The University of Alberta
ENGM 541: Modeling and Simulation of Engineering Systems

ENGM 541 & MECE 758-X5
Midterm Examination
March 3, 2010

Two hours
Open Book and Open Notes

Calculators are allowed. No computers are allowed. Any electronic device must have its communication features switched off.

There are six questions. Undergraduate students answer <u>only</u> Questions 1, 2, 3, and 4. Graduate students answer all five questions. SHOW YOUR WORK.

Question 1: (Do all four parts)

a) The element shown in Figure 1 has a constitutive relationship that can be expressed as $b = C_1 a + C_2 |a| a + C_3 a^3$,
where C_1, C_2, and C_3 are constants. Identify the loop variable and node variable and draw a graph showing the relationship.

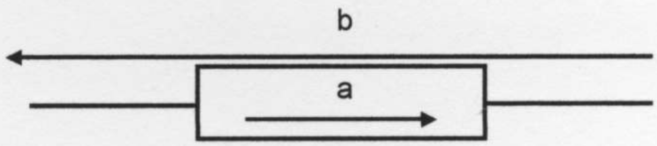


Figure 1

b) Given the set of equations for a linear equilibrium system $Ax = c$, where

$$A = \begin{bmatrix} 10 & 4 & 1 \\ 1 & 6 & -2 \\ 2 & -1 & 4 \end{bmatrix}; \quad c = \begin{Bmatrix} 99 \\ 33 \\ 28 \end{Bmatrix},$$

use the Gauss-Siedel method to solve for x, using

$$x^{(0)} = \begin{Bmatrix} 5 \\ 5 \\ 5 \end{Bmatrix}$$

for the trial solution to the system. Do at least one set of iterations.

c) For the set of equations $Ax=c$ in part b, write a set of MATLAB expressions that will
  i) define the matrix A,
  ii) define the constant coefficient column vector c, and
  iii) evaluate the vector x by inverting the matrix A.

d) A system has a nonlinear element with a constitutive relationship described by

$$q = mp^2,$$

where p is the loop variable and q is the node variable. If the loop variable is $\bar{p}$ at the normal operating point, what is the slope of the linearised relationship at the normal operating point?

Question 2: (Do both parts)

a) An equilibrium pipe network system is shown in Figure 2, with flow resistances R_1, R_2, and R_3, and pressure source P_s. Find the governing equations in terms of node variables q_1 and q_2 using a direct formulation (showing your steps), non-dimensionalise the equations using P_s and R_1 as non-dimensionalising parameters (noting that P_c is a reference pressure), and express the equations in matrix form.

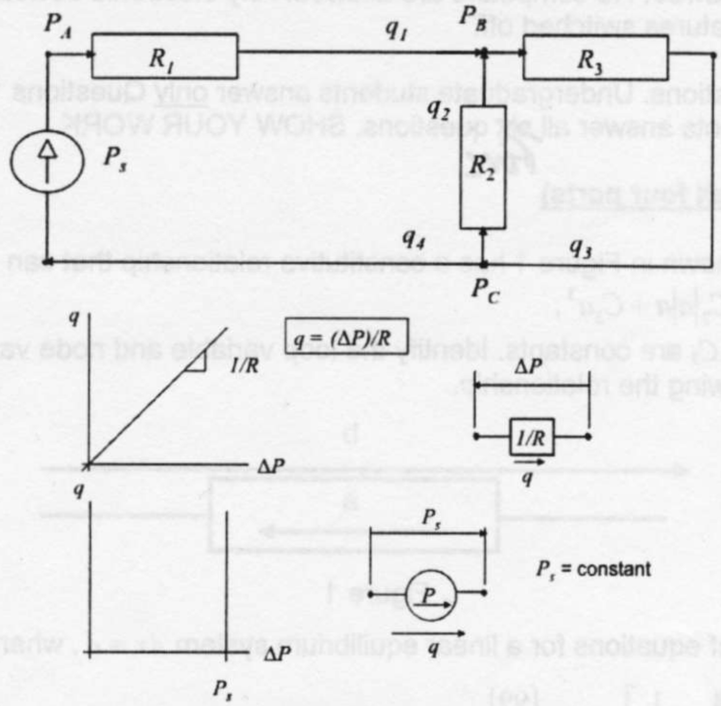


Figure 2

b) Write the non-dimensionalised coefficient matrix for the case when $R_1 = 1$, $R_2 = 2$, and $R_3 = 3$.

Question 3: (Do all five parts)

a) Express the following differential equation in first-order form:

$$\frac{d^2y}{dt^2} = \frac{dy}{dt}\left(1 - y^2\right) - y;$$

with initial conditions:

$$y\big|_{t=0} = 0.5; \quad \frac{dy}{dt}\bigg|_{t=0} = 1.$$

b) Estimate the solution for $y(t)$ using Picard's method of successive approximations. Do two sets of iterations.

c) Evaluate the solution for $y(1)$ (that is, find y at time $t = 1$) using your estimate.

d) Write the set of MATLAB expressions that would go in an m-file called `diffeqsim.m` to describe the ordinary differential equations for solving using `ode45`.

e) Write an expression for using the MATLAB ODE solver `ode45` command that calls the function developed in part d, for the interval from t=0 to t=1.

Question 4:

Write the governing equations for the system represented by the SIMULINK diagram shown in Figure 3.

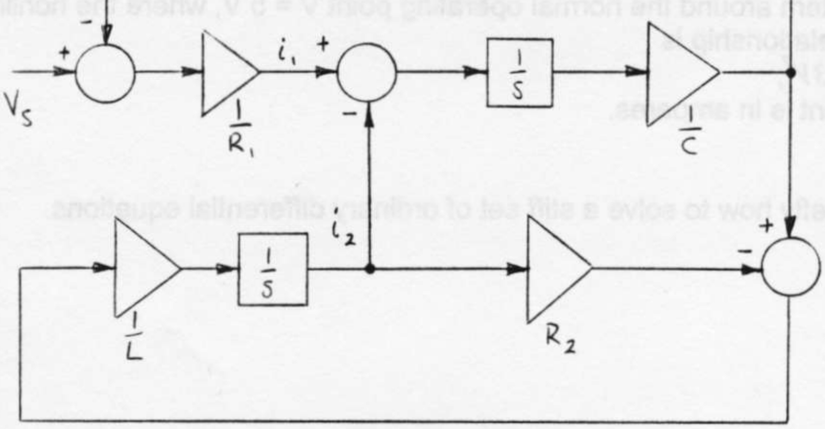


Figure 3

Question 5 (ENGM 541 graduate students do part a; MECE758 graduate students do both parts)

a) A pneumatic actuation system on a spacecraft is shown in Figure 5. It has a reservoir that acts as a pressure source P_s, with flow q going through three pipe resistances R_1, R_2, and R_3 before exiting to a vacuum. The constitutive relationship for the pipe flow resistance is $q = \Delta P/R$. Between the resistance elements are two fluid capacitors C_1 and C_2; the constitutive relationship for this element type is

$$q_c = C\frac{d}{dt}(P - P_\infty),$$

where q_c is positive going into the accumulator, P is the pressure in the capacitor, and P_∞ is the reference pressure. In this case, P_∞=0. Draw a system network diagram for an electrical system that is dynamically equivalent to this system, and label the diagram using the same variables and parameters that describe the pneumatic system.

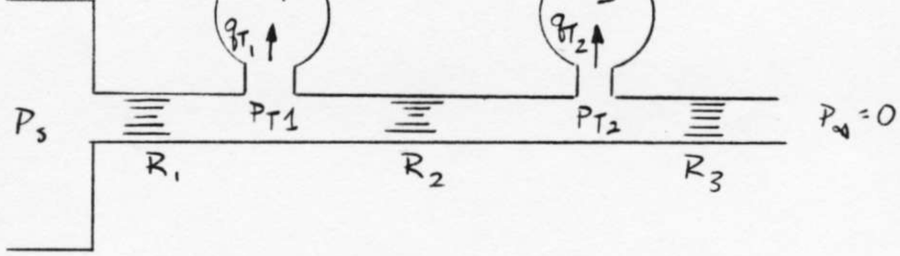


Figure 5

b) MECE 758 students only:

i) Find the linearised constitutive relationship for a nonlinear resistor element in an electrical system around the normal operating point V = 5 V, where the nonlinear constitutive relationship is

$$i_{NL} = 3V^7,$$

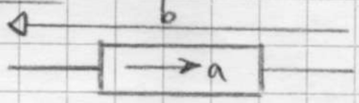
and the current is in amperes.

ii) Explain briefly how to solve a stiff set of ordinary differential equations.

ENGM541 / MECE 758 - X5 MIDTERM 2010
Solutions

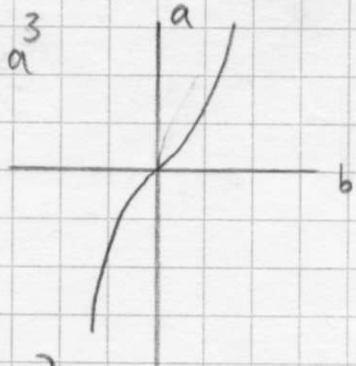
QUESTION #1

a)

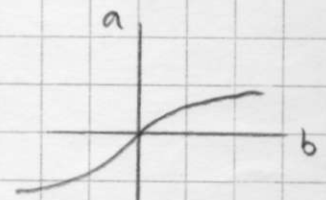


node variable a
loop variable b

$$b = C_1 a + C_2 |a| a + C_3 a^3$$



may also look like this:



depending on coefficients C_1, C_2, C_3

<or mirror images>

b) $Ax = c$, where

$$A = \begin{bmatrix} 10 & 4 & 1 \\ 1 & 6 & -2 \\ 2 & -1 & 4 \end{bmatrix} ; \quad c = \begin{Bmatrix} 99 \\ 33 \\ 28 \end{Bmatrix}$$

guess $x^{(0)} = \begin{Bmatrix} 5 \\ 5 \\ 5 \end{Bmatrix}$

check row sum criterion:

$\alpha_1 = (4+1)/10 < 1$ ok

$\alpha_2 = (1+2)/6 < 1$ ok

$\alpha_3 = (2+1)/4 < 1$ ok

first iteration:

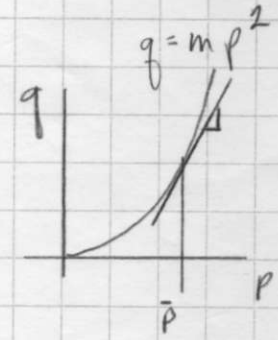
$$x_1^{(1)} = \frac{1}{a_{11}}\left[c_1 - (a_{12}x_2^{(0)} + a_{13}x_3^{(0)})\right] = \frac{1}{10}\left[99 - (4\times 5 + 1\times 5)\right] = 7.4$$

$$x_2^{(1)} = \frac{1}{a_{22}}\left[c_2 - (a_{21}x_1^{(1)} + a_{23}x_3^{(0)})\right] = \frac{1}{6}\left[33 - (1\times 7.4 - 2\times 5)\right] = 5.933$$

$$x_3^{(1)} = \frac{1}{a_{33}}\left[c_3 - (a_{31}x_1^{(1)} + a_{32}x_2^{(1)})\right] = \frac{1}{4}\left[28 - (2\times 7.4 - 1\times 5.93)\right] = 4.783$$

c)
```
A = [10 4 1; 1 6 -2; 2 -1 4];
c = [99; 33; 28];
x = A\c          or     x = inv(A)*c
```

d) $\left.\frac{dq}{dp}\right|_{\bar{p}} = 2m\bar{p}$, where $\bar{p}$ is the normal operating point.



d)

ENGM 541

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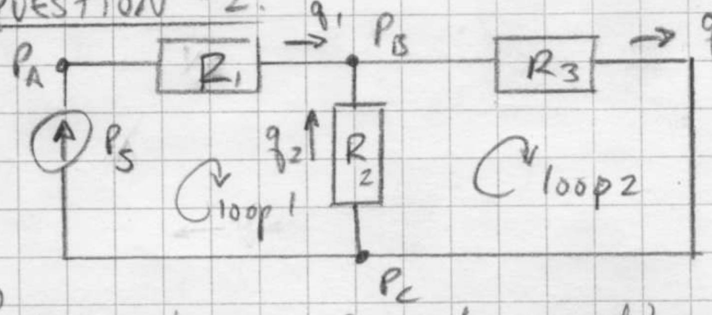
> Marks are given for showing the solution steps!

QUESTION #2:



i)

a) Solve in terms of node variables (flows) and loop variables (Pressures)

$N = 2$ (2 loops)

ii) define node variables q_1, q_2 & q_3 & check admissibility:

solve in terms of q_1 & q_2

node (A) $\quad q_1 - q_1 = 0$

node (B) $\quad q_1 + q_2 - q_3 = 0 \Rightarrow q_3 = q_1 + q_2$

node (C) $\quad q_3 - q_2 - q_1 = 0$

↳ admissible form for q_3.

trivially $q_4 = q_1$

iii) define loop variables at nodes, where P_A, P_B, & P_C are pressures defined at nodes, satisfying loop admissibility, & write loop equations:

<could also solve by using the outer loop>

loop 1:

(1) $-P_S + (P_A - P_B) + (P_B - P_C) = 0$

$(P_A - P_B) = R_1 q_1$

loop 2:

(2) $(P_C - P_B) + (P_B - P_C) = 0$

$(P_C - P_B) = R_2 q_2$

iv) substitute constitutive relationships:

$(P_B - P_C) = R_3 q_3$

third possible eqn: is $(R_1 + R_3) q + R_3 q_2 = P_S$

$-P_S + R_1 q_1 - R_2 q_2 = 0 \quad \Rightarrow R_1 q_1 - R_2 q_2 = P_S$

flow $= \dfrac{\text{Pressure}}{\text{resistance}}$

& $R_2 q_2 + R_3 q_3 = 0$

$R_2 q_2 + R_3 (q_1 + q_2) = 0 \quad \Rightarrow R_3 q_1 + (R_2 + R_3) q_2 = 0$

define $z_1 = \dfrac{R_1}{P_S} q_1$; $z_2 = \dfrac{R_1}{P_S} q_2$

so $q_1 = \dfrac{P_S}{R_1} z_1$ & $q_2 = \dfrac{P_S}{R_1} z_2$

so $R_1 \dfrac{\cancel{P_S}}{R_1} z_1 - R_2 \dfrac{\cancel{P_S}}{R_1} z_2 = \cancel{P_S}\, 1$

& $R_3 \dfrac{\cancel{P_S}}{R_1} z_1 + \dfrac{(R_2 + R_3)}{R_1} \cancel{P_S}\, z_2 = 0$

$$\Rightarrow \begin{bmatrix} 1 & -R_2/R_1 \\ R_3/R_1 & (R_2+R_3)/R_1 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

ENGM 541
MIDTERM 2010
SOLUTIONS

b) when $R_1 = 1$, $R_2 = 2$, and $R_3 = 3$, the non-dimensional set of equations becomes

$$A = \begin{bmatrix} 1 & -\frac{(2)}{1} \\ \frac{(3)}{(1)} & \frac{(2+3)}{(1)} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$$

$Az = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$; this would solve to $z = \begin{Bmatrix} 0.454 \\ -0.273 \end{Bmatrix}$ which makes sense.

QUESTION #3

a) $$\frac{d^2y}{dt^2} = \frac{dy}{dt}(1-y^2) - y \qquad y(0) = 0.5, \ \frac{dy(0)}{dt} = 1.$$

let $x_1 = y$
$x_2 = \dot{y} = \dot{x}_1$

$$\Rightarrow \boxed{\begin{array}{ll} \dot{x}_1 = x_2 & x_1(0) = 0.5 \\ \dot{x}_2 = x_2(1-x_1^2) - x_1 & x_2(0) = 1. \end{array}}$$

b) Apply Picard's method:

$$x_i^{(i+1)}(t) = x_i(0) + \int_0^t \dot{x}_i^{(i)}\, dt$$

step 1:

$$\begin{aligned} x_1^{(1)}(t) &= x_1(0) + \int_0^t \dot{x}_1^{(0)}\, dt \\ &= 0.5 + \int_0^t [x_2(0)]\, dt \\ &= 0.5 + \int_0^t [1]\, dt \\ &= 0.5 + t. \end{aligned}$$

$\dot{x}_1 = x_2$

$x_1(t) = t + \frac{1}{2}$

$$\begin{aligned} x_2^{(1)}(t) &= x_2(0) + \int_0^t \dot{x}_2^{(0)}\, dt \\ &= 1 + \int_0^t \left[x_2^{(0)}(1 - x_1^{(0)2}) - x_1^{(0)}\right] dt \\ &= 1 + \int_0^t \left\{(1)\left[1 - \left(\tfrac{1}{2}\right)^2\right] - \left(\tfrac{1}{2}\right)\right\} dt \\ &= 1 + \int_0^t \left[3/4 - 1/2\right] dt \\ &= 1 + t/4. \end{aligned}$$

$x_2 = \frac{t}{4} + 1$

step 2:

$$\begin{aligned} x_1^{(2)}(t) &= x_1(0) + \int_0^t \dot{x}_1^{(1)}\, dt \\ &= 0.5 + \int_0^t [x_2^{(1)}]\, dt \\ &= 0.5 + \int_0^t (1 + t/4)\, dt \Rightarrow \boxed{x_1^{(2)}(t) = 0.5 + t + \frac{t^2}{8}} \end{aligned}$$

c) $x_1(1) = y(1) = 0.5 + (1) + (0.125)(1)^2$
$= \boxed{1.625}$

d)
```
function dx = diffeqsim(t, x)
dx = zeros(2,1);
dx(1) = x(2);
dx(2) = x(2)*(1-x(1)*x(1)) - x(1);
```

e)
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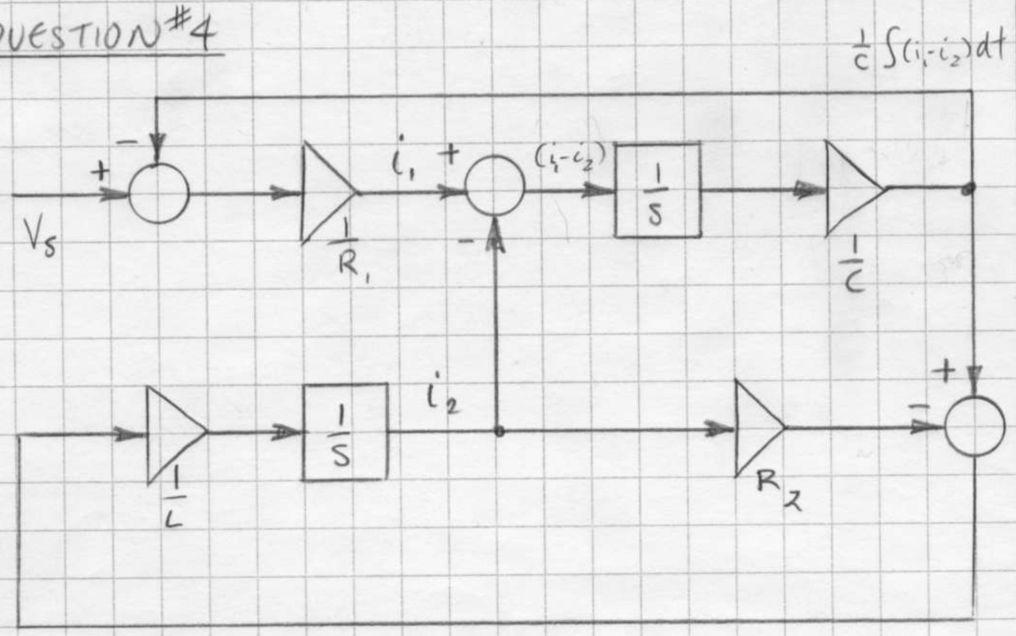
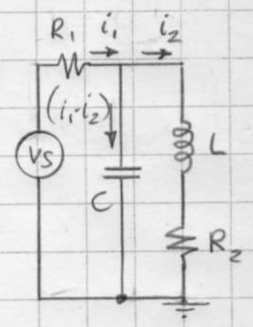
[T,X] = ode45(@diffeqsim,[0 1],[0.5 1]);
```

QUESTION #4



This Simulink diagram represents this system:



Top loop:

$$-\frac{1}{C}\int(i_1-i_2)dt + V_S = R i_1$$

$$\Rightarrow \quad R_1 i_1 + \frac{1}{C}\int(i_1-i_2)dt = V_S$$

or

$$i_1 = \frac{1}{R_1}\left[V_S - \frac{1}{C}\int(i_1-i_2)dt\right]$$

Bottom loop:

$$\frac{1}{C}\int(i_1-i_2)dt - R_2 i_2 = L\frac{di_2}{dt}$$

or

$$\frac{di_2}{dt} = \frac{1}{L}\left[\frac{1}{C}\int(i_1-i_2)dt - R_2 i_2\right]$$

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MIDTERM 2010
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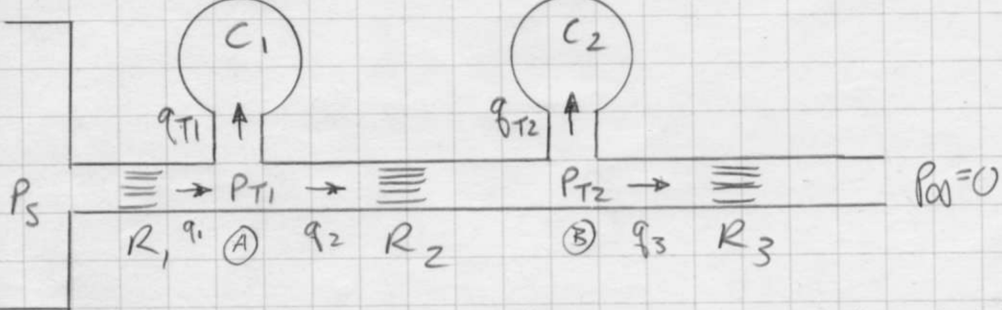
QUESTION #5 (Graduate students only)

a)



can solve interms of node variables. P_{T1} & P_{T2} satisfy adm.

NODE A:

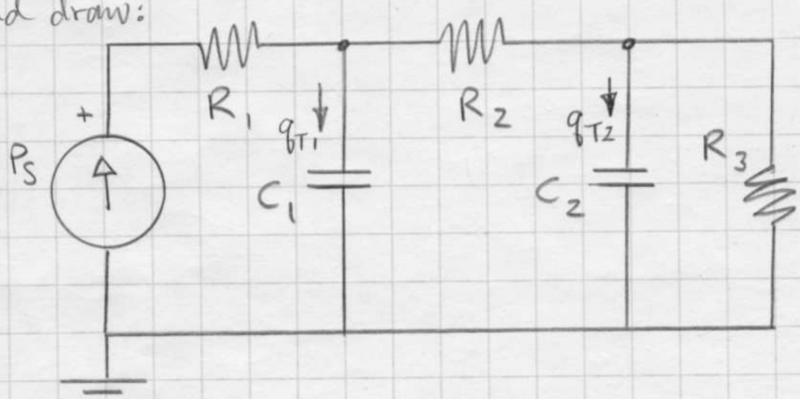
$$q_1 - q_{T1} - q_2 = 0 \qquad ①$$

NODE B: $q_2 - q_{T2} - q_3 = 0 \qquad ②$

substitute in ①: $\frac{(P_S - P_{T1})}{R_1} + C_1 \frac{d}{dt}(0 - P_{T1}) + \frac{(P_{T2} - P_{T1})}{R_1} = 0$

substitute in ②: $\frac{(P_{T1} - P_{T2})}{R_2} + C_2 \frac{d}{dt}(0 - P_{T2}) + \frac{(0 - P_{T2})}{R_3} = 0$

and draw:



b) $i_{NL} = 3v^7$

$$\Rightarrow i \approx 3\bar{v}^7 + 21 v^6\Big|_{\bar{v}} \hat{v}$$

$$\Rightarrow i \approx 234375 + 328125\,\hat{v}$$