing the quality of the public variety to $\theta_b = 0.225$. Again, we examine three ways of achieving this objective: (1) An unfunded quality increase along with a matching increase in price to $p_b = 0.225$; (2) a fully funded quality increase without a change in price, financed through an externally provided subsidy $B > 0$; and (3) an internally funded quality increase that leaves price unchanged and requires a subsidy B, funded through an income tax rate τ with an income exemption of 0.15. Panel (b) in Table 1 contains these three scenarios.

Relative to the baseline scenario the private firm's quality increases and its quantity decreases in case 1 (see Proposition 8), and the same is true in case 2 (Proposition 7). In case 3, we apply an income tax rate of $\tau = 15.8\%$ on all incomes above 0.15. Given the resulting after-tax income distribution and public variety $(p_b, \theta_b) = (0.15, 0.225)$, the private firm raises quality, cuts price, and reduces quantity relative to the baseline scenario. In the resulting market outcome the public firm incurs an operating loss of $Q_b(\theta_b - p_b) = 0.057 = B$. Relative to case 2, which has the same (p_b, θ_b) but no tax, the private firm sets a lower price and quality (Proposition 9).

Panel (b) in Figure 4 plots the indirect utility of individuals in the baseline scenario as well as the three quality scenarios. There are a number of noteworthy differences when comparing the quality scenarios with the affordability scenarios. First, while an injection

[20] This may seem unsurprising, given that the scenario involves an injection of resources into the economy. However, there is no guarantee that such an injection always leads to a Pareto improvement, and it is possible to construct examples in which some individuals are hurt by an externally funded price reduction.

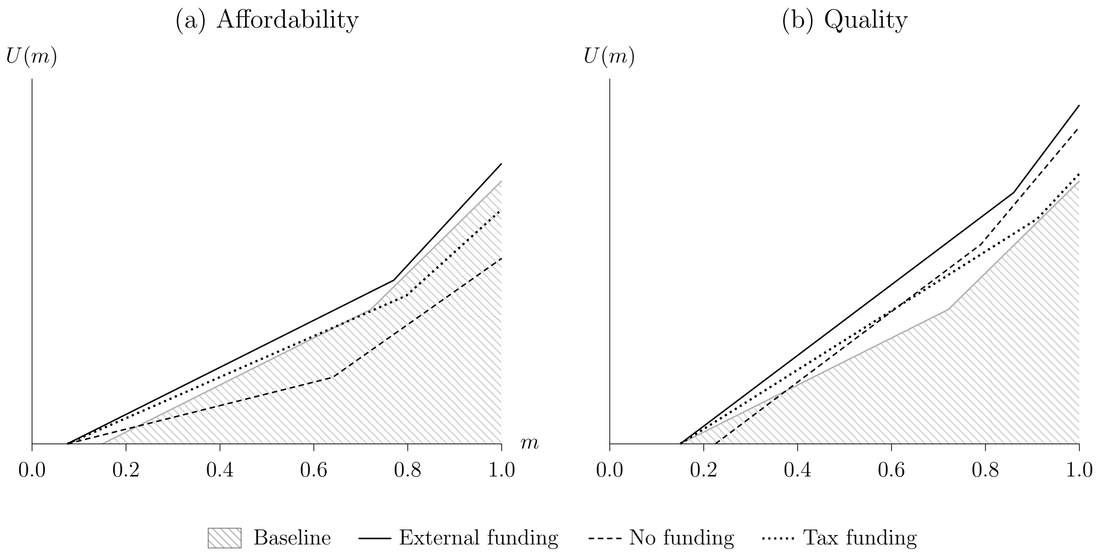

Figure 4: Distributional effects of affordability and quality improvements.

of external funds to increase quality makes all individuals better off (as was the case before), the utility gains are now especially large for individuals in the middle and the upper end of the income distribution. This is a consequence of the complementarity of income and quality in the consumers' preferences. Second, an unfunded quality increase has the opposite effect of an unfunded price reduction—poor individuals are made worse off, and richer individuals are made better off.

Lastly, in this example a tax-funded quality increase results in a Pareto-improvement over the baseline scenario, which means that the baseline scenario is inefficient given the market structure and the policy instruments we consider. Note that the size of the private firm is drastically reduced in this case, moving the market outcome close to a tax-financed state monopoly. Provision of a single variety of good Y through a tax-funded public monopoly can, indeed, Pareto-dominate a mixed duopoly in our model. While there may be long-run disadvantages of restricting competition in this manner, the example does show that maintaining high-quality public institutions—and, if necessary, funding them through progressive income taxation—can benefit even those individuals who consume higher-end private varieties.

7 Endogenous Firm Sorting and Optimal Public Policy

Our previous analysis was deliberately incomplete in two aspects, which we address in this section.

First, until this point we assumed that the private firm sells the premium variety and the public firm sells the basic variety. We argued that this sorting is often observed in mixed markets. However, imposing this ordering by assumption meant that the optimal private variety we examined represents only a local profit maximum for the private firm. We now relax this assumption and examine the private firm's global best response, i.e., when it can choose an arbitrary price and quality (Section 7.1).

Second, some of the policy changes we examined in the previous two sections could be thought of as reflecting underlying shifts in social objectives (e.g., a decrease in the public firm's price may be the result of an increased redistributive concern), while others could be thought of as reflecting changes in the government's budget constraint (e.g., an increase in the public firm's price may be the consequence of a subsidy cut during a recession). We now make the public firm's price and quality choices endogenous and explicitly dependent on its objective and constraint. To this end, we examine a Stackelberg leadership game in which the public firm maximizes a particular social welfare function in anticipation of the private firm's best response (Section 7.2).

As in the previous Section 6, we assume the Cobb-Douglas utility function $u(x, \theta) = x\theta$ and a uniform income distribution on the interval $[0, 1]$.

7.1 The private firm's global best response

Note that the public firm's budget constraint implies that $p_b \leq \theta_b$, and the private firm's profit maximization implies $p_r \geq \theta_r$. We can ignore the possibility that $p_r > p_b$ and $\theta_r < \theta_b$, as the private firm's demand would be zero in this case. Thus, we are left with two possible configurations of prices and qualities in the market:

1. $p_r > \theta_r \geq \theta_b \geq p_b$: The private firm sells the premium variety and the public firm sells the basic variety. We say that the private firm locates "above" the public firm.

2. $\theta_r < p_r \leq p_b \leq \theta_b$: The private firm sells the basic variety and the public firm sells the premium variety. We say that the private firm locates "below" the public firm.

Given the individuals' Cobb-Douglas preferences, the upper income threshold, at which a consumer is indifferent between purchasing the basic and premium variety is

$$\overline{m} \;=\; \frac{p_r\theta_r - p_b\theta_b}{\theta_r - \theta_b},$$

regardless of which firm supplies which variety. The lower income threshold, at which a consumer is indifferent between not purchasing good Y and purchasing the basic variety, is either $\underline{m} = p_b$ (if the public firm sells the basic variety) or $\underline{m} = p_r$ (if the private firm sells the basic variety). Thus, if the private firm locates above the public firm it solves

$$\max_{(p_r,\theta_r)\gg(p_b,\theta_b)} \left(1 - \frac{p_r\theta_r - p_b\theta_b}{\theta_r - \theta_b}\right)(p_r - \theta_r), \tag{14}$$

and if it locates below the public firm it solves

$$\max_{(p_r,\theta_r)\ll(p_b,\theta_b)} \left(\frac{p_r\theta_r - p_b\theta_b}{\theta_r - \theta_b} - p_r\right)(p_r - \theta_r). \tag{15}$$

A slight technicality arises when the private firm locates below the public firm and $\theta_r = 0$. In this case, all consumers with incomes in the interval $[p_r, p_b)$ are indifferent between not purchasing good Y and purchasing the private variety, as they obtain a zero utility in either case (owing to their Cobb-Douglas preferences). At the same time, these consumers would strictly prefer to purchase from the private firm if θ_r was just marginally above zero. Thus, if consumers do not purchase from the private firm when they are indifferent, the private firm's "best response" may involve an infinitesimal positive quality, which means a well-defined solution to (15) need not exist. To avoid this complication, we assume that consumers purchase from the private firm if they are indifferent between the private variety and not consuming the good.[21] The following result states that, conditional on locating below the public firm, the private firm will, indeed, supply a zero quality.

Lemma 10. *Assume that $u(x,\theta) = x\theta$ and $m \sim U[0,1]$. Fix a price-quality pair (p_b, θ_b) for the public firm, with $p_b \leq \theta_b$. If the private firm locates below the public firm, it sets $p_r = p_b/2$ and $\theta_r = 0$, and earns a profit of $p_b^2/4$.*

For a global best response, the private firm compares the maximized profits in (14) and (15), and then locates to where it earns the higher maximized profit. We can show that a location below the public firm is not a global best response if the public firm sets a sufficiently low price and quality:

Proposition 11. *Assume that $u(x,\theta) = x\theta$ and $m \sim U[0,1]$. If $p_b > 0$ is sufficiently small, and θ_b not too much larger than p_b, the private firm locates above the public firm.*

One reason why the public firm may want to set a relatively low price and quality is that it cares sufficiently about the welfare of low-income consumers. In this case, it does not want the private firm to serve the low-income segment of the market, as the private firm would offer good Y at a zero quality, resulting in a zero utility of these

[21] An alternative would be to impose a lower bound $\underline{\theta} > 0$ on the quality of each firm. If $\underline{\theta}$ is close to zero, this alternative approach would not affect our conclusions.

consumers. Instead, the public firm prefers to sell its variety at a relatively low price and a positive quality, which—if the firm's budget constraint is sufficiently tight—cannot be too far above price. In this situation, the private firm will decide to serve the high-income segment of the market with a high-price/high-quality variety. The following section explores the relationship between the public firm's objective, its constraint, and the resulting market outcome in more detail.

7.2 Stackelberg equilibrium

We now return to the public firm's problem that we introduced in Section 3.4:

$$\max_{p_b,\theta_b} \left(\int_0^\infty U(m)^{1-\psi} dF(m)\right)^{\frac{1}{1-\psi}} + \phi\pi(\theta_r, p_r) \quad \text{s.t.} \quad Q_b(\theta_b - p_b) \leq B.$$

We will numerically solve for the Stackelberg equilibrium values for both firm's prices and qualities, assuming the public firm moves first and anticipates the private firm's profit-maximizing response.

The parameters we vary are the welfare parameters ψ and ϕ, representing the government's concern for redistribution and private profits; as well as B, representing the public firm's budget constraint. For the latter two, we consider two values each: $B \in \{0, 0.02\}$ and $\phi \in \{0, 0.1\}$. Our main parameter of interest is ψ, which we vary over the range $\psi \in [0, 1)$. Note that $\psi = 0$ corresponds to a utilitarian consumer welfare function, and $\psi = 1$ to a Cobb-Douglas function.[22]

For the time being, set $\phi = 0$; that is, assume that the public firm's objective does not attach any weight to the profit of its private rival. Figure 5 (a) plots the optimal public variety (p_b, θ_b) along with the optimal private variety (p_r, θ_r), for $0 \leq \psi < 1$ and for $B = 0$ (which implies $p_b = \theta_b$).

If $\psi \leq 0.27$, the Stackelberg equilibrium involves the public firm offering the premium variety and the private firm locating below the public firm. As predicted by Lemma 10, the private firm's price is $p_r = p_b/2$ and its quality is zero. As ψ increases, p_b, θ_b, and p_r all decrease. Once ψ rises above 0.27, the equilibrium flips: The public firm lowers its price and quality discontinuously, and the private firm jumps to the high end of the price-quality spectrum. After this switch, the equilibrium remains unresponsive to

[22] There is no need to consider values $\psi \geq 1$. If $\psi \geq 1$ and a positive measure individuals do not consume good Y, consumer welfare is exactly zero (this follows from our Cobb-Douglas specification for individual utility). Thus, for all $\psi \geq 1$ the welfare maximizing public policy is to provide access to good Y to all individuals, which implies a corner solution at which $p_b = 0$.

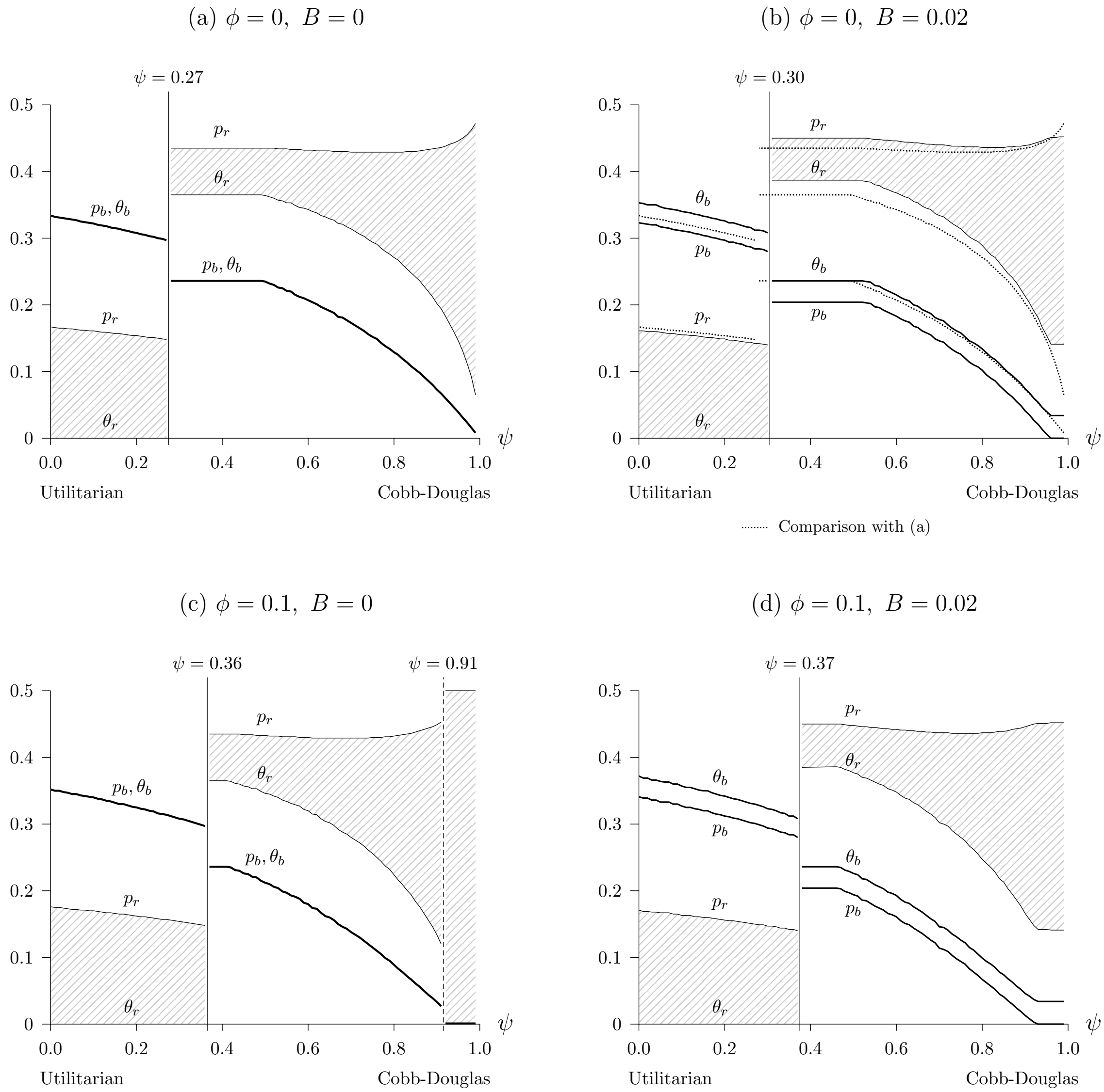

Figure 5: Stackelberg equilibrium outcomes for different values of ψ, ϕ, and B.

changes in the welfare function until $\psi = 0.5$, when p_b (and θ_b) begin to decrease again.[23] The private firm responds to these changes by decreasing its quality (see Proposition 8). The private firm's price is non-monotonic when ψ changes.

[23]The flat part of Figure 5 is best understood when considering an *decrease* in ψ in this region. The public firm would like to set a higher price and quality *if* the private firm remained above the public firm. However, if the private firm can adjust its location freely, increasing p_b and θ_b would induce the private firm to locate below, and this is not welfare-optimal unless ψ becomes sufficiently small (i.e., $\psi \leq 0.27$).

Figure 5 (b) repeats this exercise, but assumes that the public firm receives a subsidy $B = 0.02$ that allows it to make an operating loss of the same size. This means the public firm can set its quality above price. To facilitate comparison with the case where no such loss is allowed, the results from the previous case (a) are superimposed in panel (b). As before, when ψ is small the private firm locates below the public firm. In this case, the public firm allocates its subsidy primarily to increasing quality. When ψ is large enough, the private firm locates at above the public firm, and the public firm allocates its subsidy primarily to reducing price. Relative to case (a), the private firm now sets a higher price and quality, but the increase in quality is comparatively larger. Finally, as a result of being able to make an operating loss, the public firm sets a zero price when $\psi \geq 0.96$, but maintains a positive quality.

Figure 5 (c) and (d) repeat these exercises again, assuming a weight of $\phi = 0.1$ on private profits in the public firm's objective. While the main pattern remains the same, a few subtle differences emerge in comparison with the previous cases. Consider the no-subsidy case, i.e., $B = 0$. The threshold at which the equilibrium configuration flips increases to $\psi = 0.36$. For $\psi > 0.36$ the private firm locates above, and in this case p_b and θ_b drop more rapidly than in case (a), as ψ increases. At $\psi = 0.91$, a second discontinuity occurs: The interior equilibrium is replaced with a corner equilibrium, in which the public firm drops its price and quality to zero while the private firm increases its price to $p_r = 0.5$ and drops its quality to zero. The reason for this jump is that, since $B = 0$, the public firm cannot lower its price arbitrarily without reducing quality in lockstep, hence driving its customers' utilities toward zero. Once these utilities drop to a sufficiently low level, it becomes advantageous for the public firm to sacrifice consumer welfare entirely in exchange for a large increase in private profits. Relaxing the public firm's budget constraint, as in case (d), eliminates this discontinuity: The public firm now has some "breathing room" that allows it to lower its price continuously to zero without driving quality to zero. Because the utilities of low-income individuals can now remain relatively high, the public firm is less inclined to sacrifice their welfare in exchange for an increase in private profits.

8 Conclusion

Most developed countries experienced prolonged phases of public sector growth, along with an increasingly redistributive role of government. Two key manifestations of this changing role of government are the expansion of publicly provided goods and services, especially to lower-income segments of the population, and changes in the price and quality of these goods and services. These two dimensions are not independent, but linked by the government's finances. While public budgets have grown over the long run, they are not unlimited and, in general, are subject to short-run fluctuations. Thus, financial constraints are an important factor in *how* public suppliers of certain goods pursue their social welfare mandates.

To examine the relationship between government budgets, market outcomes, and social welfare, we developed a mixed duopoly model which, despite being stylized in a number of aspects (see our discussion in Section 3.5), captured several important features of markets for merit goods: Firms choose both the price and the quality of their products; consumers differ in their incomes; the public firm's objective includes a concern for redistribution; and the public firm faces a budget constraint. We now conclude this paper with a synthesis of the insights we obtained within this framework.

Throughout the paper we emphasized changes toward enhanced access to the market. In our model, such changes are the result of shifts in the social welfare function away from the utilitarian one, and are implemented by making the public variety more affordable. Our formal analysis focused on three ways of paying for such price reductions: *Externally* via an inflow of funds to the public firm from outside the model; through *cost savings* via a reduction of the quality of the publicly provided good; and *internally* via income taxation in the model. We showed that these funding mechanisms resulted in markedly different reactions by private suppliers. Public price reductions that were financed externally raised the private firm's quality, while those that were financed through cost savings lowered the private firm's quality. A particularly striking contrast emerged in our numerical example when we compared internally funded price reductions to unfunded ones (i.e., those paid for by quality reductions): The former resulted in an increased quality of the private variety at a reduced price, while the latter resulted in a reduced quality at an increased price. The resulting utility differences for the private firm's customers were large enough that even these consumers preferred to pay for the public firm's price reduction with a progressive income tax, instead of having the public firm reduce its quality.

This result underscores the important role of quality in imperfectly competitive mixed markets. Recall that, with a sufficiently strong redistributive concern, it is socially optimal in our model to let the private firm serve higher-income consumers. Due to the consumption complementarity between goods X and Y, the utilities of these consumers are relative more sensitive to quality than they are to price. Therefore, when higher-income consumers compare the private variety to the public alternative, they care less about price variations of the public variety and more about its quality variations. This means that the quality of the public firm (and not its price) is the primary disciplining force that limits the private firm's market power. Thus, a high-quality public supply not only directly benefits consumers who buy the public variety but also indirectly benefits consumers who buy the private variety, by exerting competitive pressure on the private firm. A concern for public affordability alone, without maintaining quality, weakens this disciplining channel.

Our analysis also generated insights into the effects of relaxing the public firm's budget constraint for a given social welfare function. In other words, should additional public funds, if available, be invested in the quality or affordability of publicly provided goods?[24] In our numerical analysis of the equilibrium, the allocation of incremental external funds depended on whether the private firm located above or below the public firm. If it located above the public firm, then, given a fixed welfare function, it was socially optimal to allocate most of the additional funds to making the public variety more affordable. This does contradict the aforementioned instrumental role of the public provider's quality—to the contrary, the additional funds were precisely what allowed the public firm to better pursue its redistributive goal (i.e., reduce its price) without lessening the competitive pressure faced by the private firm (i.e., maintain its quality).

Ultimately, the extent to which government should engage in redistribution is a political question. In this regard, a recent paper by Lasram and Laussel (2019) studies a mixed education system that shares some elements of our model. In particular, a public university must satisfy a budget constraint while a private university sets its tuition rate to maximize its profit. Unlike in our model, each university's quality is exogenous, and individuals differ in their abilities *ex ante* (which translates into income differences *ex post*). In this framework, Lasram and Laussel (2019) analyze the determination of the public school's tuition rate and tax-financed subsidy through majority voting.[25] Either a high tuition/low subsidy equilibrium or a low tuition/high subsidy equilibrium can arise. The policy changes we emphasized in this paper, and whose effects on the private firm we examined, may hence be interpreted as a switch from the first type of equilibrium to the second, i.e., the government taking a more redistributive role. Of course, one can also envision a reverse switch toward a political equilibrium with a *reduced* redistributive role of the state, along with a reduced participation of government in the education, healthcare, housing, and other sectors. This view of the state is often reflected in conservative budget proposals that reduce the size of public programs that provide those goods and services. Assuming that public providers are not removed from the market entirely, the effects of such policies will be opposite to those we emphasized in our analysis.

[24]For an empirical analysis of this question in the context of government-funding of higher education, see Deming and Walters (2017).

[25]Using our notation, Lasram and Laussel (2019) take θ_b as given and derive the Condorcet-winning values of (p_b, B), where p_b and B are linked through the public school's budget constraint. In contrast, we take B as given and derive the welfare-maximizing values of (p_b, θ_b).

Appendix A: Proofs

Proof of Lemma 1

We will show the following single-crossing property: If an individual with income m weakly prefers (p, θ) to $(p', \theta') \ll (p, \theta)$, then every individual with income $m' > m$ strictly prefers variety (p, θ) to (p', θ'). Suppose $u(m - p, \theta) \geq u(m - p', \theta')$. Since $u_{xx} \leq 0$ we have $u_x(m-p, \theta) \geq u_x(m-p', \theta)$, and since $u_{xy} > 0$ we have $u_x(m-p', \theta) > u_x(m-p', \theta')$. Thus, $u_x(m - p, \theta) > u_x(m - p', \theta')$, and therefore $u(m' - p, \theta) > u(m' - p', \theta')$ for all $m' > m$. The result now follows immediately. $\square$

Proof of Lemma 3

Define $P(m) = p_r(m|p_b, \theta_b) - \theta_r(m|p_b, \theta_b)$, where $\big(p_r(m|p_b, \theta_b), \theta_r(m|p_b, \theta_b)\big)$ is the point on the private firm's price-quality locus associated with marginal individual m, given (p_b, θ_b). Define $\hat{P}(m)$ in the same way, but given $(\hat{p}_b, \hat{\theta}_b)$. The firm's profit if it serves $1 - F(m)$ individuals is $\pi(m) = P(m)(1 - F(m))$ before the change, and $\hat{\pi}(m) = \hat{P}(m)(1 - F(m))$ after the change. Let $\overline{m} = \arg\max \pi(m)$ and let $\hat{m} = \arg\max \hat{\pi}(m)$.

By Topkis' Theorem, if $\hat{\pi}(m) - \pi(m)$ increases at $m = \overline{m}$ then $\hat{m} > \overline{m}$ and thus $1 - F(\hat{m}) < 1 - F(\overline{m})$. This condition can be expressed as

$$
\begin{aligned}
&- f(\overline{m})\big(\hat{P}(\overline{m}) - P(\overline{m})\big) + (1 - F(\overline{m}))\big(\hat{P}'(\overline{m}) - P'(\overline{m})\big) > 0 \\
&\qquad\qquad \Leftrightarrow \quad \hat{P}'(\overline{m}) - P'(\overline{m}) > \lambda(\overline{m})\big(\hat{P}(\overline{m}) - P(\overline{m})\big). \qquad (16)
\end{aligned}
$$

Observe that condition (16) is equivalent to an increase in the elasticity of demand: Under the original policy (p_b, θ_b), when m increases marginally the relative change in P is $P'(m)/P(m)$ and relative change in quantity is $-f(m)/(1 - F(m)) = -\lambda(m)$. At the profit maximum this elasticity equals one, so that $P'(\overline{m})/P(\overline{m}) = \lambda(\overline{m})$. Substituting this in (16) gives

$$
\hat{P}'(\overline{m}) - P'(\overline{m}) > \frac{P'(\overline{m})}{P(\overline{m})}\Big(\hat{P}(\overline{m}) - P(\overline{m})\Big) \quad \Leftrightarrow \quad \frac{\hat{P}'(\overline{m})}{\hat{P}(\overline{m})} > \frac{P'(\overline{m})}{P(\overline{m})}.
$$

Thus, under the new policy, if m is increased marginally above $\overline{m}$, the relative change in P is larger than what it was under the old policy, while the relative change in $1 - F(\overline{m})$ is still $-\lambda(\overline{m})$. It follows that, at $\overline{m}$, demand elasticity is now below one in absolute value.

Fix an individual with arbitrary income m. We make three geometric observations. First, if preferences are homothetic the iso-MRS curves in (p_r, θ_r)-space are rays through $(m, 0)$. Second, if an h.o.d.-1 utility representation is chosen, then as one moves along any iso-MRS ray the change in utility is proportional to the distance traveled along the ray. Third, regardless of preferences, as one moves along any straight line in (p_r, θ_r)-space the change in the profit margin $p_r - \theta_r$ is proportional to the distance traveled along the line.

Now apply these observations to an individual with income $\overline{m}$ and h.o.d.-1 utility. Note that the profit margin $P(\overline{m})$ is determined by the intersection of the iso-MRS ray $u_x(\overline{m}-p_r, \theta_r)/u_{\theta_r}(\overline{m}-p_r, \theta_r) = 1$ with the indifference curve $u(\overline{m}-p_r, \theta_r) - u(\overline{m}-p_b, \theta_b) = 0$. Similarly, the profit margin $\hat{P}(\overline{m})$ is determined by the intersection of the same iso-MRS ray with the indifference curve $u(\overline{m} - p_r, \theta_r) - u(\overline{m} - \hat{p}_b, \hat{\theta}_b) = 0$. It follows that

$$\hat{P}(\overline{m}) - P(\overline{m}) \;=\; k \cdot \left[u(\overline{m} - \hat{p}_b, \hat{\theta}_b) - u(\overline{m} - p_b, \theta_b)\right] \;=\; k\Delta(\overline{m})$$

for some $k < 0$, and therefore $\hat{P}'(\overline{m}) - P'(\overline{m}) \;=\; k\Delta'(\overline{m})$. Plugging these expressions into (16) gives us

$$\Delta'(\overline{m}) \;<\; \lambda(\overline{m})\Delta(\overline{m}).$$

This establishes part (b) of the result; the proof of part (a) is symmetric. □

Proof of Proposition 5

It will be convenient to view the private firm as choosing its profit margin $P = p_r - \theta_r$, instead of a marginal individual $\overline{m}$. For given P, let $(p_r(P), \theta_r(P))$ be the unique point on the price-quality locus where $p_r - \theta_r = P$. Let $m(P)$ be the associated income of the individual who is indifferent between the two firms; this is defined by the condition $u(m(P) - p_b, \theta_b) = u(m(P) - p_r(P), \theta_r(P))$. Note that $p_r(P)$, $\theta_r(P)$, and $m(P)$ do not depend on F and are all strictly increasing in P (see Section 4.1).

For a given income distribution F, let $D(P) = 1 - F(m(P))$ be the private firm's demand if it sets profit margin P. Its profit function is then $\pi(P) = PD(P)$, and at the profit maximum we have $\pi'(P) = D(P) + PD'(P) = 0$, or equivalently,

$$\frac{D'(P)}{D(P)}P \;=\; -\frac{f\big(m(P)\big)}{1 - F\big(m(P)\big)}m'(P)P \;=\; -1. \tag{17}$$

Let P^* be the solution to (17). Since $\overline{m} = m(P^*)$, we can write (17) as

$$\frac{D'(P^*)}{D(P^*)}P^* \;=\; -\lambda(\overline{m})m'(P^*)P^* \;=\; -1. \tag{18}$$

Now consider a change to income distribution $\hat{F}$. If $\hat{F}$ is less dispersed than F, we can write $\hat{F}(m) = F(m + \sigma(m))$, with $\sigma'(m) \geq 0$. Let $\hat{D}(P) = 1 - \hat{F}(m(P))$ be the private firm's demand associated with profit margin P, and let $\hat{\pi}(P) = P\hat{D}(P)$ be the profit function. We will show that $\hat{\pi}'(P^*) < 0$, or equivalently

$$\begin{aligned}\frac{\hat{D}'(P^*)}{\hat{D}(P^*)}P^* &= -\frac{f\big(\overline{m}+\sigma(\overline{m})\big)}{1-F\big(\overline{m}+\sigma(\overline{m})\big)}\big(1+\sigma'(\overline{m})\big)m'(P^*)P^* \\ &= -\lambda\big(\overline{m}+\sigma(\overline{m})\big)\big(1+\sigma'(\overline{m})\big)m'(P^*)P^* \; < \; -1. \qquad (19)\end{aligned}$$

Since $\lambda'(m) > 0 \; \forall m$, the left-hand side of (19) is strictly smaller than (18) as long as $\sigma(\overline{m}) > 0$. Note that $\hat{F}(m) = F(m + \sigma(m))$ implies $\sigma(m) = F^{-1}(\hat{F}(m)) - m$; thus, the condition $\sigma(\overline{m}) > 0$ is equivalent to

$$F^{-1}(\hat{F}(\overline{m})) - \overline{m} > 0 \quad \Leftrightarrow \quad F^{-1}(\hat{F}(\overline{m})) > \overline{m} \quad \Leftrightarrow \quad \hat{F}(\overline{m}) > F(\overline{m}).$$

If this condition holds, then $\hat{\pi}'(P^*) < \pi'(P^*) = 0$. This, in turn, implies that the private firm can increase its profit by decreasing its profit margin to below P^*. Because $p_r(P)$ and $\theta_r(P)$ are strictly increasing in P, it follows that the private firm's price and quality decrease. □

Proof of Proposition 6

The main argument is in the text already: Suppose (without loss of generality) that p_b decreases and θ_b stays the same. Then $\Delta(\overline{m}) > 0$ and $\Delta'(\overline{m}) < 0 < \lambda(\overline{m})\Delta(\overline{m})$, and Proposition 4 implies a decrease in Q_r and an increase in θ_r. At the same time, the lower threshold income $\underline{m}$, at which a consumer is indifferent between not purchasing good Y and purchasing one unit of good Y from the public provider, decreases. To see this, differentiate the condition $u(\underline{m}, 0) = u(\underline{m} - p_b, \theta_b)$ with respect to p_b, to get

$$\frac{\partial \underline{m}}{\partial p_b} = \frac{u_x(\underline{m} - p_b, \theta_b)}{u_x(\underline{m} - p_b, \theta_b) - u_x(\underline{m}, 0)} \; > \; 0.$$

Therefore, the public firm's quantity $Q_b = F(\overline{m})$ decreases. □

Proof of Proposition 7 and Proposition 8

Suppose $u(x, \theta) = x^\alpha \theta^\beta$ for $\alpha, \beta > 0$. Without loss of generality, suppose $\beta = 1 - \alpha$. We use the following technical result (the proof is given at the end of this Appendix).

Lemma 12. *Suppose $u(x, \theta) = x^\alpha \theta^{1-\alpha}$ and $m \sim U[0, M]$. At the private firm's interior profit maximum, the following holds:*

$$\overline{m} \; > \; K \equiv \frac{1}{2}\left(M + p_b + \frac{\alpha}{1-\alpha}\theta_b\right) \; > \; L \equiv \frac{\alpha}{1+\alpha}M + \frac{1}{1+\alpha}p_b.$$

To prove the Propositions, consider a change in the public firm's price and quality from (p_b, θ_b) to $(p_b + dp, \theta_b + d\theta)$. With Cobb-Douglas preferences, we can write

$$\begin{aligned}\Delta(\overline{m}) &= (\overline{m} - p_b - dp)^{\alpha}(\theta_b + d\theta)^{1-\alpha} - (\overline{m} - p_b)^{\alpha}\theta_b^{1-\alpha}, \\ \Delta'(\overline{m}) &= \alpha(\overline{m} - p_b - dp)^{\alpha-1}(\theta_b + d\theta)^{1-\alpha} - \alpha(\overline{m} - p_b)^{\alpha-1}\theta_b^{1-\alpha}.\end{aligned}$$

If $|dp|$ and $|d\theta|$ are close to zero these expressions are approximated by

$$\begin{aligned}\Delta(\overline{m}) &= -dp \cdot \alpha(\overline{m} - p_b)^{\alpha-1}\theta_b^{1-\alpha} + d\theta \cdot (1-\alpha)(\overline{m} - p_b)^{\alpha}\theta_b^{-\alpha}, \\ \Delta'(\overline{m}) &= dp \cdot \alpha(1-\alpha)(\overline{m} - p_b)^{\alpha-2}\theta_b^{1-\alpha} + d\theta \cdot \alpha(1-\alpha)(\overline{m} - p_b)^{\alpha-1}\theta_b^{-\alpha},\end{aligned}$$

and we have

$$\frac{\Delta'(\overline{m})}{\Delta(\overline{m})} = \frac{\alpha(1-\alpha)}{\overline{m} - p_b} \frac{dp \cdot \theta_b + d\theta \cdot (\overline{m} - p_b)}{-dp \cdot \alpha\theta_b + d\theta \cdot (1-\alpha)(\overline{m} - p_b)}. \tag{20}$$

Furthermore, with uniformly distributed incomes on $[0, M]$, the hazard rate at $\overline{m}$ is given by

$$\lambda(\overline{m}) = \frac{1/M}{1 - \overline{m}/M} = \frac{1}{M - \overline{m}}. \tag{21}$$

Now consider the following cases.

1. First, suppose $dp = 0$ (the case considered in Section 5.2). (20)–(21) imply that

   $$\frac{\Delta'(\overline{m})}{\Delta(\overline{m})} < \lambda(\overline{m}) \;\Leftrightarrow\; \frac{\alpha}{\overline{m} - p_b} < \frac{1}{M - \overline{m}} \;\Leftrightarrow\; \overline{m} > \frac{\alpha}{1+\alpha}M + \frac{1}{1+\alpha}p_b = L.$$

   By Lemma 12, we know this to be true. It follows that $\Delta'(\overline{m}) >/< \lambda(\overline{m})\Delta(\overline{m})$ if and only if $\Delta(\overline{m}) </> 0$, which is the case if and only if $d\theta </> 0$. Proposition 4 (a) and (d) then imply that the private firm increases/decreases its quantity and decreases/increases its quality if and only if $d\theta </> 0$.

   Without loss of generality, suppose $\partial\theta > 0$. By piecing together many small changes $d\theta$, we conclude that an increase in θ_b while p_b stays constant decreases Q_r and increases θ_r. Recall that the two firms' quantities are given by $Q_r = 1 - F(\overline{m})$ and $Q_b = F(\overline{m}) - F(\underline{m})$. With Cobb-Douglas preferences, we have $\underline{m} = p_b$ before and after the change; thus, a decrease in Q_r implies an increase in Q_b. Proposition 7 now follows.

2. Next, suppose $dp = d\theta$ (the case considered in Section 5.3). In this case, (20) becomes

$$\frac{\Delta'(\overline{m})}{\Delta(\overline{m})} = \frac{\alpha(1-\alpha)}{\overline{m}-p_b}\frac{\theta_b+\overline{m}-p_b}{-\alpha\theta_b+(1-\alpha)(\overline{m}-p_b)} = \frac{\alpha(1-\alpha)\left(\dfrac{\theta_b}{\overline{m}-p_b}+1\right)}{-\alpha\theta_b+(1-\alpha)(\overline{m}-p_b)}$$

$$< \frac{\alpha(1-\alpha)\left(\dfrac{1-\alpha}{\alpha}+1\right)}{-\alpha\theta_b+(1-\alpha)(\overline{m}-p_b)} = \frac{1-\alpha}{-\alpha\theta_b+(1-\alpha)(\overline{m}-p_b)},$$

where the inequality is due to (25). Thus, to show that $\Delta'(\overline{m})/\Delta(\overline{m}) < \lambda(\overline{m})$ it is sufficient that

$$\frac{1-\alpha}{-\alpha\theta_b+(1-\alpha)(\overline{m}-p_b)} < \frac{1}{M-\overline{m}} \quad \Leftrightarrow \quad \overline{m} > \frac{1}{2}\left(M+p_b+\frac{\alpha}{1-\alpha}\theta_b\right) = K.$$

By Lemma 12, we know this to be true. It follows that $\Delta'(\overline{m}) >/< \lambda(\overline{m})\Delta(\overline{m})$ if and only if $\Delta(\overline{m}) </> 0$. (25) implies that the marginal rate of substitution for an income-$\overline{m}$ individual who purchases from the public firm is less than one. Thus, $\Delta(\overline{m}) </> 0$ if and only if $dp = d\theta </> 0$. Proposition 4 (a) and (d) then imply that the private firm decreases/increases its quantity if and only if $dp = d\theta </> 0$.

Without loss of generality, suppose $dp = d\theta < 0$. By piecing together many small changes $(dp, d\theta)$, we conclude that a simultaneous decrease in p_b and θ_b by the same amount increases Q_r and decreases θ_r, and Proposition 8 follows. □

Proof of Proposition 9

Let $T : [0,\infty) \to [0,\infty)$ be the tax schedule, and let $\hat{F}$ be the after-tax income distribution. Since $\hat{F}(m - T(m)) = F(m)$, there exists a function $\sigma : [0,\infty) \to [0,\infty)$ such that $\hat{F}(m) = F(m + \sigma(m))$. Moreover, since T is weakly increasing, σ is weakly increasing, which implies that the net income distribution is less dispersed than the gross income distribution, according to Definition 1. For any m we can write $\sigma(m) = F^{-1}(\hat{F}(m)) - m$. Therefore, if $\sigma(\overline{m}^S) > 0$ then $\hat{F}(\overline{m}^S) > F(\overline{m}^S)$ and Proposition 5 implies that the private firm's price and quality are both smaller under income distribution $\hat{F}$ than under F. Note that T is weakly increasing implies that $\sigma(m) \geq T(m)$ $\forall m$ (see Figure 6). Thus, as long as $T(\overline{m}^S) > 0$ we have $\sigma(\overline{m}^S) > 0$, and the result follows. □

Proof of Lemma 10

Suppose the private firm locates below the public firm and let

$$\pi^{\text{below}}(p_r, \theta_r) = \left(\frac{p_r\theta_r - p_b\theta_b}{\theta_r - \theta_b} - p_r\right)(p_r - \theta_r)$$

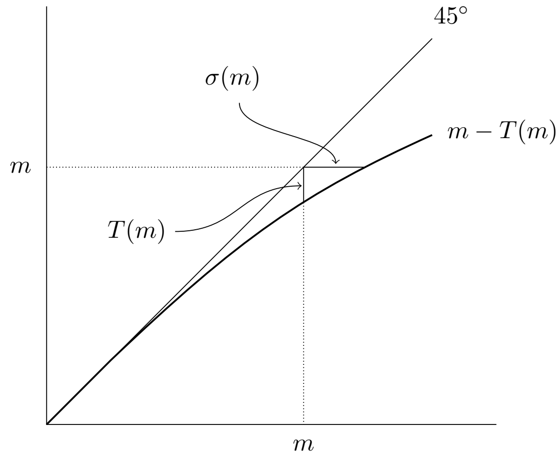

Figure 6: A non-decreasing tax schedule implies $\sigma(\overline{m}) \geq T(m)$.

be its profit function. The first-order condition for a profit maximum with respect to p_r is

$$\frac{\partial \pi^{\text{below}}}{\partial p_r} \;=\; \left(\frac{\theta_r}{\theta_r - \theta_b} - 1\right) p_r + \frac{p_r\theta_r - p_b\theta_b}{\theta_r - \theta_b} - p_r \;=\; 0,$$

which has the unique solution $p_r = p_b/2$, regardless of the values of θ_r, θ_b, or p_b. The derivative of the firm's profit with respect to θ_r is

$$\begin{aligned}
\frac{\partial \pi^{\text{below}}}{\partial \theta_r} &= \frac{p_r(\theta_r - \theta_b) - (p_r\theta_r - p_b\theta_b)}{(\theta_r - \theta_b)^2}(p_r - \theta_r) - \left(\frac{p_r\theta_r - p_b\theta_b}{\theta_r - \theta_b} - p_r\right) \\
&\overset{\text{sg}}{=} \big[p_r(\theta_r - \theta_b) - (p_r\theta_r - p_b\theta_b)\big](p_r - \theta_r) - \big[p_r\theta_r - p_b\theta_b - p_r(\theta_r - \theta_b)\big](\theta_r - \theta_b) \\
&= \theta_b(p_b - p_r)\big[p_r - \theta_b\big] \;\overset{\text{sg}}{=}\; p_r - \theta_b \;=\; p_b/2 - \theta_b \;<\; p_b - \theta_b \;\leq\; 0,
\end{aligned}$$

where the last equality uses the solution to the first-order condition with respect to p_r, and the last inequality follows from $B \geq 0$. Thus, conditional on locating below the public firm, the private firm's optimal price-quality pair is $(p_r, \theta_r) = (p_b/2, 0)$. The maximized profit resulting from this location is $(\overline{m} - \underline{m})(p_r - \theta_r) = (p_b - p_b/2)(p_b/2 - 0) = p_b^2/4$. $\square$

Proof of Proposition 11

Suppose the private firm locates above the public firm and let

$$\pi^{\text{above}}(p_r, \theta_r | p_b, \theta_b) \;=\; \left(1 - \frac{p_r\theta_r - p_b\theta_b}{\theta_r - \theta_b}\right)(p_r - \theta_r)$$

be its profit function. We will show that

$$\max_{(p_r,\theta_r)\gg(k,k)} \pi^{\text{above}}(p_r,\theta_r|k,k) \;>\; \max_{(p_r,\theta_r)\ll(k,k)} \pi^{\text{below}}(p_r,\theta_r|k,k) \tag{22}$$

for small enough $k \in (0,1)$. The same inequality will then also be true when $p_b = k$ and θ_b is slightly larger than k.

To prove that (22) holds for small k, fix $k \in (0,1)$ and note that a feasible choice for the private firm is to set

$$p_r = \overline{p} \equiv \frac{2+k}{3}, \quad \theta_r = \overline{\theta} \equiv \frac{1+2k}{3}.$$

This implies that

$$\begin{aligned}\max_{(p_r,\theta_r)\gg(k,k)} \pi^{\text{above}}(p_r,\theta_r|k,k) \;&\geq\; \pi^{\text{above}}(\overline{p},\overline{\theta}|k,k) \\ &=\; \left(1 - \frac{\overline{p}\overline{\theta}-k^2}{\overline{\theta}-k}\right)(\overline{p}-\overline{\theta}) \;=\; \frac{1}{27}(1-6k+7k^2).\end{aligned}$$

At the same time, from Lemma 10 we know that

$$\max_{(p_r,\theta_r)\ll(k,k)} \pi^{\text{below}}(p_r,\theta_r|k,k) \;=\; \frac{k^2}{4}.$$

Thus, a sufficient condition for (22) is that

$$\frac{1}{27}(1-6k+7k^2) \;>\; \frac{k^2}{4},$$

which is equivalent to $k < 12 - \sqrt{140} \approx 0.1678$. □

Proof of Lemma 12

We begin the proof with some preliminary observations. Note that the private firm sets price and quality to satisfy the conditions $I(p_r,\theta_r|\overline{m}) = 0$ and $H(p_r,\theta_r|\overline{m}) = 1$. With Cobb-Douglas preferences, these can be written as follows:

$$(\overline{m}-p_r)^{\alpha}\theta_r^{1-\alpha} - (\overline{m}-p_b)^{\alpha}\theta_b^{1-\alpha} \;=\; 0, \tag{23}$$

$$\frac{\alpha}{1-\alpha}\frac{\theta_r}{\overline{m}-p_r} \;=\; 1. \tag{24}$$

Since we are only considering interior profit maxima where $(p_r, \theta_r) \gg (p_b, \theta_b)$, (24) implies

$$\frac{\alpha}{1-\alpha}\frac{\theta_b}{\overline{m}-p_b} < 1 \quad \Leftrightarrow \quad \frac{\theta_b}{\overline{m}-p_b} < \frac{1-\alpha}{\alpha}, \tag{25}$$

and because we assume that $\theta_b \geq p_b$, (25) implies

$$\frac{p_b}{\overline{m}-p_b} < \frac{1-\alpha}{\alpha} \quad \Leftrightarrow \quad p_b < (1-\alpha)\overline{m} \quad \Leftrightarrow \quad \overline{m} > \frac{p_b}{1-\alpha}. \tag{26}$$

Solving (23)–(24) for p_r and θ_r as functions of $\overline{m}$, we get

$$\begin{aligned} p_r(\overline{m}) &= \overline{m} - \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} (\overline{m}-p_b)^{\alpha}\theta_b^{1-\alpha}, \\ \theta_r(\overline{m}) &= \left(\frac{1-\alpha}{\alpha}\right)^{\alpha} (\overline{m}-p_b)^{\alpha}\theta_b^{1-\alpha}. \end{aligned}$$

If incomes are uniformly distributed on $[0, M]$, the firm's profit, expressed as a function of a marginal individual $\overline{m}$, is given by

$$\pi(\overline{m}) = \left(1-\frac{\overline{m}}{M}\right)\big(p_r(\overline{m}) - \theta_r(\overline{m})\big) = \left(1-\frac{\overline{m}}{M}\right)\left(\overline{m} - A(\overline{m}-p_b)^{\alpha}\theta_b^{1-\alpha}\right),$$

where $A = \big[\alpha/(1-\alpha)\big]^{1-\alpha} + \big[(1-\alpha)/\alpha\big]^{\alpha}$. The first and second derivative of the firm's profit function are as follows:

$$\pi'(\overline{m}) = 1 - 2\frac{\overline{m}}{M} + \frac{A}{M}\left(\frac{\theta_b}{\overline{m}-p_b}\right)^{1-\alpha}\big[(1+\alpha)\overline{m} - \alpha M - p_b\big], \tag{27}$$

$$\begin{aligned} \pi''(\overline{m}) &= -\frac{2}{M} + \frac{A}{M}\left(\frac{\theta_b}{\overline{m}-p_b}\right)^{1-\alpha}\left(\frac{-(1-\alpha)}{\overline{m}-p_b}\big[(1+\alpha)\overline{m} - \alpha M - p_b\big] + 1 + \alpha\right) \\ &= -\frac{2}{M} + \frac{A}{M}\left(\frac{\theta_b}{\overline{m}-p_b}\right)^{1-\alpha}\left(2\alpha + (1-\alpha)\alpha\frac{M-\overline{m}}{\overline{m}-p_b}\right). \end{aligned} \tag{28}$$

We now prove Lemma 12 in two steps. Recall the definitions for K and L,

$$K \equiv \frac{1}{2}\left(M + p_b + \frac{\alpha}{1-\alpha}\theta_b\right), \quad L \equiv \frac{\alpha}{1+\alpha}M + \frac{1}{1+\alpha}p_b.$$

In Step 1, we show that $K > L$ at the private firm's profit maximum. Then, in Step 2, we show that $\overline{m} > K$.

Step 1. We show that $K > L$. Since K increases in θ_b but L does not, and $\theta_b \geq p_b$, it is sufficient to show that $K > L$ when $\theta_b = p_b$. Assuming $\theta_b = p_b$, suppose that $K \leq L$:

$$\frac{1}{2}\left(M + p_b + \frac{\alpha}{1-\alpha}p_b\right) \;=\; \frac{1}{2}\left(M + \frac{1}{1-\alpha}p_b\right) \;\leq\; \frac{\alpha}{1+\alpha}M + \frac{1}{1+\alpha}p_b$$

$$\Leftrightarrow \;\; (1-3\alpha)p_b \;\geq\; (1-\alpha)^2 M. \quad (29)$$

If $\alpha > 1/3$, the inequality in (29) would clearly be violated. If $\alpha < 1/3$, (29) implies

$$p_b \;\geq\; \frac{(1-\alpha)^2}{1-3\alpha}M \;=\; \frac{1-\alpha}{1-3\alpha}(1-\alpha)M \;>\; (1-\alpha)M \;\geq\; (1-\alpha)\overline{m}.$$

But by (26) we know this to be false, since $p_b < (1-\alpha)\overline{m}$ (otherwise, no interior optimum exists in which $(p_r, \theta_r) \gg (p_b, \theta_b)$). It follows that $K > L$.

Step 2. We show that $\overline{m} > K$. To do so, we first show that $\overline{m} \leq L$ implies $\pi'(\overline{m}) > 0$. If $\overline{m} \leq L$ then

$$(1+\alpha)\overline{m} - \alpha M - p_b \;\leq\; 0. \quad (30)$$

Moreover, (25) implies that

$$A\left(\frac{\theta_b}{\overline{m} - p_b}\right)^{1-\alpha} \;<\; A\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} \;=\; \frac{1}{\alpha}. \quad (31)$$

Together, (27), (30), and (31) imply that

$$\pi'(\overline{m}) \;\geq\; B(\overline{m}) \;\equiv\; 1 - 2\frac{\overline{m}}{M} + \frac{1}{M}\frac{1}{\alpha}\big[(1+\alpha)\overline{m} - \alpha M - p_b\big].$$

Note that

$$B'(\overline{m}) \;=\; -\frac{2}{M} + \frac{1}{M}\frac{1+\alpha}{\alpha} \;=\; \frac{1}{M}\frac{1-\alpha}{\alpha} \;>\; 0.$$

From (26) we have $\overline{m} > p_b/(1-\alpha)$, so that

$$B(\overline{m}) \;>\; B\left(\frac{p_b}{1-\alpha}\right) \;=\; 1 - 2\frac{p_b}{(1-\alpha)M} + \frac{1}{\alpha M}\left[\frac{1+\alpha}{1-\alpha}p_b - \alpha M - p_b\right] \;=\; 0.$$

We conclude that for all $\overline{m} \in \big(p_b/(1-\alpha), L\big)$, $\pi'(\overline{m}) > 0$.

Next, we proceed to show that $\overline{m} > K$. Since $\overline{m}$ decreases in p_b by Proposition 6, and because K increases in p_b, it is sufficient to show $\overline{m} > K$ when $p_b = \theta_b$. We showed already that $\pi'(\overline{m}) > 0$ for $\overline{m} \leq L$. Furthermore, (28) implies that $\pi''(\overline{m})$ is strictly decreasing in $\overline{m}$. These two properties, in turn, imply that if $\pi'(K|\theta_b = p_b) > 0$, then any profit maximum must occur at $\overline{m} > K$. Therefore, we are going to show that $\pi'(K|\theta_b = p_b) > 0$.

Define

$$t \equiv \frac{p_b}{(1-\alpha)M}, \quad k(t) \equiv \frac{1}{2}(1+t).$$

Note that $p_b > 0$ implies $t > 0$; furthermore (26) implies that at an interior optimum in which $\overline{m} < M$ we have $p_b < (1-\alpha)M$ and, thus, $t < 1$. Using $p_b = tM$ and $K = k(t)M$, we can write

$$\begin{aligned}
\pi'(K|\theta_b = p_b) &= 1 - 2\frac{k(t)M}{M} + \frac{A}{M}\left(\frac{tM}{k(t)M - tM}\right)^{1-\alpha}\big[(1+\alpha)k(t)M - \alpha M - tM\big] \\
&= -t + \frac{1}{2}A\left(\frac{2(1-\alpha)t}{1+(2\alpha-1)t}\right)^{1-\alpha}[1-\alpha+(3\alpha-1)t] \\
&= -t + \frac{1}{2}\frac{1}{\alpha^\alpha}\left(\frac{2t}{1+(2\alpha-1)t}\right)^{1-\alpha}[1-\alpha+(3\alpha-1)t] \\
&> 0 \ \Leftrightarrow \ \big[1-\alpha+(3\alpha-1)t\big] \ > \ (2\alpha t)^\alpha\big(1+(2\alpha-1)t\big)^{1-\alpha}.
\end{aligned} \tag{32}$$

Denote the difference between the left and right side of the inequality in (32) by

$$D(t) \ \equiv \ 1-\alpha+(3\alpha-1)t - (2\alpha t)^\alpha\big(1+(2\alpha-1)t\big)^{1-\alpha}.$$

Note that $D(1) = 0$; thus, in order to show that $D(t) > 0$ for $t < 1$ we will show that $D'(t) < 0$ for all $t < 1$. We have

$$\begin{aligned}
D'(t) &= 3\alpha - 1 - (2\alpha)^\alpha \alpha t^{\alpha-1}\big(1+(2\alpha-1)t\big)^{1-\alpha} - (2\alpha t)^\alpha(1-\alpha)\big(1+(2\alpha-1)t\big)^{-\alpha}(2\alpha-1) \\
&= 3\alpha - 1 - (2\alpha)^\alpha t^{\alpha-1}\big(1+(2\alpha-1)t\big)^{-\alpha}\Big[\alpha\big(1+(2\alpha-1)t\big) + (1-\alpha)(2\alpha-1)t\Big] \\
&= 3\alpha - 1 - (2\alpha)^\alpha t^{\alpha-1}\big(1+(2\alpha-1)t\big)^{-\alpha}\Big[\alpha + (2\alpha-1)t\Big].
\end{aligned}$$

Note that $D'(1) = 0$; thus, in order to show that $D'(t) < 0$ for $t < 1$ we will show that $D''(t) > 0$ for all $t < 1$. Let $w = 2\alpha - 1$ and write

$$\begin{aligned}
D''(t) &= -(2\alpha)^\alpha \frac{t^{1-\alpha}(1+wt)^\alpha - (\alpha+wt)\left[(1-\alpha)t^{-\alpha}(1+wt)^\alpha + \alpha t^{1-\alpha}w(1+wt)^{\alpha-1}\right]}{[t^{1-\alpha}(1+wt)^\alpha]^2} \\
&= -\frac{(2\alpha)^\alpha t^{-\alpha}(1+wt)^{\alpha-1}}{[t^{1-\alpha}(1+wt)^\alpha]^2}\Big[wt(1+wt) - (\alpha+wt)\big[(1-\alpha)(1+wt) + \alpha wt\big]\Big]
\end{aligned}$$

$$= -(2\alpha)^{\alpha} t^{\alpha-2}(1+wt)^{-\alpha-1}\Big[\alpha^2 - \alpha\Big] > 0.$$

This establishes (32), and hence that $\overline{m} > K$. $\square$

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