University of Alberta
ENGM 541: Modeling and Simulation of Engineering Systems
Laboratory #8

Eigenvalues

As we've seen, we can calculate eigenvalues directly for small matrices. We'll explore how eigenvalues can be found in several ways, including solution by numerical methods in MATLAB.

Let's define the matrix

$$[E] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

in the MATLAB environment:

```
E = [1 2;3 4]
```

MATLAB has a function to find the coefficients of the characteristic polynomial of a matrix. In the absence of other information, MATLAB assumes that the system of interest is $[E]\{x\} = \lambda[I]\{x\}$, for which the characteristic polynomial is:

$$(1-\lambda)(4-\lambda) - 6 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda - 2 = 0$$

The `poly` function creates a vector that includes the coefficients of the characteristic polynomial. For the system represented by the matrix $[E]$, MATLAB returns the following

```
>> p = poly(E)

p =

    1.0000   -5.0000   -2.0000
```

which agrees with the characteristic polynomial equation found above.

If we want to know in what range the eigenvalues lie for this system, we find the Gershgorin disks. The first disk is centred at h_{11} of $[H]$, where $[H]\{x\} = \lambda\{x\}$. In this case $[E] = [H]$. The radius is the sum of the absolute values of the off-diagonal terms:

$$R_1 = \sum_{i=2}^{N} |h_{1i}|$$

We know that the eigenvalues are within the disks. A check is that the sum of the eigenvalues must equal the trace of $[H]$. In this case, we find

```
>> trace(E)

ans =

     5
```

Another way to see where the eigenvalues will be is to plot the expression $x^2 - 5x - x$ in at least the range indicated by the Gershgorin disks. Let's use the range $-2 \le x \le 10$, and plot the polynomial, either by using a symbolic representation with `ezplot`

```
>> E1 ='x^2-5*x-2';
>> ezplot(E1,[-2,10])
```

```
>> grid on
```

or by generating a series of values for x and using `plot`. The eigenvalues occur when the expression has a value of zero.

There are many methods for computing zeros of functions. In this case, the order of the system is N=2, and so we have a quadratic equation to solve (which can be done by hand calculation). For higher-order polynomials we can solve this expression using the `roots` command in MATLAB and find:

```
>> roots(p)

ans =
    5.3723
   -0.3723
```

The eigenvalues of a matrix are the same as the roots of its characteristic polynomial. We thus see that [E] has two unique and real valued eigenvalues.

We can also find the eigenvalues of a matrix using the `eig` command:

```
>> eig(E)

ans =
   -0.3723
    5.3723
```

Exercise

1) Find the eigenvalues of [A], where

$$[A] = \begin{bmatrix} -149 & -50 & -154 \\ 537 & 180 & 546 \\ -27 & -9 & -25 \end{bmatrix}$$

2) Rewrite the following system of equations

$$\begin{bmatrix} (2-\lambda) & -1 \\ -1 & (1-\tfrac{2}{3}\lambda) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

in the standard form [A]{x} = λ[B]{x}, and then use `eig(A,B)` to confirm that the eigenvalues are λ= ½ and λ= 3.

Linearisation

Many systems are inherently nonlinear. In some circumstances, we will want to linearise the system in a region of its operation so that it the governing equations are easier to deal with.

The linearization procedure has five steps:

1. Derive the nonlinear model

2. Determine the normal operating point
3. Introduce the incremental variables
4. Linearise all the nonlinear terms with a Taylor Series expansion
5. Arrange the linearised equation in final form.

The first step is to generate the governing equations for the system, including any nonlinearities. We use the formulation employing loop and node variables.

Then we decide where the expected operating point will be for the loop variable x and node variable F of the nonlinear constitutive relationship of interest. This point $(\bar{x}, \bar{F})$ becomes the point through which the linearised constitutive relationship will be expressed.

Using a Taylor Series expansion of the nonlinear function, truncated after the linear term, we get the approximate linear form:

$$F_{NLS} = \bar{F}_{NLS} + \hat{F}_{NLS} \approx \bar{F}_{NLS} + k_{inc}\hat{x}$$

where $\bar{F}_{NLS}$ is the magnitude of the force at the normal operating point, and $k_{inc}\hat{x}$ is the linear approximation of the constitutive relationship, for $\hat{x} = x - \bar{x}$.
Substitute the new variables $\hat{x}$ and $\hat{F}$ into the governing equations for x and F_{NLS} to yield a set of equations in linear form.

Exercise

1) Go through Example 2.8 in the course text (Kulakowski p. 41) to linearise the mass-spring system that has a nonlinear spring with the following nonlinear constitutive relationship:

$$F_{NLS} = 2.5\sqrt{x}.$$

2) For m = 1 kg and b = 0 Ns/m, what is the natural frequency of the linearised system?

References

Kuisalaas, J. *Numerical Methods in Engineering with MATLAB.* Cambridge Press 2005.
Kulakowski, B.T., Gardner, J.F., Shearer, J.L. *Dynamic Modeling and Control of Engineering Systems*. Third Ed. Cambridge University Press, 2007.
Moler, C.B. *Numerical Computing with MATLAB*. SIAM 2004.