# Affine Yangians and deformed double current algebras in type $A$

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**Abstract**

We study the structure of Yangians of affine type and deformed double current algebras, which are deformations of the enveloping algebras of matrix $W_{1+\infty}$-algebras. We prove that they admit a PBW-type basis, establish a connection (limit construction) between these two types of algebras and toroidal quantum algebras, and we give three equivalent definitions of deformed double current algebras. We construct a Schur-Weyl functor between these algebras and rational Cherednik algebras.

## 1 Introduction

The Yangians of finite type are quantum groups, introduced by V. Drinfeld in [9], which are quantizations of the enveloping algebra of the current Lie algebra $\mathfrak{g}[v]$ of a semisimple Lie algebra $\mathfrak{g}$. The second definition of these Yangians in [10] is given in terms of a finite Cartan matrix and an infinite set of generators. If we replace it with a Cartan matrix of affine type, we obtain algebras that are called affine Yangians. We will consider only the type $A$ and $\widehat{A}$. In the second case, our definition is more general and depends on two parameters $\lambda, \beta$. (More precisely, it depends on $\frac{\lambda}{\beta}$ viewed as an element of $\mathbb{P}^1(\mathbb{C})$). These affine Yangians are deformations of the enveloping algebra of the universal central extension $\widehat{\mathfrak{sl}}_n[u^{\pm 1}, v]$ of $\mathfrak{sl}_n[u^{\pm 1}, v]$ $(= \mathfrak{sl}_n \otimes_{\mathbb{C}} \mathbb{C}[u^{\pm 1}, v])$. We will introduce a class of algebras that we will call deformed double current algebras (DDCA): they are deformations of the enveloping algebra of the universal central extension $\widehat{\mathfrak{sl}}_n[u, v]$ of $\mathfrak{sl}_n[u, v]$ $(= \mathfrak{sl}_n \otimes_{\mathbb{C}} \mathbb{C}[u, v])$.

One motivation for studying the representation theory of these algebras is that we hope that it will be easier to understand, using classical methods, than the representation theory of quantum toroidal algebras, which is still quite mysterious - for some important results, see [16],[28, 29],[18, 19]. In return, we hope that a better understanding of DDCA will help shed some light on quantum toroidal algebras, not just in type A: we expect some of our results, in particular theorem 12.1, to admit a generalization to any semisimple Lie algebra. Another motivation is that we hope to obtain a $\Gamma$-twisted version of DDCA, $\Gamma$ being a finite subgroup of $SL_2(\mathbb{C})$, which may not be possible for quantum toroidal algebras or affine Yangians (as in the theory of Cherednik algebras and symplectic reflection algebras, see [14]).

In this paper, we focus on the structure of affine Yangians and DDCA, postponing the study of their representations. Sections 3 and 4 recall all the necessary definitions concerning Yangians and Cherednik algebras. The next three concern only the affine Yangians $\widehat{\mathbf{Y}}_{\lambda,\beta}$ and its subalgebra $\mathsf{L}_{\lambda,\beta}$ considered in [17]. The main theorem about the affine Yangians is the construction in section 7 of a PBW basis, from which we can derive a few corollaries. Our approach relies on the existence of a PBW basis for Cherednik algebras and uses the Schur-Weyl functor from [17].

The second half of the paper is devoted to deformed double current algebras. After giving a first definition in section 8, we construct a Schur-Weyl functor between them and rational Cherednik algebras, which we use to obtain a PBW basis, mimicking the approach for affine Yangians. We are able to establish that they are isomorphic to the algebra $\mathsf{L}_{\lambda,\beta}$ from [17]. Therefore, specializing the parameter $\lambda$ to 0 (but with $\beta \neq 0$), we deduce that they are deformations of $\mathfrak{U}(\mathfrak{sl}_n(\mathsf{A}_\beta))$, where $\mathsf{A}_\beta$ is isomorphic to the first Weyl algebra. In section 12, we explain how they can be viewed as limit forms of affine Yangians. Afterwards, we introduce another family of algebras which are also deformations of $\mathfrak{U}(\widehat{\mathfrak{sl}}_n[u, v])$ and establish a Schur-Weyl type of equivalence between them and rational Cherednik algebras. In the last section, we prove that these algebras are isomorphic to the deformed double current algebras defined previously in section 8.

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## 3 Yangians and current algebras

Throughout this article, we will assume that $n \geq 4$, unless stated otherwise, and work always over $\mathbb{C}$. Two reasons explain this restriction: certain definitions have to be modified for $\mathfrak{sl}_2$ (for instance definition 3.1 and the one in lemma 3.2) and certain proofs perhaps could be modified for $n = 2, 3$, but they are more uniform when $n \geq 4$.

**Definition 3.1.** *[9] Let $z_\mu$ be an orthonormal basis of $\mathfrak{sl}_n$ with respect to its standard Killing form $(\cdot,\cdot)$. The Yangian $Y_\lambda, \lambda \in \mathbb{C}$, is the algebra generated by elements $z, J(z)$ for $z \in \mathfrak{sl}_n$, satisfying the following relations for $z_1, z_2, z_3 \in \mathfrak{sl}_n$:*

$$z_1 z_2 - z_2 z_1 = [z_1, z_2] \text{ (bracket in } \mathfrak{sl}_n)$$

$$J(az_1 + bz_2) = aJ(z_1) + bJ(z_2),\ a, b \in \mathbb{C},\ \ [z_1, J(z_2)] = J\big([z_1, z_2]\big)$$

$$\big[J(z_1), J([z_2, z_3])\big] + \big[J(z_3), J([z_1, z_2])\big] + \big[J(z_2), J([z_3, z_1])\big] = \lambda^2 \sum_{\sigma,\mu,\nu} \big([z_1, z_\sigma], [[z_2, z_\mu], [z_3, z_\nu]]\big)\{z_\sigma, z_\mu, z_\nu\}$$

*where $\{z_1, z_2, z_3\} = \frac{1}{24}\sum_{\pi \in S_3} z_{\pi(1)} z_{\pi(2)} z_{\pi(3)}$.*

Let $C = (c_{ij})_{1 \leq i,j \leq n-1}$ (resp. $\widehat{C} = (c_{ij})_{0 \leq i,j \leq n-1}$) be the Cartan matrix of finite (resp. affine) type $A_{n-1}$ (resp. $\widehat{A}_{n-1}$).

$$\widehat{C} = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 2 & -1 \\ -1 & 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}$$

**Definition 3.2.** *[10] Let $\lambda \in \mathbb{C}$. The Yangian $Y_\lambda$ of finite type $A_{n-1}$ can also be defined as the algebra generated by the elements $X^\pm_{i,r}, H_{i,r}, i = 1, \ldots, n-1,\ r \in \mathbb{Z}_{\geq 0}$, which satisfy the following relations :*

$$[H_{i,r}, H_{j,s}] = 0, \quad [H_{i,0}, X^\pm_{j,s}] = \pm c_{ij} X^\pm_{j,s}, \quad [X^+_{i,r}, X^-_{j,s}] = \delta_{ij} H_{i,r+s} \tag{1}$$

$$[H_{i,r+1}, X^\pm_{j,s}] - [H_{i,r}, X^\pm_{j,s+1}] = \pm\frac{\lambda}{2} c_{ij}(H_{i,r} X^\pm_{j,s} + X^\pm_{j,s} H_{i,r}) \tag{2}$$

$$[X^\pm_{i,r+1}, X^\pm_{j,s}] - [X^\pm_{i,r}, X^\pm_{j,s+1}] = \pm\frac{\lambda}{2} c_{ij}(X^\pm_{i,r} X^\pm_{j,s} + X^\pm_{j,s} X^\pm_{i,r}) \tag{3}$$

$$\sum_{\pi\in S_m}\Big[X^{\pm}_{i,r_{\pi(1)}},[X^{\pm}_{i,r_{\pi(2)}},\ldots,[X^{\pm}_{i,r_{\pi(m)}},X^{\pm}_{j,s}]\ldots]\Big]=0 \text{ where } m=1-c_{ij},r_1,\ldots,r_m,s\in\mathbb{Z}_{\geq 0} \qquad (4)$$

We will write $X_i^{\pm}$ and $H_i$ instead of $X_{i,0}^{\pm}$ and $H_{i,0}$. The set of roots of $\mathfrak{sl}_n$ will be denoted $\Delta=\{\alpha_{ij}|1\leq i\neq j\leq n\}$ with choice of positive roots $\Delta^+=\{\alpha_{ij}|1\leq i<j\leq n\}$. The longest positive root $\theta$ equals $\alpha_{1n}$. The elementary matrices will be written $E_{ij}$, so $X_i^+=E_{i,i+1},X_i^-=E_{i+1,i},H_i=E_{ii}-E_{i+1,i+1}$ for $1\leq i\leq n-1$. We set $E_\theta=E_{1n},E_{-\theta}=E_{n1}$. For $\alpha\in\Delta^+$, $X_\alpha^{\pm}$ is the standard root vector of weight $\pm\alpha$ and $X_\alpha=X_\alpha^+$; if $\alpha\in\Delta^-$, then $X_\alpha^{\pm}=X_{-\alpha}^{\mp}$ and $X_\alpha=X_{-\alpha}^-$. We may also write $E_k^+$ (resp. $E_k^-$) for $E_{k,k+1}$ (resp. $E_{k+1,k}$), $E_\alpha$ for the standard root vector of weight $\alpha\in\Delta$, $H_\theta$ for $E_{nn}-E_{11}$ and $H_{ij}$ for $E_{ii}-E_{jj}$.

The isomorphism between the two definitions of $Y_\lambda$ is given by the formulas [10]:

$$J(X_i^{\pm})\mapsto X_{i,1}^{\pm}+\lambda\omega_i^{\pm} \text{ where } \omega_i^{\pm}=\pm\frac{1}{4}\sum_{\alpha\in\Delta^+}\left([X_i^{\pm},X_\alpha^{\pm}]X_\alpha^{\mp}+X_\alpha^{\mp}[X_i^{\pm},X_\alpha^{\pm}]\right)-\frac{1}{4}(X_i^{\pm}H_i+H_iX_i^{\pm})$$

and

$$J(H_i)\mapsto H_{i,1}+\lambda\nu_i \text{ where } \nu_i=\frac{1}{4}\sum_{\alpha\in\Delta^+}(\alpha,\alpha_i)(X_\alpha^+X_\alpha^-+X_\alpha^-X_\alpha^+)-\frac{1}{2}H_i^2.$$

In view of these formulas, we will need the following notation to shorten certain expressions later: for any algebra $A$ and $a_1,a_2\in A$, we write $S(a_1,a_2)$ for $a_1a_2+a_2a_1$.

**Definition 3.3.** *Let $\lambda,\beta\in\mathbb{C}$. The affine Yangian $\widehat{\mathbf{Y}}_{\beta,\lambda}$ of type $\widehat{A}_{n-1}$ is the algebra generated by $X_{i,r}^{\pm},H_{i,r}$ for $i=0,\ldots,n-1,r\in\mathbb{Z}_{\geq 0}$, which satisfy the relations of definition 3.2 for $i,j\in\{0,\ldots,n-1\}$ except that the relations (2),(3) must be modified for $(i,j)=(1,0)$ and $(i,j)=(0,n-1)$ in the following way:*

$$[H_{j,r+1},X_{i,s}^{\pm}]-[H_{j,r},X_{i,s+1}^{\pm}]=\left(\beta-\frac{\lambda}{2}\mp\frac{\lambda}{2}\right)X_{i,s}^{\pm}H_{j,r}+\left(\frac{\lambda}{2}\mp\frac{\lambda}{2}-\beta\right)H_{j,r}X_{i,s}^{\pm} \qquad (5)$$

$$[H_{i,r+1},X_{j,s}^{\pm}]-[H_{i,r},X_{j,s+1}^{\pm}]=\left(\beta-\frac{\lambda}{2}\mp\frac{\lambda}{2}\right)H_{i,r}X_{j,s}^{\pm}+\left(\frac{\lambda}{2}\mp\frac{\lambda}{2}-\beta\right)X_{j,s}^{\pm}H_{i,r} \qquad (6)$$

$$[X_{i,r+1}^{\pm},X_{j,s}^{\pm}]-[X_{i,r}^{\pm},X_{j,s+1}^{\pm}]=\left(\beta-\frac{\lambda}{2}\mp\frac{\lambda}{2}\right)X_{i,r}^{\pm}X_{j,s}^{\pm}+\left(\frac{\lambda}{2}\mp\frac{\lambda}{2}-\beta\right)X_{j,s}^{\pm}X_{i,r}^{\pm} \qquad (7)$$

**Remark 3.1.** *It is a direct consequence of the definition of $\widehat{\mathbf{Y}}_{\lambda,\beta}$ that $[X_{i,r}^{\pm},X_{j,s}^{\pm}]=0=[H_{i,r},X_{j,s}^{\pm}]$ if $1<|j-i|<n-1$. If $\beta=\frac{\lambda}{2}$, relations (5)- (7) reduce to (2),(3). We should also note that $\widehat{\mathbf{Y}}_{\beta_1,\lambda_1}\cong\widehat{\mathbf{Y}}_{\beta_2,\lambda_2}$ if $\beta_2=\gamma\beta_1$ and $\lambda_2=\gamma\lambda_1$ for some $\gamma\neq 0$.*

In [17], we considered instead the following algebra.

**Definition 3.4.** *The loop Yangian $LY_{\lambda,\beta}$ is the quotient of $\widehat{\mathbf{Y}}_{\lambda,\beta}$ by the ideal generated by the central element $H_{0,0}+\ldots+H_{n-1,0}$.*

One useful observation is that the Yangian $Y_\lambda$ (resp. $\widehat{\mathbf{Y}}_{\lambda,\beta}$) is generated by $X_{i,r}^{\pm},H_{i,r},i=1,\ldots,n-1$ (resp. $i=0,\ldots,n-1$) with $r=0,1$ only. The other elements are obtained inductively by the formulas:

$$X_{i,r+1}^{\pm}=\pm\frac{1}{2}[H_{i,1},X_{i,r}^{\pm}]-\frac{\lambda}{2}(H_iX_{i,r}^{\pm}+X_{i,r}^{\pm}H_i),\quad H_{i,r+1}=[X_{i,r}^+,X_{i,1}^-]. \qquad (8)$$

Furthermore, the subalgebra generated by the elements with $r=0$ is isomorphic to the enveloping algebra of the Lie algebra $\mathfrak{sl}_n$ (resp. $\widehat{\mathfrak{sl}}_n[u]$, the universal central extension of $\mathfrak{sl}_n[u^{\pm 1}]$) and the subalgebra $Y_{\lambda,\beta}^0$ generated by the elements with $i\neq 0$ is an epimorphic image of $Y_\lambda$. (Actually, the PBW theorem proved in section 7 implies that $Y_{\lambda,\beta}^0\cong Y_\lambda$ - see corollary 7.1.) Therefore, the affine Yangian $\widehat{\mathbf{Y}}_{\beta,\lambda}$ contains $Y_\lambda$ and a copy of $\mathfrak{U}(\widehat{\mathfrak{sl}}_n[u^{\pm 1}])$, which together generate $\widehat{\mathbf{Y}}_{\beta,\lambda}$.

In [17], the following lemma was proved.

**Lemma 3.1.** *It is possible to define an algebra automorphism $\rho$ of $\widehat{\mathbf{Y}}_{\lambda,\beta}$ by setting*

$$\rho(H_{i,r}) = \sum_{s=0}^{r} \begin{pmatrix} r \\ s \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-s} H_{i-1,s}, \quad \rho(X_{i,r}^{\pm}) = \sum_{s=0}^{r} \begin{pmatrix} r \\ s \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-s} X_{i-1,s}^{\pm} \quad \textit{for } i \neq 0,1$$

$$\rho(H_{i,r}) = \sum_{s=0}^{r} \begin{pmatrix} r \\ s \end{pmatrix} \beta^{r-s} H_{i-1,s}, \quad \rho(X_{i,r}^{\pm}) = \sum_{s=0}^{r} \begin{pmatrix} r \\ s \end{pmatrix} \beta^{r-s} X_{i-1,s}^{\pm} \quad \textit{for } i = 0,1$$

The following subalgebra of the affine Yangians will also be of interest in view of theorem 8.1 in [17].

**Definition 3.5.** *Let $\lambda, \beta \in \mathbb{C}$. We define $\mathsf{L}_{\lambda,\beta}$ to be the subalgebra of $\widehat{\mathbf{Y}}_{\lambda,\beta}$ generated by the elements $X_{i,r}^{\pm}, H_{i,r}, X_{0,r}^{+}$ for $1 \leq i \leq n-1, r \geq 0$ and by $X_{0,r}^{-}$ for $r \geq 1$.*

We will denote by $\mathbf{K}_r(z)$ the element $z \otimes u^r$ of $\mathfrak{sl}_n[u^{\pm 1}] \subset \widehat{\mathfrak{sl}}_n[u^{\pm 1}] \subset \widehat{\mathbf{Y}}_{\lambda,\beta}$. It was noted in [17] that, because of the involution $\iota$ on $\mathsf{L}_{\lambda,\beta}$ (see proposition 8.1 in [17]), the subalgebra of $\widehat{\mathbf{Y}}_{\lambda,\beta}$ generated by the elements $X_{i,0}^{\pm}, H_{i,0}$ for $1 \leq i \leq n-1$ and by $X_{0,1}^{-}$ is isomorphic to $\mathfrak{U}(\mathfrak{sl}_n[w])$, so we can denote by $\mathbf{Q}_r(z)$ the element $z \otimes w^r$ of this copy of $\mathfrak{sl}_n[w]$ inside $\widehat{\mathbf{Y}}_{\lambda,\beta}$. In particular, $\mathbf{K}_1(E_{n1}) = X_0^{+}$ and $\mathbf{Q}_1(E_{1n}) = X_{0,1}^{-}$. We set $\mathbf{K}(z) = \mathbf{K}_1(z), \mathbf{Q}(z) = \mathbf{Q}_1(z)$.

In this paper, it will be important to have a simpler definition of the Yangians $Y_\lambda$ and $\widehat{\mathbf{Y}}_{\lambda,\beta}$ - see proposition 3.1 below. We start with a series of lemmas.

**Lemma 3.2.** *The Lie algebra $\mathfrak{sl}_n[v]$ is isomorphic to the Lie algebra $L$ generated by the elements $\mathtt{X}_{i,r}^{\pm}, \mathtt{H}_{i,r}, 1 \leq i \leq n-1, r = 0,1,$ with the relations:*

$$[\mathtt{H}_{i,r}, \mathtt{H}_{j,s}] = 0,\ r,s = 0 \textit{ or } 1 \quad [\mathtt{H}_{i,0}, \mathtt{X}_{j,s}^{\pm}] = \pm c_{ij} \mathtt{X}_{j,s}^{\pm},\ s = 0 \textit{ or } 1 \tag{9}$$

$$[\mathtt{H}_{i,1}, \mathtt{X}_{j,0}^{\pm}] = [\mathtt{H}_{i,0}, \mathtt{X}_{j,1}^{\pm}],\ [\mathtt{X}_{i,0}^{+}, \mathtt{X}_{j,0}^{-}] = \delta_{ij}\mathtt{H}_{i,0},\ [\mathtt{X}_{i,1}^{+}, \mathtt{X}_{j,0}^{-}] = [\mathtt{X}_{i,0}^{+}, \mathtt{X}_{j,1}^{-}] = \delta_{ij}\mathtt{H}_{i,1} \tag{10}$$

$$[\mathtt{X}_{i,r}^{\pm}, \mathtt{X}_{j,s}^{\pm}] = 0 \textit{ if } 1 < |i-j| < n-1, r,s = 0 \textit{ or } 1, \quad [\mathtt{X}_{i,1}^{\pm}, \mathtt{X}_{j,0}^{\pm}] = [\mathtt{X}_{i,0}^{\pm}, \mathtt{X}_{j,1}^{\pm}] \tag{11}$$

$$\left[\mathtt{X}_{i,r}^{\pm}, [\mathtt{X}_{i,r}^{\pm}, \mathtt{X}_{j,s}^{\pm}]\right] = 0 \textit{ if } (r,s) = (0,0), (0,1) \textit{ or } (1,0). \tag{12}$$

For an arbitrary associative algebra $A$, $\mathfrak{sl}_n(A)$ is defined as the derived Lie algebra $[\mathfrak{gl}_n(A), \mathfrak{gl}_n(A)]$. If $A$ is commutative, the kernel of the universal central extension $\widehat{\mathfrak{sl}}_n(A)$ of $\mathfrak{sl}_n(A)$ is isomorphic to $\Omega^1(A)/dA$, the space of 1-form on the affine variety $\mathrm{Spec}(A)$ modulo the exact forms - see [21]. As vector spaces, we can write $\widehat{\mathfrak{sl}}_n(A) \cong \mathfrak{sl}_n(A) \oplus \Omega^1(A)/dA$ and, via this identification, the bracket on $\widehat{\mathfrak{sl}}_n(A)$ is given by $[z_1 \otimes a_1, z_2 \otimes a_2] = [z_1, z_2] \otimes a_1 \cdot a_2 + (z_1, z_2) a_2 da_1$ where $(\cdot, \cdot)$ is the Killing form. We will be interested in the cases $A = \mathbb{C}[u,v]$ and $A = \mathbb{C}[u^{\pm 1}, v]$, the case $A = \mathbb{C}[u^{\pm 1}, v^{\pm 1}]$ being treated in [25].

We can put a filtration on $\widehat{\mathbf{Y}}_{\lambda,\beta}$ by giving $X_{i,r}^{\pm}, H_{i,r}$ degree $r$. The associated graded ring $gr(\widehat{\mathbf{Y}}_{\lambda,\beta})$ is an epimorphic image of $\mathfrak{U}\big(\widehat{\mathfrak{sl}}_n[u^{\pm 1}, v]\big)$. Indeed, if $\lambda = \beta = 0$, $\widehat{\mathbf{Y}}_{\lambda,\beta}$ is exactly the enveloping algebra of $\widehat{\mathfrak{sl}}_n[u^{\pm 1}, v]$: this can be proved in exactly the same way as proposition 3.5 in [25]. This means that we have a map $\widehat{\mathbf{Y}}_{\lambda=0,\beta=0} \longrightarrow \mathfrak{U}(\mathfrak{sl}_n[u^{\pm 1}, v])$ which we can restrict to $\mathsf{L}_{\lambda=0,\beta=0} \longrightarrow \mathfrak{U}(\mathfrak{sl}_n[u^{\pm 1}, v])$. Thus we see that the subalgebra $\mathsf{L}_{\lambda=0,\beta=0}$ is the enveloping algebra of a Lie algebra $\widetilde{\mathsf{L}}$ which is a central extension of $\mathfrak{sl}_n[u,w]$ where $w = u^{-1}v$. Therefore, we also have a map $\widehat{\mathfrak{sl}}_n[u,w] \longrightarrow \widetilde{\mathsf{L}}$. The Lie algebra $\widehat{\mathfrak{sl}}_n[u,w]$ can be identified with a Lie subalgebra of $\widehat{\mathfrak{sl}}_n[u^{\pm 1}, v]$ via $\mathfrak{sl}_n[u,w] \hookrightarrow \mathfrak{sl}_n[u,v], \Omega^1(\mathbb{C}[u,w])/d(\mathbb{C}[u,w]) \hookrightarrow \Omega^1(\mathbb{C}[u^{\pm 1}, v])/d(\mathbb{C}[u^{\pm 1}, v])$, and, via this embedding, $\widehat{\mathfrak{sl}}_n[u,w]$ becomes identified with $\widetilde{\mathsf{L}}$.

**Lemma 3.3.** *The Lie algebra $\widehat{\mathfrak{sl}}_n[u^{\pm 1}, v]$ is isomorphic to the algebra $\widetilde{\mathbf{L}}$ generated by $\mathtt{X}_{i,r}^{\pm}, \mathtt{H}_{i,r}, 0 \leq i \leq n-1, r = 0,1$ with the same relations as those for $L$ in lemma 3.2 extended to $0 \leq i,j \leq n-1$.*

*Proof.* This follows from lemma 3.2 by using the automorphism $\rho$ in the case $\lambda = \beta = 0$. $\square$

**Lemma 3.4.** *[13] The Lie subalgebra* $\mathbf{b}^{\pm}$ *of* $\widehat{\mathfrak{sl}}_n[u^{\pm 1}, v]$ *generated by* $X^{\pm}_{i,r}, 0 \le i \le n-1, r \ge 0$ *is isomorphic to the Lie algebra generated by these elements and satisfying only the relations*

$$[X^{\pm}_{i,r+1}, X^{\pm}_{j,s}] = [X^{\pm}_{i,r}, X^{\pm}_{j,s+1}],\ \forall i,j,\ \ [X^{\pm}_{i,r}, X^{\pm}_{j,s}] = 0 \text{ if } 1 < |i-j| < n-1 \tag{13}$$

$$[X^{\pm}_{i,r_1}, [X^{\pm}_{i,r_2}, X^{\pm}_{j,s}]] = 0 \text{ if } i-j \equiv \pm 1 \bmod n. \tag{14}$$

The Lie algebra $\widehat{\mathfrak{sl}}_n[u^{\pm 1}, v]$ is graded by giving the generators $X^{\pm}_{i,r}, H_{i,r}$ degree $r$. We have a Lie algebra monomorphism $\widehat{\mathfrak{sl}}_n[u^{\pm 1}] \longrightarrow \widehat{\mathfrak{sl}}_n[u^{\pm 1}, v]$ and we can consider the weight space decomposition of $\widehat{\mathfrak{sl}}_n[u^{\pm 1}, v]$ with respect to $\widehat{\mathfrak{d}}$, the Cartan subalgebra of $\widehat{\mathfrak{sl}}_n[u^{\pm 1}]$. We denote by $\mathbf{W}^r_\alpha$ the space of elements of $\widehat{\mathfrak{sl}}_n[u^{\pm 1}, v]$ of degree $r$ and weight $\alpha \in \widehat{\mathfrak{d}}^*$ and set $\mathbf{W}_\alpha = \sum_{r=0}^{\infty} \mathbf{W}^r_\alpha$. One can prove, exactly as in [25], that $\mathbf{W}^r_\alpha$ is one-dimensional if $r \ge 0$ and $\alpha$ is a real root and $\mathbf{W}^r_\alpha = \{0\}$ if $\alpha \ne 0$ and $\alpha$ is not a root of $\widehat{\mathfrak{d}}$. Consequently, the kernel **Ker** of the epimorphism $\widehat{\mathfrak{sl}}_n[u^{\pm 1}, v] \twoheadrightarrow \mathfrak{sl}_n \otimes_{\mathbb{C}} \mathbb{C}[u^{\pm 1}, v]$ is contained in $\oplus_{k \in \mathbb{Z}} \mathbf{W}_{k\delta}$.

**Lemma 3.5.** *The Lie algebra* $\widehat{\mathfrak{sl}}_n[u, v]$ *is isomorphic to the Lie algebra* $\mathfrak{k}$ *generated by* $X^{\pm}_{i,r}, H_{i,r}, 1 \le i \le n-1, r \ge 0$ *and* $X^{+}_{0,r}, r \ge 0$*, with the relations (1)-(7) in the case* $\lambda = \beta = 0$*, except those which involve* $X^{-}_{0,r}, H_{0,r}, r \ge 0$.

*Proof.* Let $\mathfrak{k}^{\pm}$ be the Lie subalgebra of $\mathfrak{k}$ generated by $X^{\pm}_{i,r}, r \ge 0$ with $0 \le i \le n-1$ in the "+" case and $1 \le i \le n-1$ in the "-" case, and let $\mathfrak{k}^0$ be the abelian Lie subalgebra generated by $H_{i,r}, r \ge 0, 1 \le i \le n-1$. It follows from the definition of $\mathfrak{k}$ that $\mathfrak{k} = \mathfrak{k}^- + \mathfrak{k}^0 + \mathfrak{k}^+$ and $\mathfrak{k}^+ \cong \mathbf{b}^+$ according to lemma 3.4. We have a map $f_1 : \mathfrak{k} \longrightarrow \widehat{\mathfrak{sl}}_n[u, v]$ given by, for $1 \le i \le n-1, r \ge 0$:

$$X^{+}_{i,r} \mapsto E_{i,i+1} \otimes v^r,\ \ X^{-}_{i,r} \mapsto E_{i+1,i} \otimes v^r,\ \ H_{i,r} \mapsto (E_{ii} - E_{i+1,i+1}) \otimes v^r,\ \ X^{+}_{0,r} \mapsto E_{n1} \otimes u v^r.$$

The kernel of the composite $\pi \circ f_1$ (where $\pi : \widehat{\mathfrak{sl}}_n[u, v] \longrightarrow \mathfrak{sl}_n[u, v]$) must be central because of the weight space decomposition of $\mathfrak{k}^+$ described above, so there exist also a map $f_2 : \widehat{\mathfrak{sl}}_n[u, v] \longrightarrow \mathfrak{k}$. Since $\widehat{\mathfrak{sl}}_n[u, v]$ and $\mathfrak{k}$ are perfect Lie algebras and $f_2 \circ f_1, f_1 \circ f_2$ are endomorphisms of $\mathfrak{k}$ and $\widehat{\mathfrak{sl}}_n[u, v]$, respectively, over the identity map on $\mathfrak{sl}_n[u, v]$, they must be equal to the identity according to the following well-known lemma. □

**Lemma 3.6.** *Let* $\pi : \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ *be a central extension of the Lie algebra* $\mathfrak{g}$ *with* $\widehat{\mathfrak{g}}$ *perfect. If* $\eta : \widehat{\mathfrak{g}} \longrightarrow \widehat{\mathfrak{g}}$ *is a Lie endomorphism which induces the identity map on* $\mathfrak{g}$*, then* $\eta$ *is the identity.*

**Lemma 3.7.** *The Lie algebra* $\widehat{\mathfrak{sl}}_n[u, v]$ *is isomorphic to the Lie algebra* $\mathfrak{t}$ *generated by* $\mathtt{X}^{\pm}_{i,r}, \mathtt{H}_{i,r}, 1 \le i \le n-1, r = 0, 1$ *and* $\mathtt{X}^{+}_{0,r}, r = 0, 1$ *satisfying the relations (9)-(12) for* $0 \le i \le n-1$ *except those involving* $\mathtt{X}^{-}_{0,r}, \mathtt{H}_{0,r}, r = 0, 1$.

*Proof.* We know from lemma 3.2 that the generators of $\mathfrak{t}$ with $1 \le i \le n-1$ generate a Lie subalgebra which is an epimorphic image of $\mathfrak{sl}_n[v]$, so we only have to check the relations in lemma 3.5 which involve $X^{+}_{0,r}$. We have elements $\mathtt{X}^{\pm}_{i,r}, \mathtt{H}_{i,r}$ in $\mathfrak{t}$ which are the images of $X^{\pm}_i \otimes v^r, H_i \otimes v^r \in \mathfrak{sl}_n[v]$ under $\mathfrak{sl}_n[v] \longrightarrow \mathfrak{t}$.

Define inductively $\mathtt{X}^{+}_{0,r}$ by $\mathtt{X}^{+}_{0,r} = -[\mathtt{H}_{n-1,1}, \mathtt{X}^{+}_{0,r-1}]$. Since $[\mathtt{H}_{n-1,1}, \mathtt{X}^{+}_{0,0}] = [\mathtt{H}_{1,1}, \mathtt{X}^{+}_{0,0}]$, we also have $\mathtt{X}^{+}_{0,r} = -[\mathtt{H}_{1,1}, \mathtt{X}^{+}_{0,r-1}]$. We have to verify the following relations:

1. $[\mathtt{X}^{+}_{0,r}, \mathtt{X}^{-}_{i,s}] = 0\ \forall\, 1 \le i \le n-1, \forall\, r, s \ge 0$.
2. $[\mathtt{X}^{+}_{i,r}, \mathtt{X}^{+}_{0,s}] = 0$ if $i \ne 1, n-1$.
3. $[\mathtt{X}^{+}_{i,r+1}, \mathtt{X}^{+}_{0,s}] = [\mathtt{X}^{+}_{i,r}, \mathtt{X}^{+}_{0,s+1}]$ if $i = 1, n-1$, $\forall\, r, s \ge 0$.
4. $[\mathtt{X}^{+}_{0,r}, \mathtt{X}^{+}_{0,s}] = 0\ \forall\, r, s \ge 0$.

5. $\left[\mathtt{X}^+_{i,r_1},[\mathtt{X}^+_{i,r_2},\mathtt{X}^+_{0,s}]\right]=0$ if $i=1,n-1$.

1. If $2\le i\le n-2, \mathtt{X}^-_{i,s}=\frac{1}{2^s}[\mathtt{H}_{i,1},[\mathtt{H}_{i,1},\cdots,[\mathtt{H}_{i,1},\mathtt{X}^-_{i,0}]\cdots]]$ and $\mathtt{X}^-_{1,s}=[\mathtt{H}_{2,1},[\mathtt{H}_{2,1},\ldots,[\mathtt{H}_{2,1},\mathtt{X}^-_{1,0}]\ldots]], \mathtt{X}^-_{n-1,s}=[\mathtt{H}_{n-2,1},[\mathtt{H}_{n-2,1},\ldots,[\mathtt{H}_{n-2,1},\mathtt{X}^-_{n-1,0}]\ldots]]$.

Then $[\mathtt{X}^+_{0,0},\mathtt{X}^-_{i,s}]=0$ since $[\mathtt{H}_{i,1},\mathtt{X}^+_{0,0}]$ for $2\le i\le n-2$. The general case follows by induction on $r$.

2. The proof is the same as for (1), with $\mathtt{X}^+_{i,r}=\frac{1}{2^r}[\mathtt{H}_{i,1},[\mathtt{H}_{i,1},\ldots,[\mathtt{H}_{i,1},\mathtt{X}^+_{i,0}]\ldots]]$ ($r$ times).

3. We use induction on $r$ and prove it only for $i=n-1$. Let us assume that the equality is true when $r=0$ and for arbitrary $s$. Suppose that $r\ge 1$.

$$\begin{aligned}[\mathtt{X}^+_{n-1,r+1},\mathtt{X}^+_{0,s}] &= \frac{1}{2}\left[[\mathtt{H}_{n-1,1},\mathtt{X}^+_{n-1,r}],\mathtt{X}^+_{0,s}\right]=\frac{1}{2}\left[[\mathtt{H}_{n-1,1},\mathtt{X}^+_{0,s}],\mathtt{X}^+_{n-1,r}\right]+\frac{1}{2}\left[\mathtt{H}_{n-1,1},[\mathtt{X}^+_{n-1,r},\mathtt{X}^+_{0,s}]\right]\\ &= -\frac{1}{2}[\mathtt{X}^+_{0,s+1},\mathtt{X}^+_{n-1,r}]+\frac{1}{2}\left[\mathtt{H}_{n-1,1},[\mathtt{X}^+_{n-1,r-1},\mathtt{X}^+_{0,s+1}]\right]\\ &= \frac{1}{2}[\mathtt{X}^+_{n-1,r},\mathtt{X}^+_{0,s+1}]+[\mathtt{X}^+_{n-1,r},\mathtt{X}^+_{0,s+1}]-\frac{1}{2}[\mathtt{X}^+_{n-1,r-1},\mathtt{X}^+_{0,s+2}]=[\mathtt{X}^+_{n-1,r},\mathtt{X}^+_{0,s+1}]\end{aligned}$$

We are left to prove (3) when $r=0, s\ge 0$. We use induction on $s$ and the identity $\mathtt{X}^+_{0,s+1}=-[\mathtt{H}_{1,1},\mathtt{X}^+_{0,s}]$. Then we obtain

$$[\mathtt{X}^+_{n-1,0},\mathtt{X}^+_{0,s+1}]=-\left[\mathtt{H}_{1,1},[\mathtt{X}^+_{n-1,0},\mathtt{X}^+_{0,s}]\right]=-\left[\mathtt{H}_{1,1},[\mathtt{X}^+_{n-1,1},\mathtt{X}^+_{0,s-1}]\right]=[\mathtt{X}^+_{n-1,1},\mathtt{X}^+_{0,s}].$$

4. We proceed by induction on $r+s$. (By assumption, (4) holds for $r+s=0,1$.)

$$[\mathtt{X}^+_{0,r},\mathtt{X}^+_{0,s}]=-\left[[\mathtt{X}^+_{0,r},\mathtt{H}_{n-1,1}],\mathtt{X}^+_{0,s-1}\right]-\left[\mathtt{H}_{n-1,1},[\mathtt{X}^+_{0,r},\mathtt{X}^+_{0,s-1}]\right]=-[\mathtt{X}^+_{0,r+1},\mathtt{X}^+_{0,s-1}].$$

Thus, $[\mathtt{X}^+_{0,r+1},\mathtt{X}^+_{0,s-1}]=[\mathtt{X}^+_{0,r-1},\mathtt{X}^+_{0,s+1}]$. If $r+s$ is even, we get $[\mathtt{X}^+_{0,r+s},\mathtt{X}^+_{0,0}]=[\mathtt{X}^+_{0,0},\mathtt{X}^+_{0,s+r}]$, so $[\mathtt{X}^+_{0,r+s},\mathtt{X}^+_{0,0}]=0$ and $[\mathtt{X}^+_{0,r},\mathtt{X}^+_{0,s}]=0$.

If $r+s$ is odd, we use (1) and (4) to deduce that $[\mathtt{H}_{n-1,2},\mathtt{X}^+_{0,s-2}]=-\mathtt{X}^+_{0,s}$. Proceeding by induction on $r+s$, we obtain

$$[\mathtt{X}^+_{0,r},\mathtt{X}^+_{0,s}]=-\left[[\mathtt{X}^+_{0,r},\mathtt{H}_{n-1,2}],\mathtt{X}_{0,s-2}\right]-\left[\mathtt{H}_{n-1,2},[\mathtt{X}^+_{0,r},\mathtt{X}^+_{0,s-2}]\right]=-[\mathtt{X}^+_{0,r+2},\mathtt{X}^+_{0,s-2}].$$

Therefore, supposing, without loss of generality, that $r$ is odd and $s$ is even, we obtain

$$[\mathtt{X}^+_{0,r},\mathtt{X}^+_{0,s}]=[\mathtt{X}^+_{0,r+s},\mathtt{X}^+_{0,0}]=-[\mathtt{X}^+_{0,r+s-2},\mathtt{X}^+_{0,2}]=[\mathtt{X}_{0,r+s-3},\mathtt{X}^+_{0,3}]=[\mathtt{X}^+_{0,0},\mathtt{X}^+_{0,r+s}].$$

Therefore, $[\mathtt{X}^+_{0,r+s},\mathtt{X}^+_{0,0}]=0=[\mathtt{X}^+_{0,r},\mathtt{X}^+_{0,s}]$.

5. We write $[\mathtt{X}^+_{n-1,r_1},[\mathtt{X}^+_{n-1,r_2},\mathtt{X}^+_{0,s}]]=[\mathtt{X}^+_{n-1,0},[\mathtt{X}^+_{n-1,0},\mathtt{X}^+_{0,r_1+r_2+s}]]$ using (4) and express $\mathtt{X}^+_{0,r_1+r_2+s}$ as $\mathtt{X}^+_{0,r_1+r_2+s}=(-1)^{r_1+r_2+s}[\mathtt{H}_{1,1},[\mathtt{H}_{1,1},\ldots,[\mathtt{H}_{1,1},\mathtt{X}^+_{0,0}]\ldots]]$ ($\mathtt{H}_{1,1}$ appears $r_1+r_2+s$ times). Then $[\mathtt{X}^+_{n-1,0},\mathtt{H}_{1,1}]=0$ and the result follows from the case $r_1=r_2=s=0$. The case $i=1$ is identical. □

We recall the following theorem established in [21].

**Theorem 3.1.** *[21] Let $A$ be an associative algebra over $\mathbb{C}$. The universal central extension $\widehat{\mathfrak{sl}}_n(A)$ of $\mathfrak{sl}_n(A)$ is the Lie algebra generated by elements $F_{ij}(a), 1\le i\ne j\le n, a\in A$, satisfying the following relations:*

$$F_{ij}(t_1a_1+t_2a_2)=t_1F_{ij}(a_1)+t_2F_{ij}(a_2)\quad t_1,t_2\in\mathbb{C},a_1,a_2\in A \tag{15}$$

$$[F_{ij}(a),F_{jk}(b)]=F_{ik}(ab)\ \text{if}\ i\ne j\ne k\ne i \tag{16}$$

$$[F_{ij}(a),F_{kl}(b)]=0\ \text{if}\ i\ne j\ne k\ne l\ne i \tag{17}$$

We would like to give an equivalent definition of $\widehat{\mathfrak{sl}}_n(\mathbb{C}[u,v])$. This will be useful in section 13.

**Lemma 3.8.** *The universal central extension $\widehat{\mathfrak{sl}}_n[u,v]$ can be defined as the Lie algebra $\mathfrak{S}$ generated by elements $K_{ij}(u), Q_{ij}(v)$ and $P_{ij}(w)$ with the following relations : there are Lie algebra homomorphisms $\mathfrak{sl}_n[u], \mathfrak{sl}_n[v], \mathfrak{sl}_n[w] \longrightarrow \mathfrak{S}, E_{ij}\otimes u, E_{ij}\otimes v, E_{ij}\otimes w \mapsto K_{ij}(u), Q_{ij}(v), P_{ij}(w)$, and we also have the relations*

$$[K_{ij}(u), Q_{jk}(v)] = P_{ik}(w) \text{ if } i\neq j\neq k\neq i \tag{18}$$

$$[K_{ij}(u), Q_{kl}(v)] = 0 = [P_{ij}(w), K_{kl}(u)] = [P_{ij}(w), Q_{kl}(v)] \text{ if } i\neq j\neq k\neq l\neq i \tag{19}$$

*Proof.* We have a map $\mathfrak{S} \longrightarrow \widehat{\mathfrak{sl}}_n[u,v]$ sending $K_{ij}(u)\mapsto F_{ij}(u), Q_{ij}(v)\mapsto F_{ij}(v), P_{ij}(w)\mapsto F_{ij}(uv)$ in the notation of theorem 3.1. On the other hand, $\widehat{\mathfrak{sl}}_n[u,v]$ is isomorphic to the Lie algebra $\mathfrak{t}$ (in lemma 3.7) and we have a map $\mathfrak{t}\longrightarrow\mathfrak{S}$ given by

$$\mathtt{X}^+_{i,r}\mapsto Q_{i,i+1}(v^r),\ \ \mathtt{X}^-_{i,r}\mapsto Q_{i+1,i}(v^r),\ \ \mathtt{H}_{i,r}\mapsto [Q_{i,i+1}(v^r), Q_{i+1,i}(1)],\ r=0,1, 1\leq i\leq n$$

$$\mathtt{X}^+_{0,0}\mapsto K_{n1}(u),\ \ \mathtt{X}^+_{0,1}\mapsto P_{n1}(w)$$

The composite of this map with $\mathfrak{S}\longrightarrow\widehat{\mathfrak{sl}}_n[u,v]$ is the identify. Therefore, $\mathfrak{t}\longrightarrow\mathfrak{S}$ is injective. From the definitions, it is also surjective, hence an isomorphism. □

We can now give two simpler definitions of the Yangians $Y_\lambda$ and $\widehat{\mathbf{Y}}_{\lambda,\beta}$.

**Proposition 3.1.** *The Yangian $Y_\lambda$ (resp. $\widehat{\mathbf{Y}}_{\lambda,\beta}$) can be defined as the algebra $\widetilde{Y}_\lambda$ (resp. $\widetilde{\mathbf{Y}}_{\lambda,\beta}$) generated by elements $X^\pm_{i,r}, H_{i,r}, 1\leq i\leq n-1, r=0,1$ (resp. $0\leq i\leq n-1$) satisfying the same set of relations as in definition 3.2 (resp. 3.3), except that $r$ and $s$ only take values in $\{0,1\}$: more precisely, in relation (4), we have $r_1=r_2=r, (r,s)=(0,0),(0,1),(1,0)$, whereas $r=s=0$ in relations (2),(3) (resp. also in (5),(6),(7)) and $r+s=0,1$ in the rightmost relation in (1). As for $[H_{i,r},H_{j,s}]=0$, it must hold for $r,s=0$ or $1$.*

*Proof.* We have an epimorphism $\widetilde{Y}_\lambda \twoheadrightarrow Y_\lambda$. Considering the associated graded map and using lemma 3.2, we obtain a sequence of three maps $\mathfrak{U}(\mathfrak{sl}_n[v])\twoheadrightarrow gr(\widetilde{Y}_\lambda)\twoheadrightarrow gr(Y_\lambda)$. The PBW property of $Y_\lambda$ (proved in [22]) says that the composite is an isomorphism. Therefore, $gr(\widetilde{Y}_\lambda)\twoheadrightarrow gr(Y_\lambda)$ is injective and $\widetilde{Y}_\lambda$ is isomorphic to $Y_\lambda$. The statement for the affine Yangian follows immediately from the finite case using the automorphism $\rho$. □

Another simpler definition of $Y_\lambda$, which is also valid in the $A_1$ case, was given in [23]. His definition follows directly from the one given in proposition 3.1 (when $n\geq 4$). Showing this amounts to proving that the relation $[H_{i,1},[X^+_{i,1},X^-_{i,1}]]=0$ holds in $\widetilde{Y}_\lambda$.

Later, we will also need a simpler definition of the Yangian $Y_\lambda$ which is closer to definition 3.1.

**Lemma 3.9.** *The Yangian $Y_\lambda$ is isomorphic to the algebra $\overline{Y}_\lambda$ generated by elements $\overline{X}^\pm_i, \overline{H}_i$ for $1\leq i\leq n-1$ and by $\overline{X}^{+,\pm}_0$ which satisfy the following relations: the elements with $i\neq 0$ satisfy the Serre relations for $\mathfrak{sl}_n$ and those with $i=0$ satisfy:*

$$[\overline{X}^+_1,[\overline{X}^+_1,\overline{X}^{+,-}_0]] = 0 = [\overline{X}^{+,-}_0,[\overline{X}^{+,-}_0,\overline{X}^+_1]] \text{ and the same with } \overline{X}^+_{n-1},\overline{X}^{+,+}_0 \text{ instead of } \overline{X}^+_1,\overline{X}^{+,-}_0 \tag{20}$$

$$\overline{X}^{+,+}_0 - \overline{X}^{+,-}_0 = \frac{\lambda}{2}\sum_{1\leq i\neq j\leq n-1}\big([E_{n1},E_{ij}]E_{ji} + E_{ji}[E_{n1},E_{ij}]\big) \tag{21}$$

$$[\overline{X}^{+,\pm}_0,\overline{X}^+_i] = 0 = [\overline{X}^{+,\pm}_0,\overline{X}^-_i],\ i=2,\dots,n-2 \tag{22}$$

*Proof.* Starting from definition 3.1 of $Y_\lambda$, we choose $\alpha_1, \dots, \alpha_{n-2}, \alpha_{n1}$ as a basis of simple roots for $\Delta$ and apply the Drinfeld isomorphism to $J(E_{n1})$ - see the formulas after definition 3.2. We obtain an element of $Y_\lambda$ which we denote by $X_0^{+,-}$ and which satisfies relations (20). The element $X_0^{+,+}$ is defined similarly, choosing this time $\alpha_2, \dots, \alpha_{n-1}, \alpha_{n1}$ as a basis of simple roots. Relations (21) and (22) follows from the Drinfeld isomorphism.

The elements $X_i^{\pm}, H_i, X_0^{+,\pm}$ generate $Y_\lambda$, so we have an epimorphism $\overline{Y}_\lambda \twoheadrightarrow Y_\lambda$. There are filtrations on both algebras ($X_0^{+,\pm}, \overline{X}_0^{+,\pm}$ are given degree 1) and, therefore, associated graded maps $\mathfrak{U}(\mathfrak{sl}_n[v]) \twoheadrightarrow gr(\overline{Y}_\lambda) \twoheadrightarrow gr(Y_\lambda)$. The composite is an isomorphism because of the PBW property of $Y_\lambda$ [22]. Therefore, $\overline{Y}_\lambda \xrightarrow{\sim} Y_\lambda$. □

We can simplify even more the definitions of $Y_\lambda$ and $\widehat{\mathbf{Y}}_{\lambda,\beta}$ given in proposition 3.1.

**Lemma 3.10.** *The relations* $[X_{i,1}^{\pm}, [X_i^{\pm}, X_j^{\pm}]] + [X_i^{\pm}, [X_{i,1}^{\pm}, X_j^{\pm}]] = 0$ *and* $[X_i^{\pm}, [X_i^{\pm}, X_{j,1}^{\pm}]] = 0$ *in* $\widehat{\mathbf{Y}}_{\lambda,\beta}$ *follow from the relations (2)-(3),(5)-(7) with* $r = s = 0$*, (4) with* $r_1 = r_2 = s = 0$ *and the second relation in (1) with* $s = 0$ *or* 1*.*

*Proof.* We prove it in the + case with $i = 1, j = 0$, the other cases being similar. We apply $[H_{2,1}, \cdot]$ to $[X_1^+, [X_1^+, X_0^+]] = 0$ and obtain

$$-\big[X_{1,1}^+, [X_1^+, X_0^+]\big] - \big[X_1^+, [X_{1,1}^+, X_0^+]\big] - \frac{\lambda}{2}\big[S(H_2, X_1^+), [X_1^+, X_0^+]\big] - \frac{\lambda}{2}\big[X_1^+, [S(H_2, X_1^+), X_0^+]\big] = 0.$$

This simplifies to $[X_{1,1}^+, [X_1^+, X_0^+]] + [X_1^+, [X_{1,1}^+, X_0^+]] = 0$. To obtain the relation $[X_1^+, [X_1^+, X_{0,1}^+]] = 0$, we apply instead $[H_{n-1,1}, \cdot]$. □

**Lemma 3.11.** *The relation* $[X_{i,1}^{\pm}, [X_{i,1}^{\pm}, X_{i-1}^{\pm}]] = 0$ *follows from the other relations in proposition 3.1. (The same is true for* $X_{i,1}^{\pm}, X_{i+1}^{\pm}$*.)*

*Proof.* We prove $[X_{0,1}^+, [X_{0,1}^+, X_{n-1}^+]] = 0$ only. From lemma 3.10, we know that $\big[[X_0^+, X_{n-1,1}^+], X_{n-1}^+\big] + \big[[X_0^+, X_{n-1}^+], X_{n-1,1}^+\big] = 0$, so, applying $[\cdot, X_{n-1}^-]$, we obtain

$$\big[[X_0^+, H_{n-1,1}], X_{n-1}^+\big] + \big[[X_0^+, H_{n-1}], X_{n-1,1}^+\big] + \big[[X_0^+, X_{n-1,1}^+], H_{n-1}\big] + \big[[X_0^+, X_{n-1}^+], H_{n-1,1}\big] = 0,$$

hence $[[X_0^+, H_{n-1,1}], X_{n-1}^+] + [[X_0^+, X_{n-1}^+], H_{n-1,1}] = 0$ and $2[[X_0^+, H_{n-1,1}], X_{n-1}^+] + [X_0^+, [X_{n-1}^+, H_{n-1,1}]] = 0$. Writing $[X_{n-1}^+, H_{n-1,1}] = -2X_{n-1,1}^+ - \lambda(X_{n-1}^+ H_{n-1} + H_{n-1} X_{n-1}^+)$, we conclude that

$$\big[[X_0^+, H_{n-1,1}], X_{n-1}^+\big] = [X_0^+, X_{n-1,1}^+] + \frac{\lambda}{2}[X_0^+, X_{n-1}^+ H_{n-1} + H_{n-1} X_{n-1}^+]$$

We also need that $\big[[X_0^+, H_{1,1}], [X_0^+, X_{n-1,1}^+]\big] = \Big[X_0^+, \big[[X_0^+, H_{1,1}], X_{n-1,1}^+\big]\Big] = \Big[X_0^+, \big[[X_0^+, X_{n-1,1}^+], H_{1,1}\big]\Big] = \big[[X_0^+, X_{n-1,1}^+], [X_0^+, H_{1,1}]\big]$. Comparing the first and last terms yields $[[X_0^+, H_{1,1}], [X_0^+, X_{n-1,1}^+]] = 0$. Therefore, $[X_{0,1}^+, [X_{0,1}^+, X_{n-1}^+]]$ equals

$$\begin{aligned} = \; & \Big[[X_0^+, H_{1,1}], \big[[X_0^+, H_{n-1,1}], X_{n-1}^+\big]\Big] - \Big[\beta X_0^+ H_1 + (\lambda - \beta) H_1 X_0^+, \big[[X_0^+, H_{n-1,1}], X_{n-1}^+\big]\Big] \\ & - \big[[X_0^+, H_{1,1}], [(\lambda - \beta) X_0^+ H_{n-1} + \beta H_{n-1} X_0^+, X_{n-1}^+]\big] \\ & + \big[\beta X_0^+ H_1 + (\lambda - \beta) H_1 X_0^+, [(\lambda - \beta) X_0^+ H_{n-1} + \beta H_{n-1} X_0^+, X_{n-1}^+]\big] \end{aligned}$$

which simplifies to

$$= \; -\frac{\lambda}{2}\Big(S\big([[X_0^+, H_{1,1}], X_{n-1}^+], X_0^+\big) + S\big([H_1, [[X_0^+, H_{1,1}], X_{n-1}^+]], X_0^+\big)\Big) \tag{23}$$

$$-\frac{\lambda}{2}\Big(S\big(H_1, [X_0^+, [[X_0^+, H_{1,1}], X_{n-1}^+]]\big)\Big) + \frac{\lambda^2}{4}\big[S(H_1, X_0^+), S\big([X_0^+, X_{n-1}^+], H_1\big)\big] \tag{24}$$

The terms (23) cancel each other. Since

$$\begin{aligned}[X_0^+,[[X_0^+,H_{1,1}],X_{n-1}^+]] &= [[X_0^+,H_{1,1}],[X_0^+,X_{n-1}^+]] = [X_0^+,[H_{1,1},[X_0^+,X_{n-1}^+]]] \\ &= [X_0^+,[[H_{1,1},X_0^+],X_{n-1}^+]] = 0,\end{aligned}$$

by comparing the first and last term, we see that the first expression in (24) is zero. Thus, $[X_{0,1}^+,[X_{0,1}^+,X_{n-1}^+]]$ equals

$$\begin{aligned}&= \frac{\lambda^2}{4}\Big(X_0^+[H_1,[X_0^+,X_{n-1}^+]]H_1 + [X_0^+,X_{n-1}^+][X_0^+,H_1]H_1\Big) + \frac{\lambda^2}{4}\Big([X_0^+,H_1]H_1[X_0^+,X_{n-1}^+] \\ &\quad +H_1X_0^+[H_1,[X_0^+,X_{n-1}^+]]\Big) + \frac{\lambda^2}{4}\Big(S\big([H_1,[X_0^+,X_{n-1}^+]]X_0^+,H_1\big) + S\big([X_0^+,X_{n-1}^+],H_1[X_0^+,H_1]\big)\Big) \\ &= \frac{\lambda^2}{4}[[X_0^+,X_{n-1}^+]H_1,X_0^+] + \frac{\lambda^2}{4}[X_0^+,H_1[X_0^+,X_{n-1}^+]] = 0\end{aligned}$$

□

# 4 Cherednik algebras and Schur-Weyl duality

The definitions given in this section could be stated for any Weyl group $W$. However, in this paper, we will be concerned only with the symmetric group $S_l$, so we will restrict our definitions to this case. We set $\mathfrak{h} = \mathbb{C}^l$. The symmetric group $S_l$ acts on $\mathfrak{h}$ by permuting the coordinates; associated to $\mathfrak{h}$ are two polynomial algebras: $\mathbb{C}[\mathfrak{h}] = \mathrm{Sym}(\mathfrak{h}^*) = \mathbb{C}[x_1,\ldots,x_l]$ and $\mathbb{C}[\mathfrak{h}^*] = \mathrm{Sym}(\mathfrak{h}) = \mathbb{C}[y_1,\ldots,y_l]$, where $\{x_1,\ldots,x_l\}$ and $\{y_1,\ldots,y_l\}$ are dual bases of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively. For $i \neq j$, we set $\epsilon_{ij} = x_i - x_j$, $\epsilon_{ij}^\vee = y_i - y_j$, $R = \{\epsilon_{ij} | 1 \le i \neq j \le l\}$ and $R^+ = \{\epsilon_{ij} | 1 \le i < j \le l\}$. The set $\mathcal{S} = \{x_i - x_{i+1} | 1 \le i \le l-1\}$ is a basis of simple roots. The reflection in $\mathfrak{h}$ with respect to the hyperplane $\epsilon = 0$ ($\epsilon \in \mathfrak{h}^*$) is denoted $s_\epsilon$. Let $\langle\, ,\, \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ be the canonical pairing and set $s_{ij} = s_{\epsilon_{ij}}$.

**Definition 4.1.** *[9] Let $\{u_1,\ldots,u_l\}$ be a basis of $\mathfrak{h}$. The degenerate affine Hecke algebra $\mathtt{H}_c(S_l)$ of type $\mathfrak{gl}_l$ is the algebra generated by the polynomial algebra $\mathbb{C}[u_1,\ldots,u_l]$ and the group algebra $\mathbb{C}[S_l]$ with the relations*

$$s_\epsilon \cdot u - s_\epsilon(u)\cdot s_\epsilon = -c\langle\epsilon,u\rangle \;\; \forall u \in \mathfrak{h}, \forall \epsilon \in \mathcal{S}$$

The double affine Hecke algebra $\mathbb{H}$ introduced by I. Cherednik [5] admits degenerate versions: the trigonometric one and the rational one. The extended affine Weyl group is $\widehat{S}_l = P \rtimes S_l$ where $P$ is the lattice $\oplus_{i=1}^l \mathbb{Z}x_i \subset \mathfrak{h}^*$, so its group algebra is $\mathbb{C}[\widehat{S}_l] = \mathbb{C}[X_1^{\pm 1},\ldots,X_l^{\pm 1}] \rtimes S_l$. The group $\widehat{S}_l$ is generated by $s_\epsilon$ $\forall \epsilon \in R$ and by the element $\wp = x_1 s_{12} s_{23} \cdots s_{l-1,l}$.

**Definition 4.2** (Cherednik)**.** *Let $t,c \in \mathbb{C}$. The degenerate (trigonometric) double affine Hecke algebra of type $\mathfrak{gl}_l$ is the algebra $\mathbf{H}_{t,c}(S_l)$ generated by the group algebra of the extended affine Weyl group $\mathbb{C}[\widehat{S}_l]$ and the polynomial algebra $\mathbb{C}[u_1,\ldots,u_l] = \mathrm{Sym}(\mathfrak{h})$ subject to the following relations:*

$$s_\epsilon \cdot u - s_\epsilon(u)\cdot s_\epsilon = -c\langle\epsilon,u\rangle \;\; \forall u \in \mathfrak{h}, \forall \epsilon \in \mathcal{S}$$

$$\wp u_i = u_{i+1}\wp, \;\; 1 \le i \le l-1, \quad \wp u_l = (u_1 - t)\wp$$

The rational version of the double affine Hecke algebra has been studied quite intensively in the past few years (see, for example, [1],[15]) and is usually referred to as the rational Cherednik algebra.

**Definition 4.3.** *Let $t,c \in \mathbb{C}$. The rational Cherednik algebra $\mathsf{H}_{t,c}(S_l)$ of type $\mathfrak{gl}_l$ is the algebra generated by $\mathbb{C}[\mathfrak{h}], \mathbb{C}[\mathfrak{h}^*]$ and $\mathbb{C}[S_l]$ subject to the following relations:*

$$w\cdot x\cdot w^{-1} = w(x), \qquad w\cdot y\cdot w^{-1} = w(y), \qquad \forall x \in \mathfrak{h}^*, \; \forall y \in \mathfrak{h}$$

$$[y,x] = yx - xy = t\langle y,x\rangle + c\sum_{\epsilon\in R^+}\langle\epsilon,y\rangle\langle x,\epsilon^\vee\rangle s_\epsilon$$

The elements $\mathsf{Y}_i = \frac{1}{2}(x_i y_i + y_i x_i)$ will be important later.

**Proposition 4.1.** *The algebra $\mathbf{H}_{t,c}(S_l)$ can be defined as the algebra generated by elements $X_1^{\pm 1}, \dots, X_l^{\pm 1}$, $\mathcal{Y}_1, \dots, \mathcal{Y}_l$ and $S_l$ with the relations*

$$w \cdot X_i \cdot w^{-1} = X_{w(i)}, \quad w \cdot \mathcal{Y}_i \cdot w^{-1} = \mathcal{Y}_{w(i)}, \quad [\mathcal{Y}_j, \mathcal{Y}_k] = \frac{c^2}{4} \sum_{\substack{i=1 \\ i \neq j,k}}^{l} (s_{jk} s_{ik} - s_{kj} s_{ij})$$

$$\mathcal{Y}_j X_i - X_i \mathcal{Y}_j = t\delta_{ij} X_i + \frac{c}{2} \sum_{\epsilon \in R^+} \langle \epsilon, y_j \rangle \langle x_i, \epsilon^\vee \rangle (X_i s_\epsilon + s_\epsilon X_i).$$

There exists an isomorphism $\mathbf{H}_{t,c}(S_l) \xrightarrow{\sim} \mathbb{C}[x_1^{\pm 1}, \dots, x_l^{\pm 1}] \otimes_{\mathbb{C}[\mathfrak{h}]} \mathsf{H}_{t,c}(S_l)$ which sends $\mathcal{Y}_i$ to $\mathsf{Y}_i$ and $X_i^{\pm 1}$ to $x_i^{\pm 1}$. We want to explain another connection between $\mathbf{H}_{t,c}(S_l)$ and $\mathsf{H}_{t,c}(S_l)$ which is true for Cherednik algebras attached to any Weyl group. We can filter $\mathbf{H}_{t,c}(S_l)$ by giving $\mathcal{Y}_j$ degree 1 and $X_j^{\pm 1}, \sigma \in S_l$ degree 0. Let $\mathfrak{H}_{t,c}(S_l)$ be the $\mathbb{C}[h]$-subalgebra of $\mathbf{H}_{t,c}(S_l) \otimes_{\mathbb{C}} \mathbb{C}[h]$ generated by $X_k^{\pm 1}, h\mathcal{Y}_j, \sigma \in S_l, 1 \le j,k \le l$. This is the Rees ring of $\mathbf{H}_{t,c}(S_l)$ and $\mathfrak{H}_{t,c}(S_l)/h\mathfrak{H}_{t,c}(S_l) \cong gr(\mathbf{H}_{t,c}(S_l)) \xleftarrow{\sim} \mathbb{C}[X_1^{\pm 1}, \dots, X_l^{\pm 1}, \mathcal{Y}_1, \dots, \mathcal{Y}_l] \rtimes S_l$. Consider the composite

$$\mathfrak{H}_{t,c}(S_l) \twoheadrightarrow \mathfrak{H}_{t,c}(S_l)/h\mathfrak{H}_{t,c}(S_l) \xrightarrow{\sim} \mathbb{C}[X_1^{\pm 1}, \dots, X_l^{\pm 1}, \mathcal{Y}_1, \dots, \mathcal{Y}_l] \rtimes S_l \twoheadrightarrow \mathbb{C}[\mathcal{Y}_1, \dots, \mathcal{Y}_l] \rtimes S_l,$$

where the last map is obtained by setting $X_k = 1, 1 \le k \le l$. Let $\mathbf{K}$ be the kernel of this composite and let $\mathbf{A}_{t,c}(S_l)$ be the $\mathbb{C}[h]$-subalgebra of $\mathbf{H}_{t,c}(S_l) \otimes_{\mathbb{C}} \mathbb{C}[h, h^{-1}]$ generated by $\mathfrak{H}_{t,c}(S_l)$ and $\frac{\mathbf{K}}{h}$. The following lemma is already known to others.

**Lemma 4.1.** *The algebra $\mathbf{A}_{t,c}(S_l)/h\mathbf{A}_{t,c}(S_l)$ is isomorphic to $\mathsf{H}_{t,c}(S_l)$.*

**Definition 4.4** (Cherednik)**.** *Let $q, \kappa \in \mathbb{C}^\times$. The double affine Hecke algebra $\mathbb{H}_{q,\kappa}(S_l)$ of type $\mathfrak{gl}_l$ is the unital associative algebra over $\mathbb{C}$ with generators $T_i^{\pm 1}, X_j^{\pm 1}, Y_j^{\pm}$ for $i \in \{1, \dots, l-1\}$ and $j \in \{1, \dots, l\}$ satisfying the following relations:*

$$(T_i + 1)(T_i - q^2) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$
$$T_i T_j = T_j T_i \text{ if } |i-j| > 1, \quad X_0 Y_1 = \kappa Y_1 X_0, \quad X_2 Y_1^{-1} X_2^{-1} Y_1 = q^{-2} T_1^2$$
$$X_i X_j = X_j X_i, \quad Y_i Y_j = Y_j Y_i, \quad T_i X_i T_i = q^2 X_{i+1}, \quad T_i^{-1} Y_i T_i^{-1} = q^{-2} Y_{i+1},$$
$$X_j T_i = T_i X_j, Y_j T_i = T_i Y_j \text{ if } j \neq i, i+1$$

*where $X_0 = X_1 X_2 \cdots X_l$.*

The trigonometric Cherednik algebra can be viewed as a limit (degenerate) version of the double affine Hecke algebra (or elliptic Cherednik algebra). This is explained in [6]. It was proved by I. Cherednik that the double affine Hecke algebra and its trigonometric degeneration are isomorphic after completion. His proof relied on his theory of intertwiners. Here, we present a simpler construction of $\mathbf{H}_{t,c}(S_l)$ starting from $\mathbb{H}_{q,\kappa}(S_l)$. The following lemma can be deduced from Cherednik's result that $\mathbb{H}_{q,\kappa}(S_l)[[h]] \xrightarrow{\sim} \mathbf{H}_{t,c}(S_l)[[h]]$, but it is also possible to give a more elementary proof.

**Lemma 4.2.** *Set $q = e^{\frac{c}{2}h}$,$\kappa = e^{th}$. Let $\mathbb{B}$ be the $\mathbb{C}[[h]]$-subalgebra of $\mathbb{H}(S_l)[[h]] \otimes_{\mathbb{C}[[h]]} \mathbb{C}((h))$ generated by $w \in W, X_j^{\pm 1}, \frac{Y_j^{\pm 1} - 1}{h}, 1 \le j \le l$. Then $\mathbb{B}/h\mathbb{B}$ is isomorphic to $\mathbf{H}_{t,c}(S_l)$.*

The Schur-Weyl duality established by M. Varagnolo and E. Vasserot [27] involves, on one side, a toroidal quantum algebra (a quantized version of the enveloping algebra of the universal central extension of the double loop algebra $\mathfrak{sl}_n[u^{\pm 1}, v^{\pm}]$) and, on the other side, a double affine Hecke algebra for $S_l$. Theorem 4.2 (established in [17]) provides a similar type of duality between the trigonometric Cherednik algebra $\mathbf{H}_{t,c}(S_l)$ and the loop Yangian $LY_{\lambda,\beta}$ (or $\widehat{\mathbf{Y}}_{\lambda,\beta}$), which extends the duality for the Yangian of finite type due to V. Drinfeld [9].

Before stating the more classical results on the theme of Schur-Weyl duality, we have to define the notion of module of level $l$ over $\mathfrak{sl}_n$. Set $V = \mathbb{C}^n$.

**Definition 4.5.** *A finite dimensional representation of* $\mathfrak{sl}_n$ *is of level* $l$ *if each of its irreducible components is isomorphic to a direct summand of* $V^{\otimes l}$.

**Theorem 4.1.** *[7, 9] Fix* $l \geq 1, n \geq 2$. *Let* $A$ *be one of the algebras* $\mathbb{C}[S_l], \mathtt{H}_c(S_l)$, *and let* $B$ *be the corresponding one among* $\mathfrak{Usl}_n, Y_\lambda$. *There exists a functor* $\mathcal{F}$, *which is given by* $\mathcal{F}(M) = M \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$, *from the category of finite dimensional right* $A$*-modules to the category of finite dimensional left* $B$*-modules which are of level* $l$ *as* $\mathfrak{sl}_n$*-modules. Furthermore, this functor is an equivalence of categories if* $l \leq n-1$.

**Definition 4.6.** *A module* $M$ *over* $\widehat{\mathbf{Y}}_{\lambda,\beta}$ *is called integrable if it is the direct sum of its integral weight spaces under the action of* $\widehat{\mathfrak{d}}$ *and if each generator* $X^{\pm}_{i,r}$ *acts locally nilpotently on* $M$.

The following theorem was the principal result in [17]. It is analogous to the main theorem in [27].

**Theorem 4.2.** *Suppose that* $l \geq 1, n \geq 3$ *and set* $\lambda = c, \beta = \frac{t}{2} - \frac{nc}{4} + \frac{c}{2}$. *The functor* $\mathbf{F} : M \mapsto M \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$ *sends a right* $\mathbf{H}_{t,c}(S_l)$*-module to an integrable left* $\widehat{\mathbf{Y}}_{\lambda,\beta}$*-module of level* $l$ *(as* $\mathfrak{sl}_n$*-module) with trivial central charge. Furthermore, if* $l+2 < n$, *this functor is an equivalence. The same is true if* $\mathbf{H}_{t,c}(S_l)$ *and* $\widehat{\mathbf{Y}}_{\lambda,\beta}$ *are replaced by* $\mathsf{H}_{t,c}(S_l)$ *and* $\mathsf{L}_{\lambda,\beta}$.

# 5 From quantum toroidal algebras to affine Yangians

The following definition is slightly different from the one used in [27].

**Definition 5.1.** *Let* $q_1, q_2 \in \mathbb{C}^\times$. *The toroidal quantum algebra* $\mathbb{U}_{q_1,q_2}$ *of type* $A_{n-1}$ *is the unital associative algebra over* $\mathbb{C}$ *with generators* $e_{i,r}, f_{i,r}, k_{i,r}, k_{i,0}^{-1}, i \in \{0,\ldots,n-1\}, r \in \mathbb{Z}$, *which satisfy the following relations:*

$$[k_{i,r}, k_{j,s}] = 0 \; \forall i,j \in \{0,\ldots,n-1\}, \forall r,s \in \mathbb{Z} \tag{25}$$

$$k_{i,0}e_{j,r} = q_1^{c_{ij}} e_{j,r}k_{i,0}, \; k_{i,0}f_{j,r} = q_1^{-c_{ij}} f_{j,r}k_{i,0}, \; (q_1 - q_1^{-1})[e_{i,r}, f_{j,s}] = \delta_{ij}(k^+_{i,r+s} - k^-_{i,r+s}) \tag{26}$$

*(Here,* $k^{\pm}_{i,r+s} = k_{i,r+s}$ *if* $\pm(r+s) \geq 0$ *and* $= 0$ *otherwise.)*

*The next three relations hold* $\forall i,j \in \{0,\ldots,n-1\}, \forall r,s \in \mathbb{Z}$ *except for* $(i,j) = (n-1,0),(0,1)$:

$$k_{i,r+1}e_{j,s} - q_1^{c_{ij}} k_{i,r}e_{j,s+1} = q_1^{c_{ij}} e_{j,s}k_{i,r+1} - e_{j,s+1}k_{i,r} \tag{27}$$

$$e_{i,r+1}e_{j,s} - q_1^{c_{ij}} e_{i,r}e_{j,s+1} = q_1^{c_{ij}} e_{j,s}e_{i,r+1} - e_{j,s+1}e_{i,r} \tag{28}$$

$$\{e_{i,r_1}e_{i,r_2}e_{j,s} - (q_1 + q_1^{-1})e_{i,r_1}e_{j,s}e_{i,r_2} + e_{j,s}e_{i,r_1}e_{i,r_2}\} + \{r_1 \leftrightarrow r_2\} = 0 \text{ if } i-j \equiv \pm 1 \text{ mod } n-1 \tag{29}$$

*The relations (27)-(29) hold with* $e_{i,r}$ *replaced by* $f_{i,r}$ *and* $q_1^{c_{ij}}$ *by* $q_1^{-c_{ij}}$.

*In the cases* $(i,j) = (n-1,0),(0,1)$, *we must modify the relations (27)-(29) above in the following way: we introduce a second parameter* $q_2$ *in such a way that we obtain an algebra isomorphism* $\Psi$ *of* $\mathbb{U}_{q_1,q_2}$ *given by* $e_{i,r}, f_{i,r}, k_{i,r} \mapsto q_1^r e_{i-1,r}, q_1^r f_{i-1,r}, q_1^r k_{i-1,r}$ *for* $2 \leq i \leq n-1$ *and* $e_{i,r}, f_{i,r}, k_{i,r} \mapsto q_2^r e_{i-1,r}, q_2^r f_{i-1,r}, q_2^r k_{i-1,r}$ *if* $i = 0,1$. *(We identify* $e_{-1,r}$ *with* $e_{n-1,r}$, *etc.) For instance, relation (28) for* $i=0, j=1$ *becomes*

$$q_2 e_{0,r+1}e_{1,s} - e_{0,r}e_{1,s+1} = q_1^{-1}q_2 e_{1,s}e_{0,r+1} - q_1 e_{1,s+1}e_{0,r},$$

*and with* $i = n-1, j = 0$ *we have a very similar identity:*

$$q_2 e_{n-1,r+1}e_{0,s} - e_{n-1,r}e_{0,s+1} = q_1^{-1}q_2 e_{0,s}e_{n-1,r+1} - q_1 e_{0,s+1}e_{n-1,r}.$$

The algebra $\mathbb{U}_{q_1,q_2}$ can also be defined using pairwise commuting elements $\widetilde{h}_{i,r}, 0 \leq i \leq n-1, r \in \mathbb{Z} \setminus \{0\}$, instead of the $k_{i,r}, r \neq 0$. They are related to the $k_{i,r}$ via the following equality of power series:

$$\sum_{r \geq 0} k^{\pm}_{i,r} u^{\pm r} = k^{\pm}_{i,0} \exp\left( \pm(q_1 - q_1^{-1}) \sum_{s \geq 1} \widetilde{h}_{i,s} u^{\pm s} \right).$$

They satisfy the relations $[\widetilde{h}_{i,r}, e_{j,s}] = \frac{1}{r}\frac{q_1^{rc_{ij}} - q_1^{-rc_{ij}}}{q_1 - q_1^{-1}} e_{j,r+s}$, $[\widetilde{h}_{i,r}, f_{j,s}] = -\frac{1}{r}\frac{q_1^{rc_{ij}} - q_1^{-rc_{ij}}}{q_1 - q_1^{-1}} f_{j,r+s}$, except when $(i,j) = (n-1,0),(0,1)$, in which case they have to be slightly modified.

It is possible to view the Yangian $Y_\lambda$ as a limit version of the quantum affine algebra $\dot{U}_q$ [11]. The same is true for $\widehat{\mathbf{Y}}_{\lambda,\beta}$ and $\mathbb{U}_{q_1,q_2}$. Let $\mathbb{U}[[h]]$ be the completed algebra over $\mathbb{C}[[h]]$ with parameters $q_1 = e^{\frac{\lambda}{2}h}, q_2 = e^{\beta h}$ and $k_{i,0} = \exp(\frac{h\lambda}{2}\widetilde{h}_{i,0})$, where $\widetilde{h}_{i,0}$ satisfies: $[\widetilde{h}_{i,0}, e_{j,r}] = c_{ij} e_{j,r}, [\widetilde{h}_{i,0}, f_{j,r}] = -c_{ij} f_{j,r}$. Let $\dot{U}^{ver}$ be the subalgebra of $\mathbb{U}_{q_1,q_2}$ generated by the elements $e_{i,r}, f_{i,r}, k_{i,r}, k_{i,0}^{-1}$ with $i \neq 0$ and let $\dot{U}^{hor}$ be the one generated by the elements with $r=0$. Consider the kernel $\mathbb{K}$ of the map $\mathbb{U}[[h]] \longrightarrow \mathfrak{U}(\widehat{\mathfrak{sl}}_n)$ which is the composite of the map obtained by setting $h=0$ and the one sending $\mathbb{U}_{h=0}$ to $\mathfrak{U}(\widehat{\mathfrak{sl}}_n) = \dot{U}^{hor}_{h=0}$. Let $\mathbb{A}$ be the $\mathbb{C}[[h]]$-subalgebra of $\mathbb{U}[[h]] \otimes_{\mathbb{C}[[h]]} \mathbb{C}((h))$ generated by $\mathbb{U}[[h]]$ and $\frac{\mathbb{K}}{h}$.

**Proposition 5.1.** *The quotient $\mathbb{A}/h\mathbb{A}$ is isomorphic to $\widehat{\mathbf{Y}}_{\lambda,\beta}$.*

*Proof.* To see this, let $A$ be the subalgebra of $\mathbb{A}$ generated by $\dot{U}^{ver}$ and $\frac{\mathbb{K}\cap\dot{U}^{ver}}{h}$. Since $\dot{U}^{ver}$ is a quotient of the quantum loop algebra $\dot{U}_{q_1}$, $A/hA$ is a quotient of the Yangian $Y_\lambda$ (see [11]), that is, we have an epimorphism $\zeta : Y_\lambda \longrightarrow A/hA$. The automorphism $\Psi$ of $\mathbb{U}[[h]]$ induces an automorphism, also denoted $\Psi$, on $\mathbb{A}$. It is related to the automorphism $\rho$ of $\widehat{\mathbf{Y}}_{\lambda,\beta}$ in the following way for $2 \le i \le n-1$:

$$\Psi(\zeta(X^{\pm}_{i,r})) = \zeta(\rho(X^{\pm}_{i,r})), \quad \Psi(\zeta(H_{i,r})) = \zeta(\rho(H_{i,r}))$$

$$\Psi^2(\zeta(X^{\pm}_{1,r})) = \zeta(\rho^2(X^{\pm}_{1,r})), \quad \Psi^2(\zeta(H_{1,r})) = \zeta(\rho^2(H_{1,r}))$$

From these relations, one sees that it is possible to extend $\zeta$ to $\widehat{\mathbf{Y}}_{\lambda,\beta}$ by setting $\zeta(X^{\pm}_{0,r}) = \Psi(\zeta(\rho^{-1}(X^{\pm}_{0,r})))$ and similarly for $H_{0,r}$. This extension $\zeta : \widehat{\mathbf{Y}}_{\lambda,\beta} \longrightarrow \mathbb{A}/h\mathbb{A}$ is surjective and we are left to show that it is injective.

The Schur-Weyl duality functor constructed in [27] can be extended to $\mathbb{U}[[h]]$ and $\mathbb{H}[[h]]$. Applying it to $\mathbb{H}[[h]]$ as a right module over itself, we obtain an algebra homomorphism $\Phi : \mathbb{U}[[h]] \longrightarrow \mathrm{End}_{\mathbb{C}}\big((\mathbb{H} \otimes_{\mathcal{H}} V^{\otimes l})[[h]]\big)$. We can extend it to $\mathbb{U} \otimes_{\mathbb{C}[[h]]} \mathbb{C}((h))$ and restrict it to $\mathbb{A}$, which yields $\Phi : \mathbb{A} \longrightarrow \mathrm{End}_{\mathbb{C}}\big((\mathbb{H} \otimes_{\mathcal{H}} V^{\otimes l})[[h]] \otimes_{\mathbb{C}[[h]]} \mathbb{C}((h))\big)$. It is known (see [6]) that $\mathbb{H}[[h]]$ is isomorphic to $\mathbf{H}[[h]]$ (see section 4 for the values of $q, \kappa, t, c$); using such an isomorphism or lemma 4.2, we see that $\Phi$ descends to $\Phi : \mathbb{A}/h\mathbb{A} \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbf{H} \otimes_{\mathbb{C}[S_l]} V^{\otimes l})$. The composite $\Phi \circ \zeta$ is exactly the map $\upsilon$ obtained by applying the Schur-Weyl functor to $\mathbf{H}$ viewed as a right module over itself. From corollary 7.2, we know that, given $X \in \widehat{\mathbf{Y}}_{\lambda,\beta}$ with $X$ not a multiple of $H_{0,0} + \cdots + H_{n-1,0}$, there exists $l \gg 0$ such that $\Phi \circ \zeta(X) \neq 0$. This implies that $\zeta : \widehat{\mathbf{Y}}_{\lambda,\beta} \longrightarrow \mathbb{A}/h\mathbb{A}$ is also injective, hence an isomorphism when $\beta \neq \frac{n\lambda}{4} + \frac{\lambda}{2}$. It then follows that it must be an isomorphism for any $\lambda, \beta$. ∎

# 6 Specialization at $\lambda = 0$ of $\widehat{Y}_{\lambda,\beta}$ and $\mathsf{L}_{\lambda,\beta}$

We can obtain results analogous to theorem 13.1 in [28]. In this section, we will assume that $\beta \neq 0$.

**Definition 6.1.** *Let $\widetilde{\mathfrak{sl}}_{n,\beta}$ be the complex Lie algebra generated by the elements $x^{\pm}_{i,r}, h_{i,r}$ where $i = 0, 1, \ldots, n-1$ and $r \in \mathbb{Z}_{\geq 0}$ and defined by the relations:*

$$[h_{i,r}, h_{j,s}] = 0, \quad [h_{i,r}, x^{\pm}_{j,s}] = \pm c_{ij} x^{\pm}_{j,r+s} \ \textit{if } i \neq 0 \textit{ or } i = r = 0$$

$$[x^{+}_{i,r}, x^{-}_{j,s}] = \delta_{ij} h_{i,r+s}, \quad [x^{\pm}_{i,r+1}, x^{\pm}_{j,s}] = [x^{\pm}_{i,r}, x^{\pm}_{j,s+1}] \ \textit{except if } (i,j) = (1,0) \textit{ or } (0, n-1)$$

$$[x^{\pm}_{i,r}, x^{\pm}_{j,s}] = 0 \ \textit{if } 1 < |i-j| < n-1$$

$$\sum_{\pi\in S_m}\left[x^{\pm}_{i,r_{\pi(1)}},[x^{\pm}_{i,r_{\pi(2)}},\ldots,[x^{\pm}_{i,r_{\pi(m)}},x^{\pm}_{j,s}]]\right]=0 \text{ where } m=1-c_{ij},r_1,\ldots,r_m,s\in\mathbb{Z}_{\geq 0}$$

*The next two relations hold if* $(i,j)=(1,0)$ *and* $(i,j)=(0,n-1)$.

$$[x^{\pm}_{i,r+1},x^{\pm}_{j,s}]-[x^{\pm}_{i,r},x^{\pm}_{j,s+1}]=\beta[x^{\pm}_{i,r},x^{\pm}_{j,s}]$$

$$[h^{\pm}_{i,r+1},x^{\pm}_{j,s}]-[h^{\pm}_{i,r},x^{\pm}_{j,s+1}]=\beta[h^{\pm}_{i,r},x^{\pm}_{j,s}].$$

**Definition 6.2.** *Let* $\overline{\mathfrak{sl}}_{n,\beta}$ *be the Lie subalgebra of* $\widetilde{\mathfrak{sl}}_{n,\beta}$ *generated by the elements* $x^{\pm}_{i,r},h_{i,r}$ *where* $r\in\mathbb{Z}_{\geq 0}$ *if* $i\neq 0$, $x^{+}_{0,r},r\geq 0$ *and* $x^{-}_{0,r},r\geq 1$.

The algebra $\widehat{\mathbf{Y}}_{\lambda=0\beta}$ (resp. $\mathsf{L}_{\lambda=0,\beta}$) is the universal enveloping algebra of the Lie algebra $\widetilde{\mathfrak{sl}}_{n,\beta}$ (resp. $\overline{\mathfrak{sl}}_{n,\beta}$).

**Definition 6.3.** *We denote by* $\mathbf{A}_\beta$ *(resp.* $\mathsf{A}_\beta$*) the algebra generated by the elements* $X,X^{-1}$ *and* $\partial$ *(resp.* $x$ *and* $d$*) which satisfy the relation* $\partial\cdot X-X\cdot\partial=2\beta X$ *(resp.* $d\cdot x-x\cdot d=2\beta$*).*

**Remark 6.1.** *If* $\beta_1\neq 0$ *and* $\beta_2\neq 0$*, the algebras* $\mathbf{A}_{\beta_1}$ *and* $\mathbf{A}_{\beta_2}$ *(resp.* $\mathsf{A}_{\beta_1}$ *and* $\mathsf{A}_{\beta_2}$*) are isomorphic. When* $\beta=\frac{1}{2}$, $\mathbf{A}_\beta$ *(resp.* $\mathsf{A}_\beta$*) is exactly the ring of algebraic differential operators on* $\mathbb{C}^\times$ *(resp. on the affine line* $\mathbb{C}$*). We have an embedding* $\mathsf{A}_\beta\hookrightarrow\mathbf{A}_\beta$ *given by* $x\mapsto X$ *and* $d\mapsto(\partial+\beta)X^{-1}$*; moreover* $\mathbb{C}[x,x^{-1}]\otimes_{\mathbb{C}[x]}\mathsf{A}_\beta\xrightarrow{\sim}\mathbf{A}_\beta$.

The Lie algebra $\mathfrak{sl}_n(\mathbf{A}_\beta)$ is defined as the subspace of matrices in $\mathfrak{gl}_n(\mathbf{A}_\beta)$ with trace in $[\mathbf{A}_\beta,\mathbf{A}_\beta]$, so we have the decomposition:

$$\mathfrak{sl}_n(\mathbb{C})\otimes_{\mathbb{C}}\mathbf{A}_\beta+\mathfrak{d}([\mathbf{A}_\beta,\mathbf{A}_\beta])\xrightarrow{\sim}\mathfrak{sl}_n(\mathbf{A}_\beta)$$

where $\mathfrak{d}([\mathbf{A}_\beta,\mathbf{A}_\beta])$ is the subspace of $\mathfrak{gl}_n(\mathbf{A}_\beta)$ of diagonal matrices with coefficients in $[\mathbf{A}_\beta,\mathbf{A}_\beta]$. All of this holds when $\mathbf{A}_\beta$ is replaced by $\mathsf{A}_\beta$. Note that $[\mathbf{A}_\beta,\mathbf{A}_\beta]=\mathbf{A}_\beta$ if $\beta\neq 0$, which follows from the easier observation that $\mathsf{A}_\beta=[\mathsf{A}_\beta,\mathsf{A}_\beta]$. The embedding $\mathsf{A}_\beta\hookrightarrow\mathbf{A}_\beta$ induces $\mathfrak{sl}_n(\mathsf{A}_\beta)\hookrightarrow\mathfrak{sl}_n(\mathbf{A}_\beta)$.

Our main results in this section are the next two propositions.

**Proposition 6.1** ([28])**.** *The Lie algebra* $\widetilde{\mathfrak{sl}}_{n,\beta}$ *is isomorphic to the universal central extension of* $\mathfrak{sl}_n(\mathbf{A}_\beta)$*. Its center is spanned by* $h_0+\ldots+h_{n-1}$.

**Proposition 6.2.** *The Lie algebra* $\overline{\mathfrak{sl}}_{n,\beta}$ *is isomorphic to* $\mathfrak{sl}_n(\mathsf{A}_\beta)$.

**Remark 6.2.** *When* $\beta=\frac{1}{2}$*, the universal central extension of* $\mathfrak{sl}_n(\mathbf{A}_\beta)$ *is sometimes called the matrix* $W_{1+\infty}$*-algebra. The Lie algebra* $\mathfrak{sl}_n(\mathsf{A}_\beta)$ *has no non-trivial central extension since the first cyclic homology group* $HC_1(\mathsf{A}_\beta)$ *is trivial. This is a consequence of a result in [21] which states that* $H_2(\mathfrak{sl}_n(A);\mathbb{C})\cong HC_1(A)$ *for an arbitrary associative* $\mathbb{C}$*-algebra* $A$ *and the fact that the kernel of the universal central extension of* $\mathfrak{sl}_n(A)$ *is* $H_2(\mathfrak{sl}_n(A);\mathbb{C})$ *[30]. On the other hand, it is known that* $\dim_{\mathbb{C}}HC_1(\mathbf{A}_\beta)=1$.

Proposition 6.1 can be proved using theorem 13.1 in [28] and the connection given in section 5 between $\mathbb{U}_{q_1,q_2}$ and $\widehat{\mathbf{Y}}_{\lambda,\beta}$. We could also give a direct proof which would be very similar to the proof of that theorem. Explicitly, an isomorphism $\tau$ is given by:

$$h_{i,r}\mapsto(-1)^r(E_{ii}-E_{i+1,i+1})\otimes\partial^r,\quad x^{+}_{i,r}\mapsto(-1)^rE_{i,i+1}\otimes\partial^r,x^{-}_{i,r}\mapsto(-1)^rE_{i+1,i}\otimes\partial^r \text{ for } i\neq 0$$

$$\tau(x^{+}_{0,r})\mapsto(-1)^rE_{-\theta}\otimes X(\partial+\beta)^r,\quad \tau(x^{-}_{0,r})\mapsto(-1)^rE_{\theta}\otimes(\partial+\beta)^rX^{-1}$$

$$\tau(h_{0,r})\mapsto E_{nn}\otimes(\beta-\partial)^r-(-1)^rE_{11}\otimes(\beta+\partial)^r.$$

*Proof of proposition 6.2.* Since $\tau(x^{-}_{0,1})=-E_{1n}\otimes(\partial+\beta)^rX^{-1}$, we see that $\tau(\overline{\mathfrak{sl}}_{n,\beta})\subset\mathfrak{sl}_n(\mathsf{A}_\beta)$. (See remark 6.1.) That we have an equality can be checked as in the proof in [28] of theorem 6.1. Furthermore, $\ker(\tau)\cap\overline{\mathfrak{sl}}_{n,\beta}=\{0\}$ according to proposition 6.1, so $\tau|_{\overline{\mathfrak{sl}}_{n,\beta}}$ is an isomorphism. $\square$

# 7 PBW bases for affine Yangians

The Poincaré-Birkhoff-Witt decomposition of the enveloping algebra of a Lie algebra provides a nice vector space basis and is of fundamental importance in Lie theory. In this section, we obtain a similar result for $\widehat{\mathbf{Y}}_{\lambda,\beta}$ and, consequently, for $\mathsf{L}_{\lambda,\beta}$. For Yangians of finite type, the existence of such a basis was proved in [22]. In this section, we fix $\lambda,\beta\in\mathbb{C}$, set $c=\lambda, t=2\beta+\frac{n\lambda}{2}-\lambda$ and abbreviate $\widehat{\mathbf{Y}}_{\lambda,\beta},\mathsf{L}_{\lambda,\beta},\mathbf{H}_{t,c}(S_l),\mathsf{H}_{t,c}(S_l)$ by $\widehat{\mathbf{Y}},\mathsf{L},\mathbf{H},\mathsf{H}$, respectively.

We recall that we can define a filtration on $\widehat{\mathbf{Y}}$ and $\mathsf{L}$ in the following way: we give $X_{j,r}^{\pm}$ and $H_{j,r}$ degree $r$ and define $F_i(\widehat{\mathbf{Y}})$ as the linear span of the monomials in these generators of total degree $\leq i$. We set $F_i(\mathsf{L})=\mathsf{L}\cap F_i(\widehat{\mathbf{Y}})$. We can filter $\mathbf{H}$ by giving $X_i^{\pm 1}, w\in S_l$ degree 0 and $\mathcal{Y}_j$ degree 1. This induces a filtration on $\mathbf{V}^l=\mathbf{H}\otimes_{\mathbb{C}[S_l]}V^{\otimes l}$, the elements of $V^{\otimes l}$ having degree 0.

We need to fix some notation concerning the root system of $\widehat{\mathfrak{sl}}_n[u^{\pm 1}]$. We denote by $\Delta=\{\alpha_{ij}|1\leq i\neq j\leq n\}\subset\mathfrak{d}^*$ the root system of $\mathfrak{sl}_n$ with choice of simple roots $\Pi=\{\alpha_i=\alpha_{i,i+1},1\leq i\leq n-1\}$ and by $\widehat{\Delta}\subset\mathfrak{d}^*\oplus\mathbb{C}\delta$ the root system of type $\widehat{A}_{n-1}$, which is given by $\widehat{\Delta}=\widehat{\Delta}^{re}\cup\widehat{\Delta}^{im}$, the set of real roots $\widehat{\Delta}^{re}$ being $\{\overline{\alpha}+s\delta|\overline{\alpha}\in\Delta,s\in\mathbb{Z}\}$ and the set of imaginary roots $\widehat{\Delta}^{im}=\{s\delta|s\in\mathbb{Z}\setminus\{0\}\}$ (see the notation in [20]). The set of positive roots is $\widehat{\Delta}^{+}=\{\alpha=\overline{\alpha}+s\delta|\overline{\alpha}\in\Delta,s\in\mathbb{Z}_{>0}\text{ or }s=0,\overline{\alpha}\in\Delta^+\}\cup\{s\delta|s\in\mathbb{Z}_{>0}\}$. The standard root vector of $\widehat{\mathfrak{sl}}_n[u^{\pm1}]$ corresponding to $\overline{\alpha}_{ij}+s\delta$ is $E_{ij}\otimes u^s$ and $\{H_i\otimes u^s|1\leq i\leq n-1,s\neq 0\}$ is a basis of the root space of $\widehat{\mathfrak{sl}}[u^{\pm1}]_n$ for the imaginary root $s\delta$. The simple roots for $\widehat{\Delta}$ are $\widehat{\Pi}=\{\alpha_0,\alpha_1,\ldots,\alpha_{n-1}\}$ where $\alpha_0=-\alpha_{1n}+\delta$.

Let $\alpha=\alpha_{i_1}+\alpha_{i_2}+\ldots+\alpha_{i_p}=\overline{\alpha}+s\delta\in\widehat{\Delta}^{re,+}=\widehat{\Delta}^{re}\cap\widehat{\Delta}^+,\alpha_{i_j}\in\widehat{\Pi},\overline{\alpha}\in\Delta$, be a decomposition of a positive real root $\alpha$ into a sum of simple roots such that $X_\alpha^{\pm}=[X_{i_1}^{\pm},[X_{i_2}^{\pm},\cdots,[X_{i_{p-1}}^{\pm},X_{i_p}^{\pm}]\cdots]]$ is a (non-zero) root vector of $\widehat{\mathfrak{sl}}_n[u^{\pm1}]$ of weight $\pm\alpha$. Writing $r=r_1+\ldots+r_p$ as a sum of non-negative integers, we set

$$X_{\alpha,r}^{\pm}=\Big[X_{i_1,r_1}^{\pm},[X_{i_2,r_2}^{\pm},\cdots,[X_{i_{p-1},r_{p-1}}^{\pm},X_{i_p,r_p}^{\pm}]\cdots]\Big],\quad H_{\alpha,r}^{\pm}=\pm[X_{\alpha,r}^{\pm},X_{\overline{\alpha},0}^{\mp}]\text{ if }\overline{\alpha}\in\Delta^+ \tag{30}$$

We may also write $X_{\alpha,r}$ for $X_{\alpha,r}^+$ if $\alpha\in\widehat{\Delta}^+$ and set $X_{\alpha,r}=X_{-\alpha,r}^-$ if $\alpha\in\widehat{\Delta}^-$.

One important property of the module structure on $\mathbf{V}^l$ is contained in the following two lemmas.

**Lemma 7.1.** *Let $\mathbf{h}\otimes\mathbf{v}\in F_d(\mathbf{V}^l),\mathbf{h}\in F_d(\mathbf{H}),\mathbf{v}\in V^{\otimes l}$. For $1\leq i\leq n-1$, $X_{i,r}^{\pm}(\mathbf{h}\otimes\mathbf{v})=\sum_{k=1}^l\mathbf{h}\mathcal{Y}_k^r\otimes X_i^{\pm(k)}(\mathbf{v})+\kappa$ where $\kappa\in F_{d+r-1}(\mathbf{V}^l)$ - similarly for $H_{i,r}^{\pm}$ with $X_i^{\pm(k)}$ replaced by $H_i^{(k)}$. We have also $X_{0,r}^{\pm}(\mathbf{h}\otimes\mathbf{v})=\sum_{k=1}^l\mathbf{h}\mathcal{Y}_k^rX_k^{\pm1}\otimes E_{\mp\theta}^{(k)}(\mathbf{v})+\kappa$ where $\kappa\in F_{d+r-1}(\mathbf{V}^l)$, and the same for $H_{0,r}$ with $E_{\mp\theta}$ replace by $H_\theta$, but without $X_k^{\pm1}$.*

*Proof.* We proceed by induction on $r$. First, assume that $i\neq 0$. The statement of the lemma is clearly true for $r=0,1$ (see the definition of $\mathbf{F}$ in [17]). For the inductive step, we use equation (8).

$$\begin{aligned}X_{i,r+1}^{\pm}(\mathbf{h}\otimes\mathbf{v}) &= \pm\frac{1}{2}\sum_{k=1}^l[J(H_i),\mathcal{Y}_k^r\otimes X_i^{\pm(k)}](\mathbf{h}\otimes\mathbf{v})+\kappa=\pm\frac{1}{2}\sum_{j,k=1}^l[\mathcal{Y}_j\otimes H_i^{(j)},\mathcal{Y}_k^r\otimes X_i^{\pm(k)}](\mathbf{h}\otimes\mathbf{v})+\kappa\\ &= \pm\frac{1}{2}\sum_{k=1}^l\mathbf{h}\mathcal{Y}_k^{r+1}\otimes[H_i,X_i^{\pm}]^{(k)}(\mathbf{v})+\kappa'=\sum_{k=1}^l\mathbf{h}\mathcal{Y}_k^{r+1}\otimes X_i^{\pm(k)}(\mathbf{v})+\kappa'\end{aligned}$$

where $\kappa,\kappa'\in F_{r+d}(\mathbf{V}^l)$.

We consider now the case $i=0$. The lemma is true if $r=0$ and also if $r=1$ (see section 7 in [17]). We use again induction, relation (8) and the fact that $(H_{0,1} - J(H_\theta))(\mathbf{h} \otimes \mathbf{v}) \in F_d(\mathbf{V}^l)$ - see [17].

$$\begin{aligned} X^{\pm}_{0,r+1}(\mathbf{h}\otimes\mathbf{v}) &= \pm\frac{1}{2}\sum_{k=1}^{l}[J(H_\theta), \mathcal{Y}_k^r X_k^{\pm 1} \otimes E^{(k)}_{\mp\theta}](\mathbf{h}\otimes\mathbf{v}) + \kappa \\ &= \pm\frac{1}{2}\sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k^r X_k^{\pm 1}\mathcal{Y}_k \otimes (H_\theta E_{\mp\theta})^{(k)}(\mathbf{v}) \mp \frac{1}{2}\sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k\mathcal{Y}_k^r X_k^{\pm 1} \otimes (E_{\mp\theta}H_\theta)^{(k)}(\mathbf{v}) + \kappa' \\ &= \sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k^{r+1}X_k^{\pm 1} \otimes E^{(k)}_{\mp\theta}(\mathbf{v}) + \kappa'' \text{ where } \kappa'' \in F_{r+d}(\mathbf{V}^l) \end{aligned}$$

since $\mathbf{h}\mathcal{Y}_k^r[X_k^{\pm 1}, \mathcal{Y}_k] \otimes (H_\theta E_{\mp\theta})^{(k)}(\mathbf{v}) \in F_{r+d}(\mathbf{V}^l)$. The result for $H_{i,r}$ follows from $H_{i,r} = [X^+_{i,r}, X^-_i]$. □

**Lemma 7.2.** *Let $\mathbf{h}\otimes\mathbf{v} \in F_d(\mathbf{V}^l), \mathbf{h} \in F_d(\mathbf{H}), v \in \mathbf{V}^{\otimes l}$ and $\alpha \in \widehat{\Delta}^{re,+}$. If $\alpha = \overline{\alpha} + s\delta$, then $X^{\pm}_{\alpha,r}(\mathbf{h}\otimes\mathbf{v}) = \sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k^r X_k^{\pm s} \otimes X^{\pm(k)}_{\overline{\alpha}}(\mathbf{v}) + \kappa$ where $\kappa \in F_{d+r-1}(\mathbf{V}^l)$ - similarly for $H^{\pm}_{\alpha,r}$ (if $\overline{\alpha} \in \Delta^+$) with $X^{\pm}_{\overline{\alpha}}$ replaced by $H_{\overline{\alpha}}$.*

*Proof.* We use induction on $p$ (see equation (30)), the case $p = 1$ being the content of lemma 7.1. We prove the case $s = 0$ first. Set $X^{\pm}_{\tilde{\alpha}} = [X^{\pm}_{i_2}, [X^{\pm}_{i_3}, \cdots, [X^{\pm}_{i_{p-1}}, X^{\pm}_{i_p}]]\cdots]$. For certain $\kappa, \kappa', \kappa'' \in F_{d+r-1}(\mathbf{V}^l)$,

$$\begin{aligned} X^{\pm}_{\alpha,r}(\mathbf{h}\otimes\mathbf{v}) &= \sum_{k=1}^{l}[X^{\pm}_{i_1,r_1}, \mathcal{Y}_k^{r-r_1} \otimes X^{\pm(k)}_{\tilde{\alpha}}](\mathbf{h}\otimes\mathbf{v}) + \kappa \\ &= \sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k^r \otimes [X^{\pm(k)}_{i_1}, X^{\pm(k)}_{\tilde{\alpha}}](\mathbf{v}) + \sum_{k=1}^{l}\mathbf{h}[\mathcal{Y}_k^{r-r_1}, \mathcal{Y}_k^{r_1}] \otimes X^{\pm(k)}_{\tilde{\alpha}}X^{\pm(k)}_{i_1}(\mathbf{v}) + \kappa' \\ &= \sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k^r \otimes X^{\pm(k)}_{\alpha}(\mathbf{h}\otimes\mathbf{v}) + \kappa'' \end{aligned}$$

We consider now the case $s > 0$. We will assume that $i_1 = 0$, the case $i_1 \neq 0$ being similar. As above, we write $[X^{\pm}_{i_2}, [X^{\pm}_{i_3}, \cdots, [X^{\pm}_{i_{p-1}}, X^{\pm}_{i_p}]]\cdots] = X^{\pm}_{\tilde{\alpha}}$ where $\tilde{\alpha} = \overline{\tilde{\alpha}} + (s-1)\delta$, so that $\alpha = \tilde{\alpha} + \alpha_0 = \overline{\tilde{\alpha}} + (-\theta) + s\delta$ and $\overline{\tilde{\alpha}} \in \Delta^+$. With this notation, we have $X^{\pm}_{\overline{\alpha}} = [E_{\mp\theta}, X^{\pm}_{\overline{\tilde{\alpha}}}]$.

$$\begin{aligned} X^{\pm}_{\alpha,r}(\mathbf{h}\otimes\mathbf{v}) &= \sum_{k=1}^{l}[\mathcal{Y}_k^{r_1}X_k^{\pm 1} \otimes E^{(k)}_{\mp\theta}, \mathcal{Y}_k^{r-r_1}X_k^{\pm(s-1)} \otimes X^{\pm(k)}_{\overline{\tilde{\alpha}}}](\mathbf{h}\otimes\mathbf{v}) + \kappa' \\ &= \sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k^r X_k^{\pm s} \otimes (E_{\mp\theta}X^{\pm}_{\overline{\tilde{\alpha}}})^{(k)}(\mathbf{v}) - \sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k^r X_k^{\pm s} \otimes (X^{\pm}_{\overline{\tilde{\alpha}}}E_{\mp\theta})^{(k)}(\mathbf{v}) + \kappa'' \\ &= \sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k^r X_k^{\pm s} \otimes X^{\pm(k)}_{\overline{\alpha}}(\mathbf{v}) + \kappa'' \end{aligned}$$

The result for $H^{\pm}_{\alpha,r}$ follows immediately. □

We need to construct elements in $\widehat{\mathbf{Y}}$ which specialize to central elements of $\widehat{\mathfrak{sl}}_n[u^{\pm 1}, v]$ when $\lambda = \beta = 0$. Recall that the center of $\widehat{\mathfrak{sl}}_n[u^{\pm 1}, v]$ is isomorphic to $\Omega^1(\mathbb{C}[u^{\pm 1}, v])/d(\mathbb{C}[u^{\pm 1}, v])$. A basis for this space is $\{u^s v^r du | s \in \mathbb{Z}, r \geq 1\} \cup \{u^{-1}du\}$.

It is possible to define elements $J_r(z) \in Y_\lambda$ (with $J_1(z) = J(z)$) for any $r \geq 1$ with the following properties: $J_r(E_i^\pm) - X_{i,r}^\pm \in F_{r-1}(Y_\lambda)$ and they act on $\mathcal{F}(\mathbf{H})$ by $J_r(z)(\mathbf{h} \otimes \mathbf{v}) = \sum_{k=1} \mathbf{h}\mathcal{Y}_k^r \otimes z^{(k)}(\mathbf{v}) + \kappa$ where $\kappa \in F_{r+d-2}(\mathbf{V}^l)$ if $\mathbf{h} \in F_d(\mathbf{H})$.

For $r \geq 1, s \neq 0, 1 \leq i \leq n-1$, set $\overline{C}_{i,r,s} = \frac{1}{2}[\mathbf{K}_s(H_i), J_r(H_i)]$. For $\mathbf{h} \in F_d(\mathbf{H})$, there exists an element $\kappa \in F_{d+r-2}(\mathbf{V}^l)$ such that:

$$\begin{aligned}
\overline{C}_{i,r,s}(\mathbf{h}\otimes\mathbf{v}) &= \frac{1}{2}\sum_{k=1}^{l}\mathbf{h}[\mathcal{Y}_k^r, X_k^s]\otimes(E_{ii}+E_{i+1,i+1})^{(k)}(\mathbf{v}) + \frac{1}{2}\sum_{j\neq k}\mathbf{h}[\mathcal{Y}_j^r, X_k^s]\otimes H_i^{(j)}H_i^{(k)}(\mathbf{v}) + \kappa \\
&= \frac{1}{2}\sum_{k=1}^{l}\sum_{a=0}^{s-1}\sum_{b=0}^{r-1}\mathbf{h}\mathcal{Y}_k^b X_k^a[\mathcal{Y}_k, X_k]X_k^{s-a-1}\mathcal{Y}_k^{r-b-1}\otimes(E_{ii}+E_{i+1,i+1})^{(k)}(\mathbf{v}) \\
&\quad + \frac{1}{2}\sum_{j\neq k}\sum_{a=0}^{s-1}\sum_{b=0}^{r-1}\mathbf{h}\mathcal{Y}_j^b X_k^a[\mathcal{Y}_j, X_k]X_k^{s-a-1}\mathcal{Y}_j^{r-b-1}\otimes H_i^{(k)}H_i^{(j)}(\mathbf{v}) + \kappa \\
&= \frac{trs}{2}\sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k^{r-1}X_k^s\otimes(E_{ii}+E_{i+1,i+1})^{(k)}(\mathbf{v}) \\
&\quad + \frac{c}{4}\sum_{j\neq k}\sum_{a=0}^{s-1}\sum_{b=0}^{r-1}\mathbf{h}\mathcal{Y}_k^b X_k^a(X_k+X_j)X_j^{s-a-1}\mathcal{Y}_j^{r-b-1}\otimes\left(\sum_{d=1}^{n}(E_{di}^{(k)}E_{id}^{(j)} + E_{d,i+1}^{(k)}E_{i+1,d}^{(j)})\right)(\mathbf{v}) \\
&\quad - \frac{c}{4}\sum_{j\neq k}\sum_{a=0}^{s-1}\sum_{b=0}^{r-1}\mathbf{h}\mathcal{Y}_k^{r-b-1}X_k^a(X_k+X_j)X_j^{s-a-1}\mathcal{Y}_j^b\otimes(E_{ii}^{(j)}E_{ii}^{(k)} + E_{i+1,i+1}^{(j)}E_{i+1,i+1}^{(k)} \\
&\quad - E_{i,i+1}^{(j)}E_{i+1,i}^{(k)} - E_{i+1,i}^{(j)}E_{i,i+1}^{(k)})(\mathbf{v}) + \kappa' \text{ where } \kappa' \in F_{d+r-2}(\mathbf{V}^l)
\end{aligned}$$

$$= \frac{trs}{2}\sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k^{r-1}X_k^s\otimes(E_{ii}+E_{i+1,i+1})^{(k)}(\mathbf{v})$$

$$+\frac{c}{8}\sum_{a=0}^{s-1}\sum_{b=0}^{r-1}\Big(\sum_{d=1,d\neq i,i+1}^{n}\big(S(X_{\alpha_{id}+(s-a-1)\delta,r-b-1}, X_{\alpha_{di}+(a+1)\delta,b}) + S(X_{\alpha_{id}+(s-a)\delta,r-b-1}, X_{\alpha_{di}+a\delta,b}) \tag{31}$$

$$+S(X_{\alpha_{i+1,d}+(s-a-1)\delta,r-b-1}, X_{\alpha_{d,i+1}+(a+1)\delta,b}) + S(X_{\alpha_{i+1,d}+(s-a)\delta,r-b-1}, X_{\alpha_{d,i+1}+a\delta,b})\big) \tag{32}$$

$$+4S(X_{\alpha_{i+1,i}+(a+1)\delta,r-b-1}, X_{\alpha_{i,i+1}+(s-a-1)\delta,b}) + 4S(X_{\alpha_{i+1,i}+(s-a)\delta,b}, X_{\alpha_{i,i+1}+a\delta,r-b-1})\Big)(\mathbf{h}\otimes\mathbf{v}) \tag{33}$$

$$-\frac{c}{4}\sum_{k=1}^{l}\sum_{a=0}^{s-1}\sum_{b=0}^{r-1}\mathbf{h}\mathcal{Y}_k^{r-1}X_k^s\otimes\Big(\sum_{\substack{d=1\\ d\neq i,i+1}}^{n}(E_{dd}+E_{ii}+E_{dd}+E_{i+1,i+1}) + 4E_{ii} + 4E_{i+1,i+1}\Big)^{(k)}(\mathbf{v}) + \kappa''$$

where $\kappa'' \in F_{d+r-2}(\mathbf{V}^l)$.

Set $C_{i,r,s} = \overline{C}_{i,r,s} - (31)' - (32)' - (33)'$ where $(33)'$ is the expression on line (33) without $\mathbf{h}\otimes\mathbf{v}$, etc., and set $C_{r,s} = \sum_{i=1}^{n} C_{i,r,s}$. (When $i = n, E_{d,i+1} = E_{d1}$, etc.) The element $C_{r,s}$ acts on $\mathbf{V}^l$ by

$$C_{r,s}(\mathbf{h}\otimes\mathbf{v}) = rs(t-cn)\sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k^{r-1}X_k^s\otimes\mathbf{v} + \kappa \text{ where } \kappa \in F_{d+r-2}(\mathbf{V}^l).$$

We still have to define elements $C_{r,0}$ which correspond to $u^{-1}v^r du \in \widehat{\mathfrak{sl}}_n[u^{\pm1}, v]$ when $\lambda = \beta = 0$. We would like to define elements $\widetilde{J}_r(z) \in F_r(\widehat{\mathbf{Y}})$ for $z \in \mathfrak{sl}_n$ which act on $\mathbf{V}^l$ in the following way: ($\mathbf{h} \in F_d(\mathbf{H})$)

$$\widetilde{J}_r(z)(\mathbf{h}\otimes\mathbf{v}) = \frac{1}{2}\sum_{j=1}^{l}\mathbf{h}S(X_j, \mathcal{Y}_j^r)\otimes z^{(j)}(\mathbf{v}) + \kappa$$

where $\kappa \in F_{d+r-2}(\mathbf{V}^l)$. When $r=0$, $\widetilde{J}_r(z) = \mathbf{K}_1(z)$. For $r=1$, see section 7 in [17]. Let us assume that $r \geq 1$ in the following series of computations leading to the definition of $C_{r,0}$.

Consider the element $\frac{1}{2}[J_r(H_{de}), \mathbf{K}_1(E_{de})] \in \widehat{\mathbf{Y}}$ where $H_{de} = E_{dd} - E_{ee}$. Then $\frac{1}{2}[J_r(H_{de}), \mathbf{K}_1(E_{de})](\mathbf{h} \otimes \mathbf{v})$ is equal to: ($\mathbf{h} \in F_d(\mathbf{H})$)

$$
\begin{aligned}
&= \frac{1}{2}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} \mathbf{h}[X_k, \mathcal{Y}_j^r] \otimes H_{de}^{(j)} E_{de}^{(k)}(\mathbf{v}) + \frac{1}{2}\sum_{j=1}^{l} \mathbf{h} S(X_j, \mathcal{Y}_j^r) \otimes E_{de}^{(j)}(\mathbf{v}) + \kappa,\ \kappa \in F_{d+r-2}(\mathbf{V}^l) \\
&= \frac{1}{2}\sum_{j\neq k}\sum_{a=0}^{r-1} \mathbf{h}\mathcal{Y}_j^a [X_k, \mathcal{Y}_j]\mathcal{Y}_j^{r-a-1} \otimes H_{de}^{(j)} E_{de}^{(k)}(\mathbf{v}) + \frac{1}{2}\sum_{j=1}^{l} \mathbf{h} S(X_j, \mathcal{Y}_j^r) \otimes E_{de}^{(j)}(\mathbf{v}) + \kappa \\
&= \frac{\lambda}{4}\sum_{j\neq k}\sum_{a=0}^{r-1} \mathbf{h}\left(\left(\frac{S(\mathcal{Y}_j^a, X_j)}{2}\right)\mathcal{Y}_k^{r-a-1} + \mathcal{Y}_j^a\left(\frac{S(X_k, \mathcal{Y}_k^{r-1-a})}{2}\right)\right) \otimes (E_{dd}^{(j)} E_{de}^{(k)} - E_{de}^{(j)} E_{ee}^{(k)})(\mathbf{v}) \\
&\quad + \frac{1}{2}\sum_{j=1}^{l} \mathbf{h} S(X_j, \mathcal{Y}_j^r) \otimes E_{de}^{(j)}(\mathbf{v}) + \kappa' \text{ where } \kappa' \in F_{d+r-2}(\mathbf{V}^l) \\
&= \frac{1}{2}\sum_{j=1}^{l} \mathbf{h} S(X_j, \mathcal{Y}_j^r) \otimes E_{de}^{(j)}(\mathbf{v}) + \frac{\lambda}{8}\sum_{a=0}^{r-1}\big(S(J_{r-a-1}(E_{de}), \widetilde{J}_a(H_{de})) \\
&\quad + S(J_a(H_{de}), \widetilde{J}_{r-a-1}(E_{de}))\big)(\mathbf{h} \otimes \mathbf{v}) + \kappa''
\end{aligned}
$$

Set

$$
\widetilde{J}_r(E_{de}) = \frac{1}{2}[J_r(H_{de}), \mathbf{K}_1(E_{de})] - \frac{\lambda}{8}\sum_{a=0}^{r-1}\big(S(J_{r-a-1}(E_{de}), \widetilde{J}_a(H_{de})) + S(J_a(H_{de}), \widetilde{J}_{r-a-1}(E_{de}))\big)
$$

and $\overline{C}_{i,r,0} = [\widetilde{J}_r(H_i), \mathbf{K}_{-1}(H_i)]$ where $\widetilde{J}_r(H_i) = [E_{i,i+1}, \widetilde{J}_r(E_{i+1,i})]$. Then

$$
\begin{aligned}
\overline{C}_{i,r,0}(\mathbf{h} \otimes \mathbf{v}) &= \frac{1}{2}\sum_{k=1}^{l} \mathbf{h}\left(X_k^{-1}\left(\frac{S(X_k, \mathcal{Y}_k^r)}{2}\right) - \left(\frac{S(X_k, \mathcal{Y}_k^r)}{2}\right)X_k^{-1}\right) \otimes (E_{ii} + E_{i+1,i+1})^{(k)}(\mathbf{v}) \\
&\quad + \sum_{j\neq k} \mathbf{h}\left[X_k^{-1}, \frac{S(X_j, \mathcal{Y}_j^r)}{2}\right] \otimes H_i^{(j)} H_i^{(k)}(\mathbf{v}) + \kappa \text{ where } \kappa \in F_{d+r-2}(\mathbf{V}^l) \\
&= \frac{1}{4}\sum_{k=1}^{l} \mathbf{h}\big([X_k^{-1}, \mathcal{Y}_k^r]X_k - X_k[\mathcal{Y}_k^r, X_k^{-1}]\big) \otimes (E_{ii} + E_{i+1,i+1})^{(k)}(\mathbf{v}) \\
&\quad + \frac{1}{2}\sum_{j\neq k} \mathbf{h}\left[X_k^{-1}, \frac{S(X_j, \mathcal{Y}_j^r)}{2}\right] \otimes H_i^{(j)} H_i^{(k)}(\mathbf{v}) + \kappa \\
&= \frac{1}{4}\sum_{k=1}^{l}\sum_{a=0}^{r-1} \mathbf{h} S\Big(\mathcal{Y}_k^a\big(tX_k^{-1} + \frac{c}{2}\sum_{j\neq k}(X_k^{-1} + X_j^{-1})s_{jk}\big)\mathcal{Y}_k^{r-a-1}, X_k\Big) \otimes (E_{ii} + E_{i+1,i+1})^{(k)}(\mathbf{v}) \\
&\quad - \frac{c}{8}\sum_{j\neq k}\sum_{a=0}^{r-1} \mathbf{h} S(X_j, \mathcal{Y}_j^a(X_k^{-1} + X_j^{-1})s_{jk}\mathcal{Y}_j^{r-a-1}) \otimes H_i^{(j)} H_i^{(k)}(\mathbf{v}) + \kappa \\
&= \frac{tr}{2}\sum_{k=1}^{l} \mathbf{h}\mathcal{Y}_k^{r-1} \otimes (E_{ii} + E_{i+1,i+1})^{(k)}(\mathbf{v}) + \frac{c}{8}\sum_{k=1}^{l}\sum_{a=0}^{r-1}\sum_{j\neq k} \mathbf{h}\big(\mathcal{Y}_k^a(X_k^{-1} + X_j^{-1})\mathcal{Y}_j^{r-a-1}X_j \\
&\quad + X_k\mathcal{Y}_k^a(X_k^{-1} + X_j^{-1})\mathcal{Y}_j^{r-a-1}\big) \otimes \left(\sum_{d=1}^{n}\big(E_{di}^{(k)} E_{id}^{(j)} + E_{d,i+1}^{(j)} E_{i+1,d}^{(k)}\big)\right)(\mathbf{v})
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{c}{8}\sum_{j\neq k}\sum_{a=0}^{r-1}\mathbf{h}\big(X_j\mathcal{Y}_j^a(X_k^{-1}+X_j^{-1})\mathcal{Y}_k^{r-a-1}+\mathcal{Y}_j^a(X_k^{-1}+X_j^{-1})\mathcal{Y}_k^{r-a-1}X_k\big)\otimes \\
& \big(E_{ii}^{(j)}E_{ii}^{(k)}+E_{i+1,i+1}^{(j)}E_{i+1,i+1}^{(k)}-E_{i,i+1}^{(j)}E_{i+1,i}^{(k)}-E_{i+1,i}^{(j)}E_{i,i+1}^{(k)}\big)(\mathbf{v})+\kappa',\ \kappa'\in F_{d+r-2}(\mathbf{V}^l) \\
= & \frac{tr}{2}\sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k^{r-1}\otimes(E_{ii}+E_{i+1,i+1})^{(k)}(\mathbf{v})+\frac{c}{8}\sum_{j\neq k}\sum_{a=0}^{r-1}\mathbf{h}(\mathcal{Y}_k^aX_k^{-1}\mathcal{Y}_j^{r-a-1}X_j+2\mathcal{Y}_k^a\mathcal{Y}_j^{r-a-1} \\
& +\mathcal{Y}_k^aX_k\mathcal{Y}_j^{r-1-a}X_j^{-1})\otimes\left(\sum_{d=1,d\neq i}^{n}E_{di}^{(k)}E_{id}^{(j)}+\sum_{d=1,d\neq i+1}^{n}E_{d,i+1}^{(j)}E_{i+1,d}^{(k)}\right)(\mathbf{v}) \\
& +\frac{c}{8}\sum_{j\neq k}\sum_{a=0}^{r-1}\mathbf{h}(\mathcal{Y}_k^aX_k\mathcal{Y}_j^{r-a-1}X_j^{-1}+2\mathcal{Y}_k^a\mathcal{Y}_j^{r-a-1}+\mathcal{Y}_k^aX_k^{-1}\mathcal{Y}_j^{r-a-1}X_j)\otimes(E_{i+1,i}^{(j)}E_{i,i+1}^{(k)} \\
& +E_{i,i+1}^{(j)}E_{i+1,i}^{(k)})(\mathbf{v})+\kappa'' \\
= & \frac{tr}{2}\sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k^{r-1}\otimes(E_{ii}+E_{i+1,i+1})^{(k)}(\mathbf{v}) \\
& +\frac{c}{16}\sum_{a=0}^{r-1}\Big(\sum_{\substack{d=1\\ d\neq i,i+1}}^{n}\big(S(X_{\alpha_{id}-\delta,r-1-a},X_{\alpha_{di}+\delta,a})+2S(X_{\alpha_{di},a},X_{\alpha_{id},r-a-1}) && (34) \\
& +S(X_{\alpha_{id}+\delta,r-a-1},X_{\alpha_{di}-\delta,a})+S(X_{\alpha_{i+1,d}-\delta,r-1-a},X_{\alpha_{d,i+1}+\delta,a})+2S(X_{\alpha_{i+1,d},r-a-1},X_{\alpha_{d,i+1},a}) && (35) \\
& +S(X_{\alpha_{i+1,d}+\delta,r-a-1},X_{\alpha_{d,i+1}-\delta,a})\big)+4\big(S(X_{\alpha_{i+1,i}-\delta,r-1-a},X_{\alpha_{i,i+1}+\delta,a}) && (36) \\
& +2S(X_{\alpha_{i+1,i},r-a-1},X_{\alpha_{i,i+1},a})+S(X_{\alpha_{i+1,i}+\delta,r-a-1},X_{\alpha_{i,i+1}-\delta,a})\big)\Big)(\mathbf{h}\otimes\mathbf{v}) && (37) \\
& -\frac{cr}{4}\sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k^{r-1}\otimes\left(\sum_{\substack{d=1\\ d\neq i,i+1}}^{n}(E_{dd}+E_{ii})+\sum_{\substack{d=1\\ d\neq i,i+1}}^{n}(E_{dd}+E_{i+1,i+1})+4E_{ii}+4E_{i+1,i+1}\right)^{(k)}(\mathbf{v})+\kappa'''
\end{aligned}
$$

Set $C_{i,r,0}=\overline{C}_{i,r,0}-(34)'-(35)'-(36)'-(37)'$ where $(37)'$ is the expression on line (37) but without $\mathbf{h}\otimes\mathbf{v}$, set $C_{r,0}=\sum_{i=1}^{n}C_{i,r,0}$ and $C_{0,0}=H_0+H_1+\ldots+H_{n-1}$. The element $C_{r,0}$ acts on $\mathbf{V}^l$ by

$$C_{r,0}(\mathbf{h}\otimes\mathbf{v})=r(t-cn)\sum_{k=1}^{l}\mathbf{h}\mathcal{Y}_k^{r-1}\otimes\mathbf{v}+\kappa \text{ where } \kappa\in F_{d+r-2}(\mathbf{V}^l),\mathbf{h}\in F_d(\mathbf{H}).$$

We now have all the elements that we need to construct a PBW basis for $\widehat{\mathbf{Y}}$. Set $\mathbf{B}=\{X_{\alpha,r}^{\pm},H_{i,s,r}^{\pm},H_{i,0,r}|\alpha\in\widehat{\Delta}^{re,+},r\in\mathbb{Z}_{\geq0},s\in\mathbb{Z}_{>0},1\leq i\leq n-1\}\cup\{C_{r,s}|r\in\mathbb{Z}_{\geq1},s\in\mathbb{Z}\setminus\{0\}\text{ or }s=0,r\geq0\}$ where $H_{i,s,r}^{\pm}=H_{\overline{\alpha}_i+s\delta,r}^{\pm}$.

We need a total ordering on the set $\mathbf{B}$. For instance, we could choose the following one: $X_{\alpha^1,r_1}^{-}<H_{\alpha^2,r_2}^{-}<H_{j,0,r_5}<H_{\alpha^3,r_3}^{+}<X_{\alpha^4,r_4}^{+}<C_{r,s}$ for any $\alpha^i\in\widehat{\Delta}^{+},i=1,\ldots,4,r_i\in\mathbb{Z}_{\geq0},i=1,\ldots,5,(r,s)\in\mathbb{Z}_{\geq1}\times\mathbb{Z}\setminus\{0\}\cup\mathbb{Z}_{\geq0}\times\{0\}$; $X_{\alpha^1,r_1}^{\pm}<X_{\alpha^2,r_2}^{\pm},H_{\alpha^1,r_1}^{\pm}<H_{\alpha^2,r_2}^{\pm},C_{r_1,s_1}<C_{r_2,s_2},H_{j_1,0,r_1}<H_{j_2,0,r_2}$ if $r_1<r_2$ or if $r_1=r_2$ and $\alpha^1<\alpha^2,s_1<s_2,j_1<j_2$, respectively. Set $\mathbf{B}^{LY}=\mathbf{B}\setminus\{C_{0,0}\}$.

**Theorem 7.1.** *The set of ordered monomials in the elements of* $\mathbf{B}$ *(resp.* $\mathbf{B}^{LY}$*) is a vector space basis of* $\widehat{\mathbf{Y}}_{\lambda,\beta}$ *(resp.* $LY_{\lambda,\beta}$*).*

*Proof.* The monomials in $\mathbf{B}$ span $\widehat{\mathbf{Y}}$ since $gr(\widehat{\mathbf{Y}})$ is an epimorphic image of $\mathfrak{U}(\widehat{\mathfrak{sl}}_n[u^{\pm1},v])$, so we have to prove that they are linearly independent.

We prove the theorem for $LY$ first. Suppose that we have a relation of the form

$$\sum_{\mathbf{d}\in S_1}\sum_{\substack{\mathbf{r}\in S_2(\mathbf{d}),\mathbf{I}\in S_3(\mathbf{d})\\ \mathbf{f}\in S_4(\mathbf{d})}} c(\mathbf{d},\mathbf{r},\mathbf{I},\mathbf{f})\mathbf{X}^-_{A,r^{X,-}}\cdot\mathbf{H}^-_{I,r^{H,-}}\cdot\mathbf{H}_{J,r^H}\cdot\mathbf{H}^+_{K,r^{H,+}}\cdot\mathbf{X}^+_{B,r^{X,+}}\cdot\mathbf{C}_{r^C,s^C}=0 \tag{38}$$

where $S_1, S_2(\mathbf{d}), S_3(\mathbf{d}), S_4(\mathbf{d})$ are finite sets and

$$\mathbf{X}^-_{A,r^{X,-}}=(X^-_{\alpha^1,r_1^{X,-}})^{f_1^{X,-}}\cdots(X^-_{\alpha^{d^{X,-}},r^{X,-}_{d^{X,-}}})^{f^{X,-}_{d^{X,-}}},\ \ \mathbf{H}^-_{I,r^{H^-}}=(H^-_{i_1,r_1^{H,-}})^{f_1^{H,-}}\cdots(H^-_{i_{d^{H,-}},r^{H,-}_{d^{H,-}}})^{f^{H,-}_{d^{H,-}}}$$

$$\mathbf{C}_{r^C,s^C}=(C_{r_1^C,s_1^C})^{f_1^C}\cdots(C_{r^C_{d^C},s^C_{d^C}})^{f^C_{d^C}},\ \ \mathbf{H}_J=(H_{j_1,0,r_1^H})^{f_1^H}\cdots(H_{j_{d^H},0,r^H_{d^H}})^{f^H_{d^H}}$$

$$\mathbf{X}^+_{B,r^{X,+}}=(X^+_{\beta^1,r_1^{X,+}})^{f_1^{X,+}}\cdots(X^+_{\beta^{d^{X,+}},r^{X,+}_{d^{X,+}}})^{f^{X,+}_{d^{X,+}}},\ \ \mathbf{H}^+_{K,r^{H^+}}=(H^+_{k_1,r_1^{H,+}})^{f_1^{H,+}}\cdots(H^+_{k_{d^{H,+}},r^{H,+}_{d^{H,+}}})^{f^{H,+}_{d^{H,+}}}$$

and

$$\mathbf{d}=(d^{X,-},d^{H,-},d^H,d^{H,+},d^{X,+},d^C)\in S_1\subset\mathbb{Z}^{\times 6}_{\geq 0}$$

$$\mathbf{r}=(r^{X,-},r^{H,-},r^H,r^{H,+},r^{X,+},r^C,s^C),\mathbf{I}=(A,I,J,K,B),\mathbf{f}=(f^{X,-},f^{H,-},f^H,f^{H,+},f^{X,+},f^C)$$

$$S_2(\mathbf{d})\subset\mathbb{Z}^{\times d^{X,-}}_{\geq 0}\times\mathbb{Z}^{\times d^{H,-}}_{\geq 0}\times\mathbb{Z}^{\times d^H}_{\geq 0}\times\mathbb{Z}^{\times d^{H,+}}_{\geq 0}\times\mathbb{Z}^{\times d^{X,+}}_{\geq 0}\times(\mathbb{Z}_{\geq 1}\times\mathbb{Z})^{\times d^C},$$

$$S_3(\mathbf{d})\subset(\widehat{\Delta}^+)^{\times d^{X,-}}\times([n-1]\times\mathbb{Z}_{>0})^{\times d^{H,-}}\times[n-1]^{\times d^H}\times([n-1]\times\mathbb{Z}_{>0})^{\times d^{H,+}}\times(\widehat{\Delta}^+)^{\times d^{X,+}}$$

and

$$S_4(\mathbf{d})\subset\mathbb{Z}^{\times d^{X,-}}_{\geq 0}\times\mathbb{Z}^{\times d^{H,-}}_{\geq 0}\times\cdots\times\mathbb{Z}^{\times d^{X,+}}_{\geq 0}\times\mathbb{Z}^{\times d^C}_{\geq 0}$$

$A=\{\alpha^1,\ldots,\alpha^{d^{X,-}}\}, I=\{i_1,\ldots,i_{d^{H,-}}\}, i_p=(\underline{i}_p,s_p^{H,-})\in[n-1]\times\mathbb{Z}_{>0}, B=\{\beta^1,\ldots,\beta^{d^{X,+}}\}, J=\{j_1,\ldots,j_{d^H}\}, K=\{k_1,\ldots,k_{d^{H,+}}\}, k_p=(\underline{k}_p,s_p^{H,+})\in[n-1]\times\mathbb{Z}_{>0}$ and $[n-1]=\{1,\ldots,n-1\}$. We fix a particular choice $\check{\mathbf{d}},\check{\mathbf{r}},\check{\mathbf{I}},\check{\mathbf{f}}$, of these index sets such that $c(\check{\mathbf{d}},\check{\mathbf{r}},\check{\mathbf{I}},\check{\mathbf{f}})\neq 0$ and the corresponding monomial

$$\check{\mathbf{M}}=\mathbf{X}^-_{\check{A},\check{r}^{X,-}}\cdot\mathbf{H}^-_{\check{I},\check{r}^{H,-}}\cdot\mathbf{H}_{\check{J},\check{r}^H}\cdot\mathbf{H}^+_{\check{K},\check{r}^{H,+}}\cdot\mathbf{X}^+_{\check{B},\check{r}^{X,+}}\cdot\mathbf{C}_{\check{r}^C,\check{s}^C}$$

in (38) has the following properties:

1. It has maximum value for $\sum_{g=1}^{d^{X,-}}f_g^{X,-}r_g^{X,-}+\sum_{g=1}^{d^{H,-}}f_g^{X,-}r_g^{H,-}+\sum_{g=1}^{d^H}f_g^Hr_g^H+\sum_{g=1}^{d^{H,+}}f_g^{H,+}r_g^{H,+}+\sum_{g=1}^{d^{X,+}}f_g^{X,+}r_g^{X,+}+\sum_{g=1}^{d^C}f_g^Cr_g^C$;
2. and, among these, it has maximum value for $\delta^{X,+}=\sum_{g=1}^{d^{X,+}}f_g^{X,+}$;
3. and, among these, it has maximum value for $\delta^{X,-}=\sum_{g=1}^{d^{X,-}}f_g^{X,-}$;
4. and, among these, it has maximum value for $\delta^{H,+}=\sum_{g=1}^{d^{H,+}}f_g^{H,+}$;
5. and, among these, it has maximum value for $\delta^{H,-}=\sum_{g=1}^{d^{H,-}}f_g^{H,-}$;
6. and, among these, it has maximum value for $\delta^H=\sum_{g=1}^{d^H}f_g^H$;
7. and, among these, it has maximum value for $\delta^C=\sum_{g=1}^{d^C}f_g^C$.

Set $\widehat{\delta}^{X,-}=\check{\delta}^{X,-},\widehat{\delta}^{H,-}=\widehat{\delta}^{X,-}+\check{\delta}^{H,-},\widehat{\delta}^H=\widehat{\delta}^{H,-}+\check{\delta}^H,\widehat{\delta}^{H,+}=\widehat{\delta}^H+\check{\delta}^{H,+},\widehat{\delta}^{X,+}=\widehat{\delta}^{H,+}+\check{\delta}^{X,+},\widehat{\delta}^C=\widehat{\delta}^{X,+}+\check{\delta}^C$. Consider the module $\mathbf{V}^l$ with $l\geq\widehat{\delta}^C$. We choose $\mathbf{v}_l=v^1\otimes\cdots\otimes v^l,\widetilde{\mathbf{v}}_l=\widetilde{v}^1\otimes\cdots\otimes\widetilde{v}^l\in(\mathbb{C}^n)^{\otimes l}$ to be the following elements:

If $\check{\alpha}^g=\overline{\check{\alpha}^g}+\check{s}_g^A\delta$ with $\overline{\check{\alpha}^g}=\alpha_{p_gq_g}\in\Delta$ and $\check{s}_g^A\in\mathbb{Z}_{\geq 0}, 1\leq p_g\neq q_g\leq n$, we set $v^\nu=v_{p_g},\widetilde{v}^\nu=v_{q_g}$ for $\check{f}_1^{X,-}+\ldots+\check{f}_{g-1}^{X,-}<\nu\leq\check{f}_1^{X,-}+\ldots+\check{f}_g^{X,-}$.

Set $v^\nu = v_1 + \ldots + v_n, \widetilde{v}^\nu = v_{\underline{\check{i}}_g} - v_{\underline{\check{i}}_g+1}$ for $\widehat{\delta}^{X,-} + \check{f}_1^{H,-} + \ldots + \check{f}_{g-1}^{H,-} < \nu \le \widehat{\delta}^{X,-} + \check{f}_1^{H,-} + \ldots + \check{f}_g^{H,-}$.
Set $v^\nu = v_1 + \ldots + v_n, \widetilde{v}^\nu = v_{\underline{\check{j}}_g} - v_{\underline{\check{j}}_g+1}$ for $\widehat{\delta}^{H,-} + \check{f}_1^{H} + \ldots + \check{f}_{g-1}^{H} < \nu \le \widehat{\delta}^{H,-} + \check{f}_1^{H} + \ldots + \check{f}_g^{H}$.
Set $v^\nu = v_1 + \ldots + v_n, \widetilde{v}^\nu = v_{\underline{\check{k}}_g} - v_{\underline{\check{k}}_g+1}$ for $\widehat{\delta}^{H} + \check{f}_1^{H,+} + \ldots + \check{f}_{g-1}^{H,+} < \nu \le \widehat{\delta}^{H} + \check{f}_1^{H,+} + \ldots + \check{f}_g^{H,+}$.
If $\check{\beta}^g = \overline{\check{\beta}^g} + \check{s}_g^B\delta$ with $\overline{\check{\beta}^g} = \alpha_{p_g q_g} \in \Delta$ and $\check{s}_g^B \in \mathbb{Z}_{\ge 0}$, we set $v^\nu = v_{q_g}, \widetilde{v}^\nu = v_{p_g}$ for $\widehat{\delta}^{H,+} + \check{f}_1^{X,+} + \ldots + \check{f}_{g-1}^{X,+} < \nu \le \widehat{\delta}^{H,+} + \check{f}_1^{X,+} + \ldots + \check{f}_g^{X,+}$. For $\nu > \widehat{\delta}^{X,+}$, we set $v^\nu = \widetilde{v}^\nu = v_1 + v_2 + \ldots + v_n$.

Below, we will consider the basis of $(\mathbb{C}^n)^{\otimes l}$ given by the elements of the form $\check{\mathbf{v}} = \check{v}^1 \otimes \cdots \otimes \check{v}^l$ where $\check{v}^\nu \in \{v_1, \ldots, v_n\}$ if $1 \le \nu \le \widehat{\delta}^{X,-}$ or if $\widehat{\delta}^{H,+} < \nu \le \widehat{\delta}^{X,+}$, and $\check{v}^\nu \in \{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_1 + v_2 + \ldots + v_n\}$ if $\widehat{\delta}^{X,-} < \nu \le \widehat{\delta}^{H,+}$ or if $\widehat{\delta}^{X,+} < \nu \le l$.

Because of our assumption,

$$\sum_{\mathbf{d}\in S_1} \sum_{\substack{\mathbf{r}\in S_2(\mathbf{d}),\mathbf{I}\in S_3(\mathbf{d}) \\ \mathbf{f}\in S_4(\mathbf{d})}} c(\mathbf{d},\mathbf{r},\mathbf{I},\mathbf{f}) \mathbf{X}^-_{A,r^{X,-}} \cdot \mathbf{H}^-_{I,r^{H,-}} \cdot \mathbf{H}_{J,r^H} \cdot \mathbf{H}^+_{K,r^{H,+}} \cdot \mathbf{X}^+_{B,r^{X,+}} \cdot \mathbf{C}_{r^C,s^C}(1 \otimes \mathbf{v}_l) = 0. \tag{39}$$

Set

$$\xi_{\mathcal{Y}}^{X,-} = \prod_{g=1}^{\check{d}^{X,-}} \prod_{\nu=\check{f}_1^{X,-}+\ldots+\check{f}_{g-1}^{X,-}+1}^{\check{f}_1^{X,-}+\ldots+\check{f}_g^{X,-}} \mathcal{Y}_\nu^{\check{r}_g^{X,-}}, \quad \xi_X^{X,-} = \prod_{g=1}^{\check{d}^{X,-}} \prod_{\nu=\check{f}_1^{X,-}+\ldots+\check{f}_{g-1}^{X,-}+1}^{\check{f}_1^{X,-}+\ldots+\check{f}_g^{X,-}} X_\nu^{\check{s}_g^A}$$

$$\xi_{\mathcal{Y}}^{H,-} = \prod_{g=1}^{\check{d}^{H,-}} \prod_{\nu=\widehat{\delta}^{X,-}+\check{f}_1^{H,-}+\ldots+\check{f}_{g-1}^{H,-}+1}^{\widehat{\delta}^{X,-}+\check{f}_1^{H,-}+\ldots+\check{f}_g^{H,-}} \mathcal{Y}_\nu^{\check{r}_g^{H,-}}, \quad \xi_X^{H,-} = \prod_{g=1}^{\check{d}^{H,-}} \prod_{\nu=\widehat{\delta}^{X,-}+\check{f}_1^{H,-}+\ldots+\check{f}_{g-1}^{H,-}+1}^{\widehat{\delta}^{X,-}+\check{f}_1^{H,-}+\ldots+\check{f}_g^{H,-}} X_\nu^{\check{s}_g^{H,-}}$$

$$\xi_{\mathcal{Y}}^{H} = \prod_{g=1}^{\check{d}^{H}} \prod_{\nu=\widehat{\delta}^{H,-}+f_1^{H}+\ldots+f_{g-1}^{H}+1}^{\widehat{\delta}^{H,-}+\check{f}_1^{H}+\ldots+\check{f}_g^{H}} \mathcal{Y}_\nu^{\check{r}_g^{H}}$$

$$\xi_{\mathcal{Y}}^{H,+} = \prod_{g=1}^{\check{d}^{H,+}} \prod_{\nu=\widehat{\delta}^{H}+\check{f}_1^{H,+}+\ldots+\check{f}_{g-1}^{H,+}+1}^{\widehat{\delta}^{H}+\check{f}_1^{H,+}+\ldots+\check{f}_g^{H,+}} \mathcal{Y}_\nu^{\check{r}_g^{H,+}}, \quad \xi_X^{H,+} = \prod_{g=1}^{\check{d}^{H,+}} \prod_{\nu=\widehat{\delta}^{H}+\check{f}_1^{H,+}+\ldots+\check{f}_{g-1}^{H,+}+1}^{\widehat{\delta}^{H}+\check{f}_1^{H,+}+\ldots+\check{f}_g^{H,+}} X_\nu^{\check{s}_g^{H,+}}$$

$$\xi_{\mathcal{Y}}^{X,+} = \prod_{g=1}^{\check{d}^{X,+}} \prod_{\nu=\widehat{\delta}^{H,+}+\check{f}_1^{X,+}+\ldots+\check{f}_{g-1}^{X,+}+1}^{\widehat{\delta}^{H,+}+\check{f}_1^{X,+}+\ldots+\check{f}_g^{X,+}} \mathcal{Y}_\nu^{\check{r}_g^{X,+}}, \quad \xi_X^{X,+} = \prod_{g=1}^{\check{d}^{X,+}} \prod_{\nu=\widehat{\delta}^{H,+}+\check{f}_1^{X,+}+\ldots+\check{f}_{g-1}^{X,+}+1}^{\widehat{\delta}^{H,+}+\check{f}_1^{X,+}+\ldots+\check{f}_g^{X,+}} X_\nu^{\check{s}_g^{B}}$$

$$\xi_{\mathcal{Y}}^{C} = \prod_{g=1}^{\check{d}^{C}} \prod_{\nu=\widehat{\delta}^{X,+}+\check{f}_1^{C}+\ldots+\check{f}_{g-1}^{C}+1}^{\widehat{\delta}^{X,+}+\check{f}_1^{C}+\ldots+\check{f}_g^{C}} \mathcal{Y}_\nu^{\check{r}_g^{C}}, \quad \xi_X^{C} = \prod_{g=1}^{\check{d}^{C}} \prod_{\nu=\widehat{\delta}^{X,+}+\check{f}_1^{C}+\ldots+\check{f}_{g-1}^{C}+1}^{\widehat{\delta}^{X,+}+\check{f}_1^{C}+\ldots+\check{f}_g^{C}} X_\nu^{\check{s}_g^{C}}$$

Set $\xi_{\mathcal{Y}} = \xi_{\mathcal{Y}}^C \xi_{\mathcal{Y}}^{X,+} \xi_{\mathcal{Y}}^{H,+} \xi_{\mathcal{Y}}^H \xi_{\mathcal{Y}}^{H,-} \xi_{\mathcal{Y}}^{X,-}$ and $\xi_X = \xi_X^C \xi_X^{X,+} \xi_X^{H,+} \xi_X^{H,-} \xi_X^{X,-}$ and consider the coefficient of $\xi_{\mathcal{Y}} \xi_X \otimes \widetilde{\mathbf{v}}$ on the left-hand side of equality (39). Applying our particular choice of monomial $\check{\mathbf{M}}$ to $1 \otimes \mathbf{v}$ and writing down the element of $\mathbf{V}^l$ thus obtained as a sum of basis elements of the type $m(\mathcal{Y}_1, \ldots, \mathcal{Y}_l) m(X_1^{\pm 1}, \ldots, X_l^{\pm 1}) \otimes \check{\mathbf{v}}$, where $m(\mathcal{Y}_1, \ldots, \mathcal{Y}_l)$ and $m(X_1^{\pm 1}, \ldots, X_l^{\pm 1})$ are monomials, we see that, in $\check{\mathbf{M}}(1 \otimes \mathbf{v})$, the element $\xi_{\mathcal{Y}} \xi_X \otimes \widetilde{\mathbf{v}}$ appears with coefficient equal to $\check{a} c(\check{\mathbf{d}}, \check{\mathbf{r}}, \check{\mathbf{I}}, \check{\mathbf{f}}) l^{\check{e}}$ where $\check{a}$ is a non-zero scalar (which can be expressed in terms of $t, c, n$ and the different values of $\check{r}, \check{s}, \check{f}$) and $\check{e}$ is equal to the multiplicity of $C_{1,0}$ in $\check{\mathbf{M}}$. (Here, we use our assumption that $t \neq cn$.) Moreover, the only other monomials in (38) which can produce a non-zero scalar multiple of $\xi_{\mathcal{Y}} \xi_X \otimes \widetilde{\mathbf{v}}_l$ when applied to $1 \otimes \mathbf{v}_l$ must differ from $\check{\mathbf{M}}$ only by the multiplicity of $C_{1,0}$.

Now choose any $l_1 > l$. We can apply the left-hand side of (38) to $1 \otimes \mathbf{v}_{l_1}$ and expand the elements of $\mathbf{V}^l$ as a sum of basis vectors as above. The element $\xi_{\mathcal{Y}} \xi_X \otimes \widetilde{\mathbf{v}}_{l_1}$ will appear in $\check{\mathbf{M}}(1 \otimes \mathbf{v})$ with coefficient equal to $\check{a} c(\check{\mathbf{d}}, \check{\mathbf{r}}, \check{\mathbf{I}}, \check{\mathbf{f}}) l_1^{\check{e}}$. Therefore, we can view the coefficient of $\xi_{\mathcal{Y}} \xi_X \otimes \widetilde{\mathbf{v}}_l$ in (39) as a polynomial in $l$.

Since this polynomial must be zero for infinitely many values of $l$ because of the vanishing of (39), it must vanish identically, hence $c(\check{\mathbf{d}}, \check{\mathbf{r}}, \check{\mathbf{I}}, \check{\mathbf{f}}) = 0$. We can repeat this argument to conclude that all the coefficients in relation (38) are zero.

We must now extend our proof from $LY$ to $\widehat{\mathbf{Y}}$. We will follow some of the ideas in [3]. We need to consider a completion of $\widehat{\mathbf{Y}}$. For $k \geq 0$, we denote by $\mathbf{Y}_k$ the span of all monomials of the form given in (38) with $\max\{\max(A), \max(H,-), \max(H,+), \max(B), \max(C)\} \geq k$, where $\max(A) = \max\{s_g^A, g = 1, \ldots, d^{X,-}\}$, $\max(H,\pm) = \max\{s_g^{H,\pm}, g = 1, \ldots, d^{H,\pm}\}$, $\max(B) = \max\{s_g^B, g = 1, \ldots, d^{X,+}\}$, $\max(C) = \max\{s_g^C, g = 1, \ldots, d^C\}$. We let $\overline{\mathbf{Y}}$ be the completion of $\widehat{\mathbf{Y}}$ with respect to the system of neighborhoods of 0 given by the $\mathbf{Y}_k$'s.

We can define an algebra homomorphism $\Delta$ from $\widehat{\mathbf{Y}}$ to the completed tensor product $\overline{\mathbf{Y}}\widehat{\otimes}\overline{\mathbf{Y}}$ in the following way. It is the usual coproduct on $\mathfrak{U}(\widehat{\mathfrak{sl}}_n[u^{\pm 1}]) \subset \widehat{\mathbf{Y}}$ and, for $1 \leq i \leq n-1$,

$$\begin{aligned}\Delta(H_{i,1}) &= H_{i,1} \otimes 1 + 1 \otimes H_{i,1} + \lambda H_i \otimes H_i + \lambda \sum_{\alpha\in\Delta^+}\sum_{s\geq 1}(\alpha,\alpha_i)(E_\alpha u^{-s}) \otimes (E_{-\alpha}u^s) \\ &\quad -\lambda \sum_{\alpha\in\Delta^+}\sum_{s\geq 0}(\alpha,\alpha_i)(E_{-\alpha}u^{-s}) \otimes (E_\alpha u^s)\end{aligned}$$

$$\begin{aligned}\Delta(X_{i,1}^+) &= X_{i,1}^+ \otimes 1 + 1 \otimes X_{i,1}^+ + \lambda \sum_{\alpha\in\Delta^+}\sum_{s\geq 0}\left([E_i^+, E_{-\alpha}]u^{-s}\right) \otimes (E_\alpha u^s) \\ &\quad -\lambda \sum_{\alpha\in\Delta^+}\sum_{s\geq 1}(E_\alpha u^{-s}) \otimes \left([E_i^+, E_{-\alpha}]u^s\right)\end{aligned}$$

$$\begin{aligned}\Delta(X_{i,1}^-) &= X_{i,1}^- \otimes 1 + 1 \otimes X_{i,1}^- + \lambda \sum_{s\geq 0}(E_{i+1,i}u^{-s}) \otimes (H_i u^s) - \lambda \sum_{s\geq 1}(H_i u^{-s}) \otimes (E_{i+1,i}u^s) \\ &\quad +\lambda \sum_{\alpha\in\Delta^+}\sum_{s\geq 0}\left([E_i^-, E_{-\alpha}]u^{-s}\right) \otimes (E_\alpha u^s) - \lambda \sum_{\alpha\in\Delta^+}\sum_{s\geq 1}(E_\alpha u^{-s}) \otimes \left([E_i^-, E_{-\alpha}]u^s\right)\end{aligned}$$

The automorphism $\rho$ of $\widehat{\mathbf{Y}}$ can be extended to an automorphism of $\overline{\mathbf{Y}}\widehat{\otimes}\overline{\mathbf{Y}}$ which we denote by $\overline{\rho}$. The maps $\Delta$ and $\rho, \overline{\rho}$ are related in the following way :

$$\overline{\rho}(\Delta(X_{i,r}^\pm)) = \Delta(\rho(X_{i,r}^\pm)), \ \ \overline{\rho}(\Delta(H_{i,r})) = \Delta(\rho(H_{i,r})) \text{ for } i \neq 0,1, \ r = 0,1 \tag{40}$$

$$\overline{\rho}^2(\Delta(X_{1,r}^\pm)) = \Delta(\rho^2(X_{1,r}^\pm)), \ \ \overline{\rho}^2(\Delta(H_{1,r})) = \Delta(\rho^2(H_{1,r})) \text{ for } r = 0,1 \tag{41}$$

It is possible to extend $\Delta$ to all of $\widehat{\mathbf{Y}}$ by setting

$$\Delta(X_{0,r}^\pm) = \overline{\rho}^{-1}\Big(\Delta\big(\rho(X_{0,r}^\pm)\big)\Big), \ \ \Delta(H_{0,r}^\pm) = \overline{\rho}^{-1}\Big(\Delta\big(\rho(H_{0,r})\big)\Big) \text{ for } r = 0,1.$$

We also need to construct a representation $E$ of $\overline{\mathbf{Y}}$ on which the central element $C_{0,0}$ acts by a non-zero scalar. We denote by $\overline{\mathfrak{U}}(\widehat{\mathfrak{gl}}_n[s^{\pm 1}])$ the completion of $\mathfrak{U}(\widehat{\mathfrak{gl}}_n[s^{\pm 1}])$ with respect to the topology defined by the system neighborhoods of zero similar to the one for $\widehat{\mathbf{Y}}$. We can define an algebra homomorphism $ev : \widehat{\mathbf{Y}} \longrightarrow \overline{\mathfrak{U}}(\widehat{\mathfrak{gl}}_n[s^{\pm 1}])$ in the following way: for $1 \leq i \leq n-1$, it is given by

$$\begin{aligned}ev(H_{i,1}) &= H_i + \frac{\lambda}{2}\sum_{k<i} S(E_{ki}, E_{ik}) + \lambda \sum_{k\neq i}\sum_{s\geq 1}(E_{ki}u^s)\cdot(E_{ik}u^{-s}) \\ &\quad -\frac{\lambda}{2}\sum_{k<i+1} S(E_{k,i+1}, E_{i+1,k}) - \lambda \sum_{k\neq i+1}\sum_{s\geq 1}(E_{k,i+1}u^s)\cdot(E_{i+1,k}u^{-s}) \\ &\quad +\lambda \sum_{s\geq 0}(E_{ii}u^s)\cdot(E_{ii}u^{-s}) - \lambda \sum_{s\geq 0}(E_{i+1,i+1}u^s)\cdot(E_{i+1,i+1}u^{-s}) - \frac{\lambda}{2}(E_{ii}^2 - E_{i+1,i+1}^2 - H_i^2)\end{aligned}$$

The formula for $ev(X_{i,1}^{\pm})$ can be deduced from this one. The map $ev$ is related to $\rho$ in the same way as $\Delta$ in the equalities (40), (41) and can be extended similarly to all of $\widehat{\mathbf{Y}}$.

Given a representation $E$ of $\overline{\mathfrak{U}}(\widehat{\mathfrak{gl}}_n[s^{\pm 1}])$, we can pull it back to $\overline{\mathbf{Y}}$ via $ev$ and get a representation which we denote also by $E$.

Starting with a relation similar to (38), but this time with monomials including powers of $C_{0,0}$, we can apply the same argument as above to prove that the monomials are linearly independent, using the coproduct $\Delta$ to turn $E \otimes \mathbf{V}^l$ into a representation of $\widehat{\mathbf{Y}}$ with $E$ suitably chosen and with non-trivial central charge.

We have thus proved that, if $\beta \neq \frac{n\lambda}{4} + \frac{\lambda}{2}$, then $\mathfrak{U}(\widehat{\mathfrak{sl}}_n[u^{\pm 1}, v]) \xrightarrow{\sim} gr(\widehat{\mathbf{Y}})$. It follows that this must be true for all values of $\lambda, \beta \in \mathbb{C}$ by upper-semicontinuity, which completes the proof of theorem 7.1. □

**Corollary 7.1.** *Fix $j \in \{0, \dots, n-1\}$. The elements $X_{i,r}^{\pm}, H_{i,r}$ with $i \neq j, r \in \mathbb{Z}_{\geq 0}$ generate a subalgebra $Y_{\lambda,\beta}^{j}$ of $\widehat{\mathbf{Y}}_{\lambda,\beta}$ (or of $LY_{\lambda,\beta}$) isomorphic to $Y_\lambda$.*

*Proof.* Using the automorphism $\rho$, we can reduce to the case $j = 0$. The proof of theorem 7.1 implies that $Y_{\lambda,\beta}^{0}$ has a PBW basis exactly like the basis for $Y_\lambda$ constructed in [22]. Therefore, the natural map $Y_\lambda \twoheadrightarrow Y_{\lambda,\beta}^{0}$ must be an isomorphism. □

The main ingredient in the proof of theorem 7.1 can be stated explicitly in the following way. (The next corollary, in the case of Yangians of finite type, has been known for a long time [4].)

**Corollary 7.2.** *Suppose that $\beta \neq \frac{n\lambda}{4} + \frac{\lambda}{2}$. Let $\Phi_l : LY \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbf{V}^l)$ be the $LY$-module structure map of $\mathbf{V}^l$. Given $X \in LY, X \neq 0$, there exists an $l \gg 0$ such that $\Phi_l(X) \neq 0$.*

**Corollary 7.3.** *The canonical maps $\mathfrak{U}(\widehat{\mathfrak{sl}}_n[u^{\pm 1}, v]) \longrightarrow gr(\widehat{\mathbf{Y}})$ and $\mathfrak{U}(\widehat{\mathfrak{sl}}_n[t_1, t_2]) \longrightarrow gr(\mathsf{L})$ where $t_1 = u, t_2 = u^{-1}v$ are isomorphisms.*

As a consequence of corollary 7.2, we can prove that $\mathsf{L}$ (and therefore $\widehat{\mathbf{Y}}$) contains infinitely many copies of $Y_\lambda$. This is in accordance with the following observation made in [14]. Let $\gamma_1, \gamma_2, \in \mathbb{C}$; we have an algebra embedding $\iota : \mathtt{H}_c \longrightarrow \mathsf{H}_{t,c}(S_l)$ that sends $\tilde{u}_i = u_i + \frac{c}{2}\sum_{j \neq i} \mathrm{sign}(j-i)s_{ij}$ to $\widetilde{\mathfrak{u}}_i = \gamma_1 x_i + \gamma_2 y_i + \mathcal{Y}_i$ and $\mathtt{H}_c \supset \mathbb{C}[S_l] \xrightarrow{\sim} \mathbb{C}[S_l] \subset \mathsf{H}_{t,c}(S_l)$. (In [14], $\mathcal{Y}_i$ is replaced by $\mathcal{U}_i$ and $\tilde{u}_i$ by $u_i$.) Consider the elements $\chi_i^{\pm} = \gamma_1 \mathbf{K}(X_i^{\pm}) + \gamma_2 \mathbf{Q}(X_i^{\pm}) + J(X_i^{\pm}), \mathcal{H}_i = \gamma_1 \mathbf{K}(H_i) + \gamma_2 \mathbf{Q}(H_i) + J(H_i), i = 1, \dots, n-1$, of $\mathsf{L}$. Set $\mathsf{V}^l = \mathsf{H} \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$. Since the subalgebra of $\mathsf{H}$ generated by $\widetilde{\mathfrak{z}}_1, \dots, \widetilde{\mathfrak{z}}_l$ and $S_l$ is isomorphic to $\mathtt{H}_c$, we are led to assert the following proposition. (It was also suggested in [2].)

**Proposition 7.1.** *The subalgebra $Y^{\gamma_1,\gamma_2}$ of $\mathsf{L}$ generated by $X_i^{\pm}, H_i, \chi_i^{\pm}$ and by $\mathcal{H}_i$ for $1 \leq i \leq n-1$. is isomorphic to $Y_\lambda$.*

*Proof.* Let $\Psi_l(z) \in \mathrm{End}_{\mathbb{C}}(\mathsf{V}^l)$ be given by $\Psi_l(z)(\mathbf{h} \otimes \mathbf{v}) = \sum_{k=1}^{l} \mathbf{h}\widetilde{\mathfrak{u}}_k \otimes z^{(k)}(\mathbf{v}), \forall z \in \mathfrak{sl}_n$. We know from theorem 1 in [9] and the observation from [14] recalled in the previous paragraph that we have an algebra homomorphism $\psi_l : Y_\lambda \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathsf{V}^l)$ given by $\psi_l(z) = z$ and $\psi_l(J(z)) = \Psi_l(z)$. An analog of corollary 7.2 holds for $\psi_l$.

We know that $\mathsf{V}^l$ is a module over $\mathsf{L}$ and that, if we denote by $\varphi_l : \mathsf{L} \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathsf{V}^l)$ the algebra structure map, then $\varphi_l(\chi_i^{\pm}) = \psi_l(J(X_i^{\pm}))$ and $\varphi_l(\mathcal{H}_i) = \psi_l(J(H_i))$. Corollary 7.2 allows us to conclude the proof. □

# 8 Deformed double current algebras in type $A$

In section 5, we explained how affine Yangians are related to quantum toroidal algebras. Starting with the affine Yangians and applying similar ideas, we arrive at a new class of algebras that we call deformed double current algebras (of type $A$), as explained in section 12.

**Definition 8.1.** *Let $\lambda, \beta \in \mathbb{C}$. We defined $\mathfrak{D}_{\lambda,\beta}$ to be the algebra generated by elements $\mathsf{X}^{\pm}_{i,0}, \mathsf{X}^{\pm}_{i,1}, \mathsf{H}_{i,0}, \mathsf{H}_{i,1}$ for $1 \leq i \leq n-1$ and by $\mathsf{X}^{+}_{0,0}, \mathsf{X}^{+,+}_{0,1}, \mathsf{X}^{+,-}_{0,1}$, which satisfy the following relations:*

*(A) The elements with $i \neq 0$ satisfy those in definition 3.2 of $\mathfrak{U}(\mathfrak{sl}_n[v])$ and those with $r=0$ satisfy the Serre relations for $\mathfrak{U}(\mathfrak{sl}_n[u])$, so we have homomorphisms $\mathfrak{U}(\mathfrak{sl}_n[u]) \longrightarrow \mathfrak{D}_{\lambda,\beta}, \mathfrak{U}(\mathfrak{sl}_n[v]) \longrightarrow \mathfrak{D}_{\lambda,\beta}$ and elements $E_{ij}, \mathfrak{K}_s(E_{ij}), \mathfrak{Q}_r(E_{ij}) \in \mathfrak{D}_{\lambda,\beta}$ corresponding to the elementary matrices in $\mathfrak{sl}_n$ and to $E_{ij} \otimes u^s, E_{ij} \otimes v^r$, respectively.*

*(B) We have $[\mathsf{H}_{i,0}, \mathsf{X}^{+,\pm}_{0,1}] = c_{i0}\mathsf{X}^{+,\pm}_{0,1}$ for $i \neq 0$. The elements with $i=0$ satisfy the following relations among themselves:*

$$[\mathsf{X}^{+,+}_{0,1}, \mathsf{X}^{+}_{0,0}] = 2\lambda E_{-\theta}\mathsf{X}^{+}_{0,0}, \quad [\mathsf{X}^{+,+}_{0,1}, E_{-\theta}] = \lambda E^2_{-\theta} \text{ and the same with } \mathsf{X}^{+,-}_{0,1} \text{ instead of } \mathsf{X}^{+,+}_{0,1} \tag{42}$$

$$\mathsf{X}^{+,+}_{0,1} - \mathsf{X}^{+,-}_{0,1} = \frac{\lambda}{2} \sum_{1 \leq i \neq j \leq n-1} S([E_{-\theta}, E_{ij}], E_{ji}) \tag{43}$$

*(C) When $k = 2, \ldots, n-2$, we have*

$$[\mathsf{X}^{+}_{0,0}, \mathsf{X}^{\pm}_{k,0}] = 0 = [\mathsf{X}^{+,\pm}_{0,1}, \mathsf{X}^{\pm}_{k,0}], \; [\mathsf{X}^{+,+}_{0,1}, \mathsf{X}^{+}_{k,1}] = -\frac{\lambda}{2} \sum_{2 \leq i \leq n-1} S([E_{n1}, E_{1i}], [\mathsf{X}^{+}_{k,1}, E_{i1}]) \tag{44}$$

$$[\mathsf{X}^{+}_{0,0}, \mathsf{X}^{\pm}_{k,1}] = \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{n1}, E_{ij}], [E_{ji}, E^{\pm}_{k}]) \tag{45}$$

$$\left[\mathsf{X}^{+}_{k,r}, [\mathsf{X}^{+}_{k,r}, \mathsf{X}^{+}_{0,0}]\right] = 0 = \left[\mathsf{X}^{+}_{0,0}, [\mathsf{X}^{+}_{0,0}, \mathsf{X}^{+}_{k,r}]\right] \text{ for } r = 0, 1 \tag{46}$$

$$\left[\mathsf{X}^{+,\pm}_{0,1}, [\mathsf{X}^{+,\pm}_{0,1}, \mathsf{X}^{+}_{k,0}]\right] = 0 = \left[\mathsf{X}^{+}_{k,0}, [\mathsf{X}^{+}_{k,0}, \mathsf{X}^{+,\pm}_{0,1}]\right] \tag{47}$$

*(D) We have some more complicated relations in the cases $i = 0, j = n-1$ and $i = 0, j = 1$.*

$$[\mathsf{X}^{+}_{0,0}, \mathsf{X}^{-}_{n-1,1}] = -\lambda E_{n,n-1}E_{n1}, \quad [\mathsf{X}^{+,+}_{0,1}, X^{-}_{n-1,0}] = 0, \quad [\mathsf{X}^{+}_{0,0}, \mathsf{X}^{-}_{1,1}] = -\lambda E_{n1}E_{21}, \quad [\mathsf{X}^{+,-}_{0,1}, X^{-}_{1,0}] = 0 \tag{48}$$

$$[\mathsf{X}^{+}_{n-1,1}, \mathsf{X}^{+}_{0,0}] - [\mathsf{X}^{+}_{n-1,0}, \mathsf{X}^{+,-}_{0,1}] = (\beta - \lambda)E_{-\theta}\mathsf{X}^{+}_{n-1,0} - \beta\mathsf{X}^{+}_{n-1,0}E_{-\theta} \tag{49}$$

$$[\mathsf{X}^{+}_{1,1}, \mathsf{X}^{+}_{0,0}] - [\mathsf{X}^{+}_{1,0}, \mathsf{X}^{+,+}_{0,1}] = (\beta - \lambda)\mathsf{X}^{+}_{1,0}E_{-\theta} - \beta E_{-\theta}\mathsf{X}^{+}_{1,0} \tag{50}$$

$$\left[\mathsf{X}^{+,+}_{0,1}, [\mathsf{X}^{+,+}_{0,1}, \mathsf{X}^{+}_{n-1,0}]\right] = \left[\mathsf{X}^{+}_{n-1,0}, [\mathsf{X}^{+}_{n-1,0}, \mathsf{X}^{+,+}_{0,1}]\right] = 0 = \left[\mathsf{X}^{+,-}_{0,1}, [\mathsf{X}^{+,-}_{0,1}, \mathsf{X}^{+}_{1,0}]\right] = \left[\mathsf{X}^{+}_{1,0}, [\mathsf{X}^{+}_{1,0}, \mathsf{X}^{+,-}_{0,1}]\right] \tag{51}$$

$$\left[\mathsf{X}^{+}_{1,1}, [\mathsf{X}^{+}_{1,1}, \mathsf{X}^{+}_{0,0}]\right] = 2\lambda[E_{-\theta}, \mathsf{X}^{+}_{1,0}]\mathsf{X}^{+}_{1,1}, \quad \left[\mathsf{X}^{+}_{n-1,1}, [\mathsf{X}^{+}_{n-1,1}, \mathsf{X}^{+}_{0,0}]\right] = 2\lambda\mathsf{X}^{+}_{n-1,1}[\mathsf{X}^{+}_{n-1,0}, E_{-\theta}] \tag{52}$$

$$\left[\mathsf{X}^{+}_{0,0}, [\mathsf{X}^{+}_{0,0}, \mathsf{X}^{+}_{1,1}]\right] = -2\lambda[E_{-\theta}, \mathsf{X}^{+}_{1,0}]\mathsf{X}^{+}_{0,0}, \quad \left[\mathsf{X}^{+}_{0,0}, [\mathsf{X}^{+}_{0,0}, \mathsf{X}^{+}_{n-1,1}]\right] = 2\lambda[\mathsf{X}^{+}_{n-1}, E_{-\theta}]\mathsf{X}^{+}_{0,0} \tag{53}$$

**Remark 8.1.** *We set $\mathsf{X}^{\pm}_{i} = \mathsf{X}^{\pm}_{i,0}, \mathsf{H}_i = \mathsf{H}_{i,0}$. The elements $\mathsf{X}^{\pm}_{i}$ with $i \neq 0$ and $\mathsf{X}^{\pm,+}_{0,1}$ (or $\mathsf{X}^{\pm,-}_{0,1}$) generate a subalgebra of $\mathfrak{D}_{\lambda,\beta}$ which is a quotient of the Yangian $Y_\lambda$, see lemma 3.9. (The main theorem of section 10shows that it is isomorphic to $Y_\lambda$.) In particular, we can define elements $\mathfrak{J}(z)$ as the images of $J(z)$ under $Y_\lambda \longrightarrow \mathfrak{D}_{\lambda,\beta}$. The algebra $\mathfrak{D}_{\lambda=0,\beta=0}$ is the enveloping algebra of $\widehat{\mathfrak{sl}}_n[u,v]$: see lemma 3.7.*

# 9 Schur-Weyl functor for $\mathfrak{D}_{\lambda,\beta}$

Since $\mathfrak{D}_{\lambda,\beta}$ is isomorphic to $\mathsf{L}_{\lambda,\beta}$ as proved in section 11, we have a Schur-Weyl functor $\mathfrak{F}$ relating $\mathsf{H}_{t,c}(S_l)$-modules to $\mathfrak{D}_{\lambda,\beta}$-modules. In this section, we simply give the formulas for $\mathfrak{F}$.

We define elements $\omega^{+,\pm}_0$ by

$$\omega^{+,\pm}_0 = \mp\frac{1}{4}\sum_{j=2}^{n-1}(E_{nj}E_{j1} + E_{j1}E_{nj}) - \frac{1}{4}(E_{n1}H_\theta + H_\theta E_{n1})$$

and note that $\omega_0^{+,-} = [E_{n-1,1}, \omega_{n-1}^-]$ and $\omega_0^{+,+} = [\omega_1^-, E_{n2}]$.

Fix $t, c \in \mathbb{C}$ and set $\lambda = c, \beta = \frac{t}{2} - \frac{nc}{4} + \frac{c}{2}$. Let $M$ be a rightmodule over $\mathsf{H}_{t,c}(S_l)$ and set $\mathfrak{F}(M) = M \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$. We let the elements $\mathsf{X}_{i,r}^{\pm}, \mathsf{H}_{i,r}$ for $1 \leq i \leq n-1, r = 0, 1$ act on $\mathfrak{F}(M)$ in the following way:

$$\mathsf{X}_{i,r}^{\pm}(m \otimes \mathbf{v}) = \sum_{j=1}^{l} m y_j^r \otimes E_i^{\pm,(j)}(\mathbf{v}), \quad \mathsf{H}_{i,r}(m \otimes \mathbf{v}) = \sum_{j=1}^{l} m y_j^r \otimes H_i^{(j)}(\mathbf{v})$$

It is easy to see that the relations in definition 8.1 involving the elements with $i, j \neq 0$ are all satisfied. We now set

$$\mathsf{X}_0^{+}(m \otimes \mathbf{v}) = \sum_{k=1}^{l} m x_k \otimes E_{-\theta}^{(k)}(\mathbf{v}), \mathsf{X}_{0,1}^{+,\pm}(m \otimes \mathbf{v}) = \sum_{j=1}^{l} m \mathsf{Y}_j \otimes E_{-\theta}^{(j)}(\mathbf{v}) - \lambda \omega_0^{+,\pm}(m \otimes \mathbf{v})$$

**Theorem 9.1.** *These formulas give $M \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$ a structure of left module over $\mathfrak{D}_{\lambda,\beta}$. Thus we have a functor $\mathfrak{F} : \mathsf{H}_{t,c}(S_l) - \mathrm{mod}_R \longrightarrow \mathfrak{D}_{\lambda,\beta} - \mathrm{mod}_L$.*

*Proof.* We leave it to the reader to check that all the relations in definition 8.1 are satisfied. □

# 10 PBW bases of deformed double current algebras

We would like to prove that $\mathfrak{D}_{\lambda,\beta}$ has a basis of PBW type, following the same approach as in section 7. We fix $\lambda, \beta, t, c$ such that $c = \lambda, t = 2\beta - c + \frac{nc}{2}$ and abbreviate $\mathsf{H}_{t,c}(S_l)$ by $\mathsf{H}$, $\mathfrak{D}_{\lambda,\beta}$ by $\mathfrak{D}$. For $1 \leq i \leq n-1$, we set $\mathsf{X}_{i,r}^{\pm} = \frac{1}{2^r}[\mathsf{H}_{i,1}, [\mathsf{H}_{i,1}, \cdots, [\mathsf{H}_{i,1}, \mathsf{X}_i^{\pm}] \cdots]]$ where $\mathsf{H}_{i,1}$ appears $r$ times. In this section, we will need elements $\mathsf{X}_{0,r}^{+}$ which we define inductively by $\mathsf{X}_{0,r}^{+} = \frac{1}{2}[\mathsf{X}_{0,r-1}^{+}, \mathsf{H}_{n-1,1} + \mathsf{H}_{1,1}], r \geq 1$.

We consider the following "set of roots" for the Lie algebra $\mathfrak{sl}_n[u]$: $\breve{\Delta} = \breve{\Delta}^{re} \cup \breve{\Delta}^{im}$ is the subset of $\widehat{\Delta}$ given by $\breve{\Delta}^{re} = \{\alpha = \overline{\alpha} + s\delta | \overline{\alpha} \in \Delta, s \in \mathbb{Z}_{\geq 0}\}$ and $\breve{\Delta}^{im} = \{s\delta | s \in \mathbb{Z}_{>0}\}$. We set $\breve{\Delta}^{+} = \breve{\Delta} \cap \widehat{\Delta}^{+}$, $\breve{\Delta}^{-} = \breve{\Delta} \cap \widehat{\Delta}^{-} = \{\overline{\alpha} \in \Delta^{-}\}$ and $\breve{\Pi} = \widehat{\Pi}$.

Let $\alpha = \alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_p} = \overline{\alpha} + s\delta \in \breve{\Delta}^{re,+}, \alpha_{i_j} \in \breve{\Pi}, \overline{\alpha} \in \Delta, s \geq 0$, be a decomposition of a positive real root $\alpha$ into a sum of simple roots such that $X_{\alpha}^{\pm} = [X_{i_1}^{\pm}, [X_{i_2}^{\pm}, \cdots, [X_{i_{p-1}}^{\pm}, X_{i_p}^{\pm}] \cdots]]$ is a (non-zero) root vector of $\mathfrak{sl}_n[u]$ of weight $\pm\alpha = \overline{\alpha} + s\delta, \overline{\alpha} \in \Delta, s \geq 0$. (If $s > 0$, $X_{\alpha}^{-}$ is not defined.) Writing $r$ as a sum of non-negative integers $r = r_1 + \ldots + r_p$, we set

$$\mathsf{X}_{\alpha,r}^{\pm} = \left[\mathsf{X}_{i_1,r_1}^{\pm}, \left[\mathsf{X}_{i_2,r_2}^{\pm}, \cdots, [\mathsf{X}_{i_{p-1},r_{p-1}}^{\pm}, \mathsf{X}_{i_p,r_p}^{\pm}] \cdots\right]\right], \quad \mathsf{H}_{\alpha,r} = [\mathsf{X}_{\alpha,r}^{+}, \mathsf{X}_{\overline{\alpha},0}^{-}] \text{ if } \overline{\alpha} \in \Delta^{+} \text{ and } s > 0. \tag{54}$$

Using the filtration on $\mathsf{H}$ obtained by giving $x \in \mathfrak{h}^*, \sigma \in S_l$ degree 0 and $y \in \mathfrak{h}$ degree 1, we obtain a filtration $F_{\bullet}(\mathsf{V}^l)$ on $\mathsf{V}^l$. There is a filtration on $\mathfrak{D}_{\lambda,\beta}$ obtained by giving $\mathsf{X}_{i,r}^{\pm}, \mathsf{H}_{i,r}$ degree $r$ for $r = 0, 1$. We now prove a series of lemmas which are analogous to, but simpler than, those in the proof of the PBW property of affine Yangians.

**Lemma 10.1.** *Let $\mathsf{h} \otimes \mathbf{v} \in F_d(\mathsf{V}^l), \mathsf{h} \in F_d(\mathsf{H}), \mathbf{v} \in V^{\otimes l}$. We have $\mathsf{X}_{0,r}^{+}(\mathsf{h} \otimes \mathbf{v}) = \sum_{k=1}^{l} \mathsf{h} y_k^r x_k \otimes E_{n1}^{(k)}(\mathbf{v}) + \kappa$ where $\kappa \in F_{d+r-1}(\mathsf{V}^l)$.*

*Proof.* We proceed by induction on $r$, the lemma being true for $r = 0, 1$.

$$\begin{aligned}\mathsf{X}^+_{0,r}(\mathsf{h}\otimes\mathbf{v}) &= \frac{1}{2}\sum_{j\neq k}\mathsf{h}[y_j, y_k^{r-1}x_k]\otimes E_{n1}^{(k)}(H_1+H_{n-1})^{(j)}(\mathbf{v}) \\ &\quad +\frac{1}{2}\sum_{k=1}^{l}\mathsf{h}(y_k y_k^{r-1}x_k + y_k^{r-1}x_k y_k)\otimes E_{n1}^{(k)}(\mathbf{v})+\kappa \\ &= \sum_{k=1}^{l}\mathsf{h}y_k^r x_k\otimes E_{n1}^{(k)}(\mathbf{v})+\kappa'.\end{aligned}$$

□

**Lemma 10.2.** *If $\alpha=\overline{\alpha}+s\delta$ with $s>0$, then $\mathsf{X}^+_{\alpha,r}(\mathsf{h}\otimes\mathbf{v})=\sum_{k=1}^{l}\mathsf{h}y_k^r x_k^s\otimes E_{\overline{\alpha}}^{(k)}(\mathbf{v})+\kappa$ where $\kappa\in F_{d+r-1}(\mathsf{V}^l)$ - similarly for $\mathsf{H}_{\alpha,r}$ if $\overline{\alpha}\in\Delta^+$ with $E_{\overline{\alpha}}$ replaced by $H_{ij}$ if $\overline{\alpha}=\alpha_{ij}$.*

*Proof.* We use induction on $p$ (see equation (54)). We need only consider the case $i_1=0$. We write $\widetilde{\alpha}=\alpha-\alpha_0$ and $\widetilde{\alpha}=\overline{\widetilde{\alpha}}+(s-1)\delta$, so that $\alpha=\widetilde{\alpha}+\alpha_0=\overline{\widetilde{\alpha}}+(-\theta)+s\delta$ and $\overline{\widetilde{\alpha}}\in\Delta^+\cup\{0\}$. With this notation, we have $E_{\overline{\alpha}}=[E_{n1},E_{\overline{\widetilde{\alpha}}}]$. We find that:

$$\begin{aligned}\mathsf{X}^+_{\alpha,r}(\mathsf{h}\otimes\mathbf{v}) &= \sum_{k=1}^{l}[\mathsf{X}^+_{0,r_1}, y_k^{r-r_1}x_k^{s-1}\otimes E_{\overline{\widetilde{\alpha}}}^{(k)}](\mathsf{h}\otimes\mathbf{v})+\kappa \\ &= \left(\sum_{k=1}^{l}\mathsf{h}y_k^{r-r_1}x_k^{s-1}y_k^{r_1}x_k\otimes(E_{n1}E_{\overline{\widetilde{\alpha}}})^{(k)}(\mathbf{v})-\sum_{k=1}^{l}\mathsf{h}y_k^{r_1}x_k y_k^{r-r_1}x_k^{s-1}\otimes(E_{\overline{\widetilde{\alpha}}}E_{n1})^{(k)}(\mathbf{v})\right)+\kappa' \\ &= \sum_{k=1}^{l}\mathsf{h}y_k^r x_k^s\otimes[E_{n1},E_{\overline{\widetilde{\alpha}}}]^{(k)}(\mathbf{v})+\kappa''=\sum_{k=1}^{l}\mathsf{h}y_k^r x_k^s\otimes E_{\overline{\alpha}}^{(k)}(\mathbf{v})+\kappa''\end{aligned}$$

The result for $\mathsf{H}_{\alpha,r}$ follows immediately. □

We now have to define elements $\mathfrak{C}_{r,s}$ which, when $\lambda=\beta=0$, span the center of $\widehat{\mathfrak{sl}}_n[u,v]$. We proceed as in section 7. Recall that this center is isomorphic to $\Omega^1(\mathbb{C}[u,v])/d(\mathbb{C}[u,v])\cong\{u^{s-1}v^r du|r,s\geq 1\}$.

For $r,s\geq 1, 1\leq i\leq n-1$, set $\overline{\mathfrak{C}}_{i,r,s}=\frac{1}{2}[\mathfrak{K}_s(H_i),H_{i,r}]$ and set $\overline{\mathfrak{C}}_{n,r,s}=\frac{1}{2}[\mathfrak{K}_s(H_\theta),\mathfrak{Q}_r(H_\theta)]$. Then $\overline{\mathfrak{C}}_{i,r,s}(\mathsf{h}\otimes\mathbf{v})$ is equal to:

$$\begin{aligned}&\frac{1}{2}\sum_{k=1}^{l}\mathsf{h}[y_k^r,x_k^s]\otimes(E_{ii}+E_{i+1,i+1})^{(k)}(\mathbf{v})+\frac{1}{2}\sum_{j\neq k}\mathsf{h}[y_j^r,x_k^s]\otimes H_i^{(k)}H_i^{(j)}(\mathbf{v}) \\ &\qquad = \frac{1}{2}\sum_{k=1}^{l}\sum_{a=0}^{s-1}\sum_{b=0}^{r-1}\mathsf{h}y_k^b x_k^a[y_k,x_k]x_k^{s-a-1}y_k^{r-b-1}\otimes(E_{ii}+E_{i+1,i+1})^{(k)}(\mathbf{v}) \\ &\qquad\quad +\frac{1}{2}\sum_{j\neq k}\sum_{a=0}^{s-1}\sum_{b=0}^{r-1}\mathsf{h}y_j^b x_k^a[y_j,x_k]x_k^{s-a-1}y_j^{r-b-1}\otimes H_i^{(k)}H_i^{(j)}(\mathbf{v}) \\ &\qquad = \frac{1}{2}\sum_{k=1}^{l}\sum_{a=0}^{s-1}\sum_{b=0}^{r-1}\mathsf{h}y_k^b x_k^a\left(t+c\sum_{j\neq k}s_{jk}\right)x_k^{s-a-1}y_k^{r-b-1}\otimes(E_{ii}+E_{i+1,i+1})^{(k)}(\mathbf{v}) \\ &\qquad\quad -\frac{c}{2}\sum_{j\neq k}\sum_{a=0}^{s-1}\sum_{b=0}^{r-1}\mathsf{h}y_j^b x_k^a s_{jk}x_k^{s-a-1}y_j^{r-b-1}\otimes H_i^{(k)}H_i^{(j)}(\mathbf{v})\end{aligned}$$

$$
\begin{aligned}
= & \frac{trs}{2}\sum_{k=1}^{l}\mathsf{h}y_k^{r-1}x_k^{s-1}\otimes(E_{ii}+E_{i+1,i+1})^{(k)}(\mathbf{v}) \\
& +\frac{c}{2}\sum_{j\neq k}\sum_{a=0}^{s-1}\sum_{b=0}^{r-1}\mathsf{h}y_k^b x_k^a x_j^{s-a-1}y_j^{r-b-1}\otimes\left(\sum_{d=1}^{n}(E_{id}^{(j)}E_{di}^{(k)}+E_{i+1,d}^{(j)}E_{d,i+1}^{(k)})\right)(\mathbf{v}) \\
& -\frac{c}{2}\sum_{j\neq k}\sum_{a=0}^{s-1}\sum_{b=0}^{r-1}\mathsf{h}y_j^b x_k^a x_j^{s-a-1}y_k^{r-b-1}\otimes(E_{ii}^{(k)}E_{ii}^{(j)}+E_{i+1,i+1}^{(k)}E_{i+1,i+1}^{(j)}-E_{i,i+1}^{(k)}E_{i+1,i}^{(j)}-E_{i+1,i}^{(k)}E_{i,i+1}^{(j)})(\mathbf{v}) \\
& +\kappa \text{ where } \kappa\in F_{d+r-2}(\mathbf{V}^l) \\
= & \frac{trs}{2}\sum_{k=1}^{l}\mathsf{h}y_k^{r-1}x_k^{s-1}\otimes(E_{ii}+E_{i+1,i+1})^{(k)}(\mathbf{v}) \\
& +\frac{c}{2}\sum_{j\neq k}\sum_{a=0}^{s-1}\sum_{b=0}^{r-1}\mathsf{h}y_k^b x_k^a y_j^{r-b-1}x_j^{s-a-1}\otimes\left(\sum_{d=1,d\neq i}^{n}E_{id}^{(j)}E_{di}^{(k)}+\sum_{d=1,d\neq i+1}^{n}E_{i+1,d}^{(j)}E_{d,i+1}^{(k)}\right)(\mathbf{v}) \\
& +\frac{c}{2}\sum_{j\neq k}\sum_{a=0}^{s-1}\sum_{b=0}^{r-1}\mathsf{h}y_j^b x_j^{s-a-1}y_k^{r-b-1}x_k^a\otimes(E_{i,i+1}^{(k)}E_{i+1,i}^{(j)}+E_{i+1,i}^{(k)}E_{i,i+1}^{(j)})(\mathbf{v})+\kappa' \\
= & \frac{trs}{2}\sum_{k=1}^{l}\mathsf{h}y_k^{r-1}x_k^{s-1}\otimes(E_{ii}+E_{i+1,i+1})^{(k)}(\mathbf{v}) && (55) \\
& +\frac{c}{4}\sum_{a=0}^{s-1}\sum_{b=0}^{r-1}\Big(\sum_{d=1,d\neq i,i+1}^{n}\big(S(\mathsf{X}_{\alpha_{id}+(s-a-1)\delta,r-b-1},\mathsf{X}_{\alpha_{di}+a\delta,b}) && (56) \\
& +S(\mathsf{X}_{\alpha_{i+1,d}+(s-a-1)\delta,r-b-1},\mathsf{X}_{\alpha_{d,i+1}+a\delta,b})\big)+2S(\mathsf{X}_{\alpha_{i+1,i}+a\delta,r-b-1},\mathsf{X}_{\alpha_{i,i+1}+(s-a-1)\delta,b})\Big)(\mathsf{h}\otimes\mathbf{v}) && (57) \\
& -\frac{crs}{4}\sum_{k=1}^{l}\mathsf{h}y_k^{r-1}x_k^{s-1}\otimes\Big(\sum_{\substack{d=1\\ d\neq i,i+1}}^{n}2E_{dd}+(n+2)E_{ii}+(n+2)E_{i+1,i+1}\Big)^{(k)}(\mathbf{v})+\kappa'' && (58)
\end{aligned}
$$

where $\kappa''\in F_{d+r-2}(\mathsf{V}^l)$. Set $\mathfrak{C}_{i,r,s}=\overline{\mathfrak{C}}_{i,r,s}-(56)'-(57)'$, where $(57)'$ is the expression on line (57) but without $\mathsf{h}\otimes\mathbf{v}$, and set $\mathfrak{C}_{r,s}=\sum_{i=1}^{n}\mathfrak{C}_{i,r,s}$. (When $i=n, E_{d,i+1}=E_{d1}$.) The element $\mathfrak{C}_{r,s}$ acts on $\mathsf{V}^l$ by

$$
\mathfrak{C}_{r,s}(\mathsf{h}\otimes\mathbf{v})=rs(t-cn)\sum_{k=1}^{l}\mathsf{h}y_k^{r-1}x_k^{s-1}\otimes\mathbf{v}+\kappa \text{ where } \kappa\in F_{d+r-2}(\mathsf{V}^l).
$$

Set $\mathfrak{B}=\{\mathsf{X}_{\alpha,r}^{\pm},\mathsf{H}_{i,s,r}|\alpha\in\check{\Delta},\ r,s\in\mathbb{Z}_{\geq0},1\leq i\leq n-1\}\cup\{\mathfrak{C}_{r,s}|r,s\in\mathbb{Z}_{\geq1}\}$ where $\mathsf{H}_{i,s,r}=\mathsf{H}_{\alpha_i+s\delta,r}$. We can put a total ordering on the set $\mathfrak{B}$ as for $\mathbf{B}$ and we have the following analogue of theorem 7.1.

**Theorem 10.1.** *The set of ordered monomials in the elements of $\mathfrak{B}$ forms a vector space basis of $\mathfrak{D}$.*

*Proof.* The proof is very similar to the case of affine Yangians . First, we assume that $\beta\neq\frac{n\lambda}{4}+\frac{\lambda}{2}$. As a vector space, $\mathsf{V}^l\cong\mathbb{C}[y_1,\ldots,y_l]\otimes_{\mathbb{C}}\mathbb{C}[x_1,\ldots,x_l]\otimes_{\mathbb{C}}V^{\otimes l}$, which follows from the PBW property of $\mathsf{H}$ - see [14]. We have an epimorphism $\mathfrak{U}(\widehat{\mathfrak{sl}}_n[u,v])\twoheadrightarrow gr(\mathfrak{D}_{\lambda,\beta})$. Therefore, monomials in the elements of $\mathfrak{B}$ span $\mathfrak{D}$, so the main difficulty is to prove that they are linearly independent.

Suppose that we have a relation of the form ($S_1,S_2(\mathbf{d}),S_3(\mathbf{d}),S_4(\mathbf{d})$ are finite sets)

$$
\sum_{\mathbf{d}\in S_1}\sum_{\substack{\mathbf{r}\in S_2(\mathbf{d}),\mathbf{I}\in S_3(\mathbf{d})\\ \mathbf{f}\in S_4(\mathbf{d})}}c(\mathbf{d},\mathbf{r},\mathbf{I},\mathbf{f})\mathsf{X}_{A,r^-}^{-}\cdot\mathsf{H}_{J,r^H}\cdot\mathsf{X}_{B,r^+}^{+}\cdot\mathsf{C}_{r^C,s^C}=0 \tag{59}
$$

where

$$
\mathsf{X}_{A,r^-}^{-}=(\mathsf{X}_{\alpha^1,r_1^-}^{-})^{f_1^-}\cdots(\mathsf{X}_{\alpha^{d^-},r_{d^-}^-}^{-})^{f_{d^-}^-},\quad \mathsf{H}_{J,r^H}=(\mathsf{H}_{j_1,r_1^H})^{f_1^H}\cdots(\mathsf{H}_{j_{d^H},r_{d^H}^H})^{f_{d^H}^H}
$$

$$\mathsf{X}^+_{B,r^+} = (\mathsf{X}^+_{\beta^1,r^+_1})^{f^+_1}\cdots(\mathsf{X}^+_{\beta^{d^+},r^+_{d^+}})^{f^+_{d^+}},\ \ \mathsf{C}_{r^C,s^C} = (\mathfrak{C}_{r^C_1,s^C_1})^{f^C_1}\cdots(\mathfrak{C}_{r^C_{d^C},s^C_{d^C}})^{f^C_{d^C}}$$

and

$$\mathbf{d} = (d^-,d^H,d^+,d^C)\in S_1\subset\mathbb{Z}^{\times 4}_{\geq 0}, \mathbf{I} = (A,J,B)\in S_3(\mathbf{d})\subset(\Delta^+)^{\times d^-}\times([n-1]\times\mathbb{Z}_{\geq 0})^{\times d^H}\times(\widehat{\Delta}^+)^{\times d^+},$$

$$\mathbf{r} = (r^-,r^H,r^+,r^C,s^C)\in S_2(\mathbf{d})\subset\mathbb{Z}^{\times d^-}_{\geq 0}\times\mathbb{Z}^{\times d^H}_{\geq 0}\times\mathbb{Z}^{\times d^+}_{\geq 0}\times(\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 1})^{\times d^C}$$

$$S_4(\mathbf{d})\subset\mathbb{Z}^{\times d^-}_{\geq 0}\times\mathbb{Z}^{\times d^H}_{\geq 0}\times\mathbb{Z}^{\times d^+}_{\geq 0}\times\mathbb{Z}^{\times d^C}_{\geq 0},\ \mathbf{f} = (f^-,f^H,f^+,f^C)$$

and $A = \{\alpha^1,\ldots,\alpha^{d^-}\}, J = \{j_1,\ldots,j_{d^H}\}, j_i = (\underline{j}_i, s^0_i)\in[n-1]\times\mathbb{Z}_{\geq 0}, B = \{\beta^1,\ldots,\beta^{d^+}\},\ [n-1] = \{1,\ldots,n-1\}$.

We fix a particular choice $\check{\mathbf{d}},\check{\mathbf{r}},\check{\mathbf{I}},\check{\mathbf{f}}$ of these index sets such that $c(\check{\mathbf{d}},\check{\mathbf{r}},\check{\mathbf{I}},\check{\mathbf{f}})\neq 0$ and the corresponding monomial

$$\check{\mathbf{M}} = \mathsf{X}^-_{\check{A},\check{r}^-}\cdot\mathsf{H}_{\check{J},\check{r}^H}\cdot\mathsf{X}^+_{\check{B},\check{r}^+}\cdot\mathsf{C}_{\check{r}^C,\check{s}^C}$$

in (59) has the following properties:

1. It has maximum value for $\sum_{g=1}^{d^-} f^-_g r^-_g + \sum_{g=1}^{d^H} f^H_g r^H_g + \sum_{g=1}^{d^+} f^+_g r^+_g + \sum_{g=1}^{d^C} f^C_g (r^C_g - 1)$;
2. and, among these, it has maximum value for $\delta^+ = \sum_{g=1}^{d^+} f^+_g$;
3. and, among these, it has maximum value for $\delta^- = \sum_{g=1}^{d^-} f^-_g$;
4. and, among these, it has maximum value for $\delta^H = \sum_{g=1}^{d^H} f^H_g$;
5. and, among these, it has maximum value for $\delta^C = \sum_{g=1}^{d^C} f^C_g$.

Set $\widehat{\delta}^- = \check{\delta}^-, \widehat{\delta}^H = \widehat{\delta}^- + \check{\delta}^H, \widehat{\delta}^+ = \widehat{\delta}^H + \check{\delta}^+, \widehat{\delta}^C = \widehat{\delta}^+ + \check{\delta}^C$. Consider the module $\mathsf{V}^l$ for $l\geq\widehat{\delta}^C$. We choose $\mathbf{v}_l = v^1\otimes\cdots\otimes v^l, \widetilde{\mathbf{v}}_l = \widetilde{v}^1\otimes\cdots\otimes\widetilde{v}^l\in(\mathbb{C}^n)^{\otimes l}$ to be the following elements:

If $\check{\alpha}^g = \overline{\alpha^g}$ with $\overline{\alpha^g} = \alpha_{p_g q_g}\in\Delta^+$, set $v^\nu = v_{p_g}, \widetilde{v}^\nu = v_{q_g}$ for $\check{f}^-_1 + \ldots + \check{f}^-_{g-1} < \nu\leq\check{f}^-_1 + \ldots + \check{f}^-_g$.
Set $v^\nu = v_1 + \ldots + v_n, \widetilde{v}^\nu = v_{\check{\underline{j}}_g} - v_{\check{\underline{j}}_g+1}$ for $\widehat{\delta}^- + \check{f}^H_1 + \ldots + \check{f}^H_{g-1} < \nu\leq\widehat{\delta}^- + \check{f}^H_1 + \ldots + \check{f}^H_g$.
If $\check{\beta}^g = \overline{\check{\beta}^g} + \check{s}_g\delta$ with $\overline{\check{\beta}^g} = \alpha_{p_g q_g}\in\Delta$ and $\check{s}_g\in\mathbb{Z}_{\geq 0}$, we set $v^\nu = v_{q_g}, \widetilde{v}^\nu = v_{p_g}$ for $\widehat{\delta}^H + \check{f}^+_1 + \ldots + \check{f}^+_{g-1} < \nu\leq\widehat{\delta}^H + \check{f}^+_1 + \ldots + \check{f}^+_g$.
For $\nu > \widehat{\delta}^+$, we set $v^\nu = \widetilde{v}^\nu = v_1 + v_2 + \ldots + v_n$.

Below, we will consider the basis of $(\mathbb{C}^n)^{\otimes l}$ given by the elements $\check{\mathbf{v}} = \check{v}^1\otimes\cdots\otimes\check{v}^l$ where $\check{v}^\nu\in\{v_1,\ldots,v_n\}$ if $1\leq\nu\leq\widehat{\delta}^-$ or if $\widehat{\delta}^H < \nu\leq\widehat{\delta}^+$, and $\check{v}^\nu\in\{v_1 - v_2, v_2 - v_3,\ldots,v_{n-1} - v_n, v_1 + v_2 + \ldots + v_n\}$ if $\widehat{\delta}^- < \nu\leq\widehat{\delta}^H$ or if $\widehat{\delta}^+ < \nu\leq l$. In particular, the elements $\mathbf{v}_l,\widetilde{\mathbf{v}}_l$ above belong to this basis.

Because of our assumption,

$$\sum_{\mathbf{d}\in S_1}\sum_{\substack{\mathbf{r}\in S_2(\mathbf{d}),\mathbf{I}\in S_3(\mathbf{d})\\ \mathbf{f}\in S_4(\mathbf{d})}} c(\mathbf{d},\mathbf{r},\mathbf{I},\mathbf{f})\mathsf{X}^-_{A,r^-}\cdot\mathsf{H}_{J,r^H}\cdot\mathsf{X}^+_{B,r^+}\cdot\mathsf{C}_{r^C,s^C}(1\otimes\mathbf{v}_l) = 0 \tag{60}$$

Set

$$\mathfrak{E}^-_y = \prod_{g=1}^{\check{d}^-}\prod_{\nu=\check{f}^-_1+\ldots+\check{f}^-_{g-1}+1}^{\check{f}^-_1+\ldots+\check{f}^-_g} y_\nu^{\check{r}^-_g},\quad \mathfrak{E}^H_y = \prod_{g=1}^{\check{d}^H}\prod_{\nu=\widehat{\delta}^-+\check{f}^H_1+\ldots+\check{f}^H_{g-1}+1}^{\widehat{\delta}^-+\check{f}^H_1+\ldots+\check{f}^H_g} y_\nu^{\check{r}^H_g},\quad \mathfrak{E}^H_x = \prod_{g=1}^{\check{d}^H}\prod_{\nu=\widehat{\delta}^-+\check{f}^H_1+\ldots+\check{f}^H_{g-1}+1}^{\widehat{\delta}^-+\check{f}^H_1+\ldots+\check{f}^H_g} x_\nu^{\check{s}^H_g}$$

$$\mathfrak{E}_y^+ = \prod_{g=1}^{\check{d}^+} \prod_{\nu=\widehat{\delta}^H+\check{f}_1^+ + \ldots + \check{f}_{g-1}^+ +1}^{\widehat{\delta}^H+\check{f}_1^+ + \ldots + \check{f}_g^+} y_\nu^{\check{r}_g^+}, \quad \mathfrak{E}_x^+ = \prod_{g=1}^{\check{d}^+} \prod_{\nu=\widehat{\delta}^H+\check{f}_1^+ + \ldots + \check{f}_{g-1}^+ +1}^{\widehat{\delta}^H+\check{f}_1^+ + \ldots + \check{f}_g^+} x_\nu^{\check{s}_g^+}$$

$$\mathfrak{E}_y^C = \prod_{g=1}^{\check{d}^C} \prod_{\nu=\widehat{\delta}^+ +\check{f}_1^C + \ldots + \check{f}_{g-1}^C +1}^{\widehat{\delta}^+ +\check{f}_1^C + \ldots + \check{f}_g^C} y_\nu^{\check{r}_g^C - 1}, \quad \mathfrak{E}_x^C = \prod_{g=1}^{\check{d}^C} \prod_{\nu=\widehat{\delta}^+ +\check{f}_1^C + \ldots + \check{f}_{g-1}^C +1}^{\widehat{\delta}^+ +\check{f}_1^C + \ldots + \check{f}_g^C} x_\nu^{\check{s}_g^C - 1}$$

Set $\mathfrak{E}_y = \mathfrak{E}_y^- \mathfrak{E}_y^H \mathfrak{E}_y^+ \mathfrak{E}_y^C, \mathfrak{E}_x = \mathfrak{E}_x^H \mathfrak{E}_x^+ \mathfrak{E}_x^C$ and consider the coefficient of $\mathfrak{E}_y \mathfrak{E}_x \otimes \widetilde{\mathbf{v}}_l$ on the left-hand side of equality (60). Applying our particular choice of monomial $\check{\mathfrak{M}}$ to $1 \otimes \mathbf{v}_l$ and writing down the element of $\mathsf{V}^l$ thus obtained as a sum of basis elements of the type $m(y_1, \ldots, y_l) m(x_1, \ldots, x_l) \otimes \check{\mathbf{v}}$, where $m(y_1, \ldots, y_l)$ and $m(x_1, \ldots, x_l)$ are monomials, we see that, in $\check{\mathfrak{M}}(1 \otimes \mathbf{v})$, the element $\mathfrak{E}_y \mathfrak{E}_x \otimes \widetilde{\mathbf{v}}_l$ appears with coefficient equal to $\check{a} c(\check{\mathbf{d}}, \check{\mathbf{r}}, \check{\mathbf{I}}, \check{\mathbf{f}}) l^{\check{e}}$ where $\check{a}$ is a non-zero scalar (which can be expressed in terms of $t, c, n$ and the different values of $\check{r}, \check{s}, \check{f}$) and $\check{e}$ is equal to the multiplicity of $\mathfrak{C}_{1,1}$ in $\check{\mathfrak{M}}$. (Here, we use our assumption that $t \neq cn$.) Moreover, the only other monomials in (59) which can produce a non-zero scalar multiple of $\mathfrak{E}_y \mathfrak{E}_x \otimes \widetilde{\mathbf{v}}_l$ must differ from $\check{\mathfrak{M}}$ only by the multiplicity of $\mathfrak{C}_{1,1}$. The rest of the proof is exactly the same as for the proof of theorem 7.1. □

The main idea in the proof of theorem 10.1 is the content of our first corollary.

**Corollary 10.1.** *Assume that* $\beta \neq \frac{n\lambda}{4} + \frac{\lambda}{2}$. *Let* $\mathfrak{P}_l : \mathfrak{D} \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathsf{V}^l)$ *be the* $\mathfrak{D}$*-module structure map of* $\mathsf{V}^l$. *Given* $X \in \mathfrak{D}, X \neq 0$, *there exists an* $l \gg 0$ *such that* $\mathfrak{P}_l(X) \neq 0$.

**Corollary 10.2.** *We have an isomorphism* $\mathfrak{U}(\widehat{\mathfrak{sl}}_n[u,v]) \xrightarrow{\sim} gr(\mathfrak{D})$.

As a corollary of the proof of theorem 10.1, we can show that $\mathfrak{D}$ contains infinitely many copies of the Yangian $Y_\lambda$. This is also a consequence of theorem 11.1 and proposition 7.1.

## 11 Isomorphism between $\mathfrak{D}_{\lambda,\beta}$ and $\mathsf{L}_{\lambda,\beta}$

We would like to define an algebra homomorphism $\mathsf{f} : \mathfrak{D}_{\lambda,\beta} \longrightarrow \mathsf{L}_{\lambda,\beta}$ by the formulas:

$$\mathsf{f}(\mathsf{X}_{i,0}^\pm) = X_{i,0}^\pm, \; i = 1, \ldots, n-1, \;\; \mathsf{f}(\mathsf{X}_0^+) = X_0^+$$
$$\mathsf{f}(\mathsf{X}_{0,1}^{+;-}) = [X_{n-1,1}^-, E_{n-1,1}] - 2\lambda\omega_0^{+,-}, \;\; \mathsf{f}(\mathsf{X}_{0,1}^{+;+}) = [E_{n2}, X_{1,1}^-] - 2\lambda\omega_0^{+,+}$$
$$\mathsf{f}(\mathsf{X}_{i,1}^+) = \big[E_{i1}, [X_{0,1}^-, E_{n,i+1}]\big], \; \mathsf{f}(\mathsf{X}_{i,1}^-) = \big[E_{i+1,1}, [X_{0,1}^-, E_{ni}]\big] \text{ for } i = 2, \ldots, n-2$$
$$\mathsf{f}(\mathsf{X}_{1,1}^+) = [X_{0,1}^-, E_{n2}], \;\; \mathsf{f}(\mathsf{X}_{1,1}^-) = [E_{21}, [X_{0,1}^-, E_{n1}]]$$
$$\mathsf{f}(\mathsf{X}_{n-1,1}^+) = [E_{n-1,1}, X_{0,1}^-], \;\; \mathsf{f}(\mathsf{X}_{n-1,1}^-) = \big[[E_{n1}, X_{0,1}^-], E_{n,n-1}\big]$$

**Remark 11.1.** *In the proof of theorem 11.1 below, the following observation will be very useful. Writing* $X_{n-1,1}^- = J(E_{n,n-1}) - \lambda\omega_{n-1}^-$, *we find that*

$$\mathsf{f}(\mathsf{X}_{0,1}^{+;-}) = J(E_{n1}) - \frac{\lambda}{4}\sum_{j=2}^{n-1} S(E_{nj}, E_{j1}) + \frac{\lambda}{4} S(E_{n1}, H_\theta)$$

*Similarly, one can check that*

$$\mathsf{f}(\mathsf{X}_{0,1}^{+;+}) = J(E_{n1}) + \frac{\lambda}{4}\sum_{j=2}^{n-1} S(E_{nj}, E_{j1}) + \frac{\lambda}{4} S(E_{n1}, H_\theta).$$

*For an interpretation of these formulas, see the proof of lemma 3.9.*

**Theorem 11.1.** *The map* $\mathsf{f}$ *extends to a well-defined algebra isomorphism* $\mathfrak{D}_{\lambda,\beta} \xrightarrow{\sim} \mathsf{L}_{\lambda,\beta}$.

*Proof.* We have to check that relations (42)-(53) are satisfied.

$$[\mathsf{f}(\mathsf{X}_{0,1}^{+,+}), \mathsf{f}(\mathsf{X}_0^+)] = \big[[E_{n2}, X_{1,1}^-] - 2\lambda\omega_0^{+,+}, X_0^+\big] = -2\lambda[\omega_0^{+,+}, X_0^+] = 2\lambda E_{n1} X_0^+ = 2\lambda \mathsf{f}(E_{n1})\mathsf{f}(\mathsf{X}_0^+).$$

The rest of (42)-(43) is easy to verify and so are the first two relations in (44).

For $k = 2, \dots, n-2$,

$$\begin{aligned}
[\mathsf{f}(\mathsf{X}_{0,1}^{+,+}), \mathsf{f}(\mathsf{X}_{k,1}^+)] &= \Big[[E_{n2}, X_{1,1}^-], \big[E_{k1}, [X_{0,1}^-, E_{n,k+1}]\big]\Big] - 2\lambda[\omega_0^{+,+}, \mathsf{f}(\mathsf{X}_{k,1}^+)] \\
&= \Big[\big[[E_{n2}, X_{1,1}^-], E_{k1}\big], [X_{0,1}^-, E_{n,k+1}]\Big] + \Big[E_{k1}, \big[[[E_{n2}, X_{1,1}^-], X_{0,1}^-], E_{n,k+1}\big]\Big] \qquad (61) \\
&\quad -2\lambda[\omega_0^{+,+}, \mathsf{f}(\mathsf{X}_{k,1}^+)] \qquad (62)
\end{aligned}$$

We compute the second term in (61):

$$\begin{aligned}
\Big[E_{k1}, \big[[[E_{n2}, X_{1,1}^-], X_{0,1}^-], E_{n,k+1}\big]\Big] &= \Big[E_{k1}, \big[[[E_{n2}, X_{0,1}^-], X_{1,1}^-], E_{n,k+1}\big]\Big] \\
&\quad + \Big[E_{k1}, \big[[E_{n2}, [X_{1,1}^-, X_{0,1}^-]], E_{n,k+1}\big]\Big] \\
&= \Big[E_{k1}, \big[[[E_{n2}, X_{0,1}^-], E_{n,k+1}], X_{1,1}^-\big]\Big] \\
&\quad + \Big[\big[E_{n2}, [E_{k1}, [X_{1,1}^-, X_{0,1}^-]]\big], E_{n,k+1}\Big] \qquad (63)
\end{aligned}$$

The first term is zero since $\big[[E_{n2}, X_{0,1}^-], E_{n,k+1}\big] = 0$.

As for the term on line (63), we can write:

$$\big[E_{k1}, [X_{1,1}^-, X_{0,1}^-]\big] = \big[E_{k1}, [E_{21}, X_{0,2}^-]\big] + \frac{\lambda}{2}[E_{k1}, S(X_1^-, X_{0,1}^-)] = \frac{\lambda}{2} S\big(X_1^-, [E_{k1}, X_{0,1}^-]\big)$$

since

$$\big[E_{k1}, [E_{21}, X_{0,2}^-]\big] = \Big[\big[E_{k2}, [E_{21}, X_{0,2}^-]\big], E_{21}\Big] = \big[[E_{k1}, X_{0,2}^-], E_{21}\big] = \big[E_{k1}, [X_{0,2}^-, E_{21}]\big],$$

which implies that $[E_{k1}, [E_{21}, X_{0,2}^-]] = 0$. Thus, (63) simplifies to

$$\Big[E_{k1}, \big[[[E_{n2}, X_{1,1}^-], X_{0,1}^-], E_{n,k+1}\big]\Big] = \frac{\lambda}{2}\Big(E_{n1}\big[[X_{k1}, X_{0,1}^-], E_{n,k+1}\big] + \big[[X_{k1}, X_{0,1}^-], E_{n,k+1}\big]E_{n1}\Big)$$

and (61)+(62) becomes

$$\begin{aligned}
[\mathsf{f}(\mathsf{X}_{0,1}^{+,+}), \mathsf{f}(\mathsf{X}_{k,1}^+)] &= -\lambda\Big[\big[[E_{n2}, \omega_1^-], E_{k1}\big], [X_{0,1}^-, E_{n,k+1}]\Big] - 2\lambda[\omega_0^{+,+}, \mathsf{f}(\mathsf{X}_{k,1}^+)] \\
&\quad + \frac{\lambda}{2}\Big(E_{n1}\big[[X_{k1}, X_{0,1}^-], E_{n,k+1}\big] + \big[[X_{k1}, X_{0,1}^-], E_{n,k+1}\big]E_{n1}\Big) \\
&= -\lambda\mathbf{Q}(E_{n,k+1})E_{k1} - \lambda E_{n1}\mathbf{Q}(E_{k,k+1}) + \lambda\mathbf{Q}(E_{n,k+1})E_{k1} - \lambda E_{n,k+1}\mathbf{Q}(E_{k1}) \\
&\quad + \lambda E_{n1}\mathbf{Q}(E_{k,k+1}) \\
&= -\lambda E_{n,k+1}\mathbf{Q}(E_{k1}) = -\lambda \mathsf{f}(E_{n,k+1})\mathsf{f}\big([\mathsf{X}_{k,1}^+, E_{k+1,1}]\big)
\end{aligned}$$

This proves that the third equality in (44) holds.

$$\begin{aligned}
[\mathsf{f}(\mathsf{X}_0^+), \mathsf{f}(\mathsf{X}_{k,1}^+)] &= \Big[X_0^+, \big[E_{k1}, [X_{0,1}^-, E_{n,k+1}]\big]\Big] = \big[E_{k1}, [H_{0,1}, E_{n,k+1}]\big] \\
&= [E_{k1}, [[H_{0,1}, E_{n,n-1}], E_{n-1,k+1}]] = [E_{k1}, [X_{n-1,1}^- + \frac{\lambda}{2}S(X_{n-1}^-, H_0), E_{n-1,k+1}]] \\
&= -\lambda\big[E_{k1}, [\omega_{n-1}^-, E_{n-1,k+1}]\big] - \frac{\lambda}{2}S(E_{k1}, E_{n,k+1}) = -\frac{\lambda}{2}S(\mathsf{f}(E_{k1}), \mathsf{f}(E_{n,k+1})) \qquad (64)
\end{aligned}$$

This proves equation (45) in the + case and (46) follows from this one: the non-trivial case is the equality $[\mathsf{f}(\mathsf{X}_{k,1}^+),[\mathsf{f}(\mathsf{X}_{k,1}^+),\mathsf{f}(\mathsf{X}_0^+)]]=0$, which is a consequence of (64) and the fact that the subalgebra $\mathsf{L}_Y$ of $\mathsf{L}$ is isomorphic to $\mathfrak{U}(\mathfrak{sl}_n[s])$. (This is explained in [17]; $\mathsf{L}_Y$ is defined as the subalgebra of $\mathsf{L}$ generated by $X_{0,1}^-$ and by $X_{i,r}^\pm, H_{i,r}, 1\leq i\leq n-1, r\geq 0$.)

$$\begin{aligned}[\mathsf{f}(\mathsf{X}_0^+),\mathsf{f}(\mathsf{X}_{n-1,1}^-)] &= \Big[X_0^+,[[E_{n1},X_{0,1}^-],E_{n,n-1}]\Big]=\big[[E_{n1},H_{0,1}],E_{n,n-1}\big]=\big[E_{n1},[H_{0,1},E_{n,n-1}]\big]\\ &= \big[E_{n1},X_{n-1,1}^-+\frac{\lambda}{2}S(H_0,X_{n-1}^-)\big]=-\lambda E_{n,n-1}E_{n1}=-\lambda\mathsf{f}(\mathsf{X}_{n-1}^-)\mathsf{f}(E_{n1})\end{aligned}$$

$$\begin{aligned}[\mathsf{f}(\mathsf{X}_0^+),\mathsf{f}(\mathsf{X}_{1,1}^-)] &= \Big[X_0^+,[E_{21},[X_{0,1}^-,E_{n1}]]\Big]=\big[E_{21},[H_{0,1},E_{n1}]\big]=\big[[E_{21},H_{0,1}],E_{n1}\big]\\ &= -\big[X_{1,1}^-+\frac{\lambda}{2}S(X_1^-,H_0),E_{n1}\big]=-\lambda E_{n1}E_{21}=-\lambda\mathsf{f}(E_{n1})\mathsf{f}(X_1^-)\end{aligned}$$

We have just proved that the first and third relations in (48) are satisfied. The other two can be easily checked.

The expression $[\mathsf{f}(\mathsf{X}_{n-1,1}^+),\mathsf{f}(X_0^+)]-[\mathsf{f}(\mathsf{X}_{n-1}^+),\mathsf{f}(\mathsf{X}_{0,1}^{+;-})]$ is equal to

$$\begin{aligned}&= \big[[E_{n-1,1},X_{0,1}^-],X_0^+\big]-\big[X_{n-1}^+,[X_{n-1,1}^-,E_{n-1,1}]-2\lambda\omega_0^{+,-}\big]\\ &= -\big[E_{n-1,2},[E_{21},H_{0,1}]\big]-J(E_{n-1,1})+\lambda[E_{n-1,n},\omega_0^{+,-}]\\ &= [E_{n-1,2},X_{1,1}^-]+\beta E_{n-1,1}H_0+(\lambda-\beta)H_0E_{n-1,1}-J(E_{n-1,1})+\lambda[E_{n-1,n},\omega_0^{+,-}]\\ &= -\lambda[E_{n-1,2},\omega_1^-]-\beta E_{n-1,1}+\lambda H_0E_{n-1,1}+\lambda[E_{n-1,n},\omega_0^{+,-}]\\ &= -\frac{\lambda}{2}S(E_{n-1,n},E_{n1})-\frac{\lambda}{2}S(E_{n-1,1},H_0)-\beta E_{n-1,1}\\ &\quad+\frac{\lambda}{2}(H_0E_{n-1,1}+E_{n-1,1}H_0)+\frac{\lambda}{2}[H_0,E_{n-1,1}]\\ &= -\frac{\lambda}{2}(E_{n-1,n}E_{n1}+E_{n1}E_{n-1,n})-\beta E_{n-1,1}+\frac{\lambda}{2}E_{n-1,1}\\ &= -\lambda E_{n1}E_{n-1,n}-\beta E_{n-1,1}\end{aligned}$$

This proves that relation (49) is satisfied. The proof for (50) is identical. As for the relations in (51), they follow immediately from remark 11.1.

Now we check (52) for $\mathsf{X}_{1,1}^+$:

$$\begin{aligned}\big[\mathsf{f}(\mathsf{X}_{1,1}^+),[\mathsf{f}(\mathsf{X}_{1,1}^+),\mathsf{f}(\mathsf{X}_0^+)]\big] &= \Big[[X_{0,1}^-,E_{n2}],\big[[X_{0,1}^-,E_{n2}],X_0^+\big]\Big]-\big[[X_{0,1}^-,E_{n2}],[H_{0,1},E_{n2}]\big]\\ &= -\Big[\big[[X_{0,1}^-,E_{n2}],H_{0,1}\big],E_{n2}\Big]=\Big[\big[[[X_{0,1}^-,X_{n-1}^-],H_{0,1}],E_{n-1,2}\big],E_{n2}\Big]\end{aligned}\tag{65}$$

We know that $[X_{0,1}^-,[X_{0,1}^-,X_{n-1}^-]]=0$ in $\mathsf{L}_{\lambda,\beta}$, so applying $[X_{0,0}^+,\cdot]$ to it yields $2[H_{0,1},[X_{0,1}^-,X_{n-1}^-]]-[[H_{0,1},X_{0,1}^-],X_{n-1}^-]=0$, hence $2\big[H_{0,1},[X_{0,1}^-,X_{n-1}^-]\big]+2[X_{0,2}^-,X_{n-1}^-]+\lambda[S(H_0,X_{0,1}^-),X_{n-1}^-]=0$. From this equation, we get

$$\begin{aligned}\Big[\big[[X_{0,1}^-,X_{n-1}^-],H_{0,1}\big],E_{n-1,2}\Big] &= \frac{\lambda}{2}\big[[S(H_0,X_{0,1}^-),X_{n-1}^-],E_{n-1,2}\big]+[X_{0,2}^-,E_{n2}]\\ &= \frac{\lambda}{2}[S(H_0,X_{0,1}^-),E_{n2}]+[X_{0,2}^-,E_{n2}]\end{aligned}$$

Using this in equation (65) and the fact that $[[X_{0,2}^-,X_{n-1}^-],X_{n-1}^-]=0$ gives

$$[\mathsf{f}(\mathsf{X}_{1,1}^+),[\mathsf{f}(\mathsf{X}_{1,1}^+),\mathsf{f}(\mathsf{X}_0^+)]]=\lambda S\big([H_0,E_{n2}],[X_{0,1}^-,E_{n2}]\big)=2\lambda\mathsf{f}(E_{n2})\mathsf{f}(X_{1,1}^+).$$

This proves the first relation in (52); the second one can be established in a similar manner.

Let us now consider (53) in the case of $\mathsf{X}_{1,1}^{+}$:

$$\begin{aligned}
\big[\mathsf{f}(\mathsf{X}_0^+),[\mathsf{f}(\mathsf{X}_0^+),\mathsf{f}(X_{1,1}^+)]\big] &= \Big[X_0^+,\big[X_0^+,[X_{0,1}^-,E_{n2}]\big]\Big] = \Big[X_0^+,\big[[H_{0,1},E_{n,n-1}],E_{n-1,2}\big]\Big] \\
&= \Big[\big[X_0^+,X_{n-1,1}^- + \frac{\lambda}{2}S(H_0,X_{n-1}^-)\big],E_{n-1,2}\Big] \\
&= -\lambda[S(X_0^+,X_{n-1}^-),E_{n-1,2}] = -2\lambda\mathsf{f}\big([E_{n1},\mathsf{X}_1^+]\big)\mathsf{f}(\mathsf{X}_0^+)
\end{aligned}$$

In conclusion, $\mathsf{f}$ is a well defined algebra homomorphism. Since $\mathsf{L}$ is generated by $X_{i,0}^{\pm}, 0\le i\le n-1$ and $X_{0,1}^-$, the map $\mathsf{f}$ is surjective. It respects the filtrations on $\mathfrak{D}$ and $\mathsf{L}$, so it induces an algebra homomorphism $gr(\mathsf{f}) : gr(\mathfrak{D}) \longrightarrow gr(\mathsf{L})$, which must be an isomorphism because of the PBW property of both algebras. Therefore, $\mathsf{f}$ is injective. □

# 12 From affine Yangians to deformed double current algebras

As explained in section 5, affine Yangians can be viewed as limit forms of quantum toroidal algebras. In this section, we want to explain how to obtain deformed double current algebras from affine Yangians via a similar procedure.

We fix $\lambda,\beta$. Let us start with $\widehat{\mathbf{Y}}$ and its usual filtration. Let $\widetilde{\mathbf{R}}$ be the subring of $\widehat{\mathbf{Y}}\otimes_{\mathbb{C}}\mathbb{C}[h]$ generated by $h^rX_{i,r}^{\pm}, h^rH_{i,r}, 0\le i\le n-1, r\ge 0$. Set $\widehat{\mathbf{R}}=\widetilde{\mathbf{R}}/h\widetilde{\mathbf{R}}$, so $\widehat{\mathbf{R}}\cong gr(\widehat{\mathbf{Y}})\cong\mathfrak{U}(\widehat{\mathfrak{sl}}_n[u^{\pm1},v])$. Thus, we have a map $\widehat{\mathbf{R}}\twoheadrightarrow\mathfrak{U}(\mathfrak{sl}_n[u^{\pm1},v])$.

Consider the composite $\widetilde{\mathbf{R}}\twoheadrightarrow\widehat{\mathbf{R}}\twoheadrightarrow\mathfrak{U}(\mathfrak{sl}_n[u^{\pm1},v])\twoheadrightarrow\mathfrak{U}(\mathfrak{sl}_n[v])$, where the last map is obtained by setting $u=1$. Let $\mathbf{K}$ be its kernel. Let $\mathbf{R}$ be the $\mathbb{C}[h]$-subalgebra of $\widehat{\mathbf{Y}}\otimes_{\mathbb{C}[h]}\mathbb{C}[h,h^{-1}]$ generated by $\widetilde{\mathbf{R}}$ and $\frac{\mathbf{K}}{h}$.

**Theorem 12.1.** *The algebra $\mathbf{R}/h\mathbf{R}$ is isomorphic to the deformed double current algebra $\mathfrak{D}$.*

*Proof.* Our strategy is to define a surjective map $\varphi:\mathfrak{D}\longrightarrow\mathbf{R}/h\mathbf{R}$ explicitly by the formulas below and then to show that it is injective by using the functor $\mathfrak{F}$ and corollary 10.1.

Set $\mathfrak{X}_{i,r}^{\pm}=h^rX_{i,r}^{\pm}$ and $\mathfrak{H}_{i,r}=h^rH_{i,r}$. The map $\varphi$ is defined in terms of the generators of $\mathfrak{D}$ in the following way. For $1\le i\le n-1, r=0,1$, we set $\varphi(\mathsf{X}_{i,r}^{\pm})=\mathfrak{X}_{i,r}^{\pm}, \varphi(\mathsf{H}_{i,r})=\mathfrak{H}_{i,r}$ and set

$$\varphi(\mathsf{X}_0^+)=\frac{\mathfrak{X}_{0,0}^+-E_{-\theta}}{h},\quad \varphi(\mathsf{X}_{0,1}^{+,-})=\frac{\mathfrak{X}_{0,1}^+-[\mathfrak{X}_{n-1,1}^-,E_{n-1,1}]}{h},\quad \varphi(\mathsf{X}_{0,1}^{+,+})=\frac{\mathfrak{X}_{0,1}^+-[E_{n2},\mathfrak{X}_{1,1}^-]}{h}$$

We must show that $\varphi$ extends to an algebra homomorphism on all of $\mathfrak{D}$. It is not difficult to see that $\varphi$ yields algebra maps $\mathfrak{U}(\mathfrak{sl}_n[u]),\mathfrak{U}(\mathfrak{sl}_n[v])\longrightarrow\mathbf{R}/h\mathbf{R}$. The relations involving $H_{i,0}$ for $i\ne 0$ are easy to check. In the following computations, the expressions on the right-hand side belong to $\mathbf{R}/h\mathbf{R}$: we first treat them as elements of $\widehat{\mathbf{Y}}\otimes_{\mathbb{C}}\mathbb{C}[h,h^{-1}]$ and, after simplification, we consider their images in the quotient $\mathbf{R}/h\mathbf{R}$.

$$\begin{aligned}
\big[\varphi(\mathsf{X}_{0,1}^{+,+}),\varphi(\mathsf{X}_0^+)\big] &= \frac{1}{h}\big[X_{0,1}^+-[E_{n2},X_{1,1}^-],X_0^+-E_{n1}\big] \\
&= \frac{1}{h}\Big([X_{0,1}^+,X_0^+]-\big[E_{n2},[X_{1,1}^-,X_0^+]\big]-[X_{0,1}^+,E_{n1}]+\big[[E_{n2},X_{1,1}^-],E_{n1}\big]\Big) \\
&= \frac{1}{h}\Big(\lambda(X_0^+)^2+\big[[E_{n2},X_{1,1}^-],E_{n1}\big]\Big)=\frac{\lambda}{h}\Big((X_0^+)^2-\lambda\big[[E_{n2},\omega_1^-],E_{n1}\big]\Big) \\
&= \frac{\lambda}{h}\left((X_0^+)^2-E_{n1}^2\right)=\lambda\frac{(X_0^+-E_{n1})}{h}(X_0^++E_{n1})=2\lambda\varphi(\mathsf{X}_0^+)\varphi(E_{n1})
\end{aligned}$$

since $X_0^+ \equiv E_{n1}$ in $\mathbf{R}/h\mathbf{R}$. The other relation in (42) is easier.

$$\begin{aligned}\varphi(\mathsf{X}_{0,1}^{+;+}) - \varphi(\mathsf{X}_{0,1}^{+;-}) &= [X_{n-1,1}^-, E_{n-1,1}] - [E_{n2}, X_{1,1}^-] = \lambda[E_{n2}, \omega_1^-] - \lambda[\omega_{n-1}^-, E_{n-1,1}] \\ &= \frac{\lambda}{2}\sum_{j=2}^{n-1}(E_{nj}E_{j1} + E_{j1}E_{nj})\end{aligned}$$

The first two relations in (44) are not difficult to obtain, and for the third one we just have to compute $\lambda[[E_{n2}, \omega_k^+], hX_{1,1}^-]$. As for (45), we simply use $[X_0^+, X_{k,1}^+] = 0$ and compute $[E_{n1}, \omega_k^\pm]$. Using (44) and (45), relations (46) and (47) are easy to obtain.

The first relation in (48) says that $\left[\frac{X_0^+ - E_{n1}}{h}, \mathfrak{X}_{n-1,1}^-\right] = -[E_{n1}, X_{n-1,1}^-] = \lambda[E_{n1}, \omega_{n-1}^-] = -\lambda E_{n,n-1}E_{n1}$ and the other ones are as simple. We now turn to relations (49)-(53).

We find that $[\varphi(\mathsf{X}_{n-1,1}^+), \varphi(\mathsf{X}_0^+)] - [\varphi(\mathsf{X}_{n-1}^+), \varphi(\mathsf{X}_{0,1}^{+;-})]$ equals

$$\begin{aligned}&= [X_{n-1,1}^+, X_0^+ - E_{n1}] - \left[E_{n-1,n}, X_{0,1}^+ - [X_{n-1,1}^-, E_{n-1,1}]\right] \\ &= (\beta-\lambda)X_0^+X_{n-1}^+ - \beta X_{n-1}^+X_0^+ - [X_{n-1,1}^+, E_{n1}] + \left[E_{n-1,n}, [X_{n-1,1}^-, E_{n-1,1}]\right] \\ &= (\beta-\lambda)X_0^+X_{n-1}^+ - \beta X_{n-1}^+X_0^+ - \left[[X_{n-1,1}^+, E_{n,n-1}], E_{n-1,1}\right] \\ &\quad + \left[[E_{n-1,n}, X_{n-1,1}^-], E_{n-1,1}\right] \\ &= (\beta-\lambda)\varphi(E_{n1})\varphi(\mathsf{X}_{n-1}^+) - \beta\varphi(\mathsf{X}_{n-1}^+)\varphi(E_{n1})\end{aligned}$$

The equality corresponding to (50) via $\varphi$ can be checked following the same steps. We now turn to relation (52) with $[\varphi(\mathsf{X}_{n-1,1}^+), [\varphi(\mathsf{X}_{n-1,1}^+), \varphi(\mathsf{X}_0^+)]]$ and find that it is equal to

$$\begin{aligned}&= \left[\mathfrak{X}_{n-1,1}^+, [X_{n-1,1}^+, X_0^+ - E_{n1}]\right] = -h\left[X_{n-1,1}^+, [X_{n-1,1}^+, E_{n1}]\right] = -h\left[X_{n-1,1}^+, \left[[X_{n-1,1}^+, E_{n,n-1}], E_{n-1,1}\right]\right] \\ &= -h\left[X_{n-1,1}^+, [H_{n-1,1}, E_{n-1,1}]\right] = -h\left[X_{n-1,1}^+, \left[X_{n-2,1}^- + \frac{\lambda}{2}S(H_{n-1}, X_{n-2}^-), E_{n-2,1}\right]\right] \\ &= -\frac{\lambda h}{2}[X_{n-1,1}^+, S(H_{n-1}, E_{n-1,1})] = \lambda h S(X_{n-1,1}^+, E_{n-1,1}) = 2\lambda\varphi(\mathsf{X}_{n-1,1}^+)\varphi(E_{n-1,1})\end{aligned}$$

In a similar way, one can check that $[\varphi(\mathsf{X}_{1,1}^+), [\varphi(\mathsf{X}_{1,1}^+), \varphi(\mathsf{X}_0^+)]] = 2\lambda\varphi(E_{n2})\varphi(\mathsf{X}_{1,1}^+)$. The second and fourth relations in (51) can be deduced without difficulty from the defining relations of $\widehat{\mathbf{Y}}$. The first relation in (51) is a bit more complicated to check.

$$\begin{aligned}\left[\varphi(\mathsf{X}_{0,1}^{+;+}), [\varphi(\mathsf{X}_{0,1}^{+;+}), \varphi(\mathsf{X}_{n-1}^+)]\right] &= \left[\varphi(\mathsf{X}_{0,1}^{+;+}), \frac{1}{h}\left[X_{0,1}^+ - [E_{n2}, X_{1,1}^-], X_{n-1}^+\right]\right] \\ &= \frac{1}{h^2}\left[X_{0,1}^+ - [E_{n2}, X_{1,1}^-], [X_{0,1}^+, X_{n-1}^+] + [E_{n-1,2}, X_{1,1}^-]\right] = 0 \qquad (66)\end{aligned}$$

The vanishing of the expression on the last line requires some explanations. From the definition of $\widehat{\mathbf{Y}}$, we know that $[X_{0,1}^+, [X_{0,1}^+, X_{n-1}^+]] = 0$ and $[X_{0,1}^+, [E_{n-1,2}, X_{1,1}^-]] = 0$ since $[X_{0,1}^+, X_{1,1}^-] = 0$, $[X_{0,1}^+, E_{n-1,2}] = 0$. Moreover, $[[E_{n2}, X_{1,1}^-], X_{1,1}^-] = 0$ since $[[X_2^-, X_{1,1}^-], X_{1,1}^-] = 0$. It is also the case that $[[E_{n2}, X_{1,1}^-], E_{n-1,2}] = 0$: indeed, if $n > 4$ (the case $n = 4$ is simpler) :

$$[[E_{n2}, X_{1,1}^-], E_{n-1,2}] = \left[\left[E_{n3}, [E_{32}, X_{1,1}^-]\right], E_{n-1,2}\right] = \left[E_{n3}, \left[[E_{32}, X_{1,1}^-], E_{n-1,2}\right]\right]$$

and

$$\begin{aligned}\big[[E_{32}, X^-_{1,1}], E_{n-1,2}\big] &= \big[E_{32}, [X^-_{1,1}, E_{n-1,2}]\big] = \Big[E_{32}, \big[E_{n-1,3}, [X^-_{1,1}, E_{32}]\big]\Big] \\ &= -\big[E_{n-1,2}, [X^-_{1,1}, E_{32}]\big] = -\big[[E_{32}, X^-_{1,1}], E_{n-1,2}\big]\end{aligned}$$

Comparing the first and last term, we conclude that $[[E_{32}, X^-_{1,1}], E_{n-1,2}] = 0$, hence $[[E_{n2}, X^-_{1,1}], E_{n-1,2}] = 0$.

To prove equality (66), we are left to check that $[[E_{n2}, X^-_{1,1}], [X^+_{0,1}, X^+_{n-1}]] = 0$. From the relations in $\widehat{\mathbf{Y}}$, we have $[E_{n2}, X^+_{0,1}] = [X^-_{1,1}, X^+_{0,1}] = 0$. Moreover, $[X^+_{0,1}, [[E_{n2}, X^-_{1,1}], X^+_{n-1}]] = -[X^+_{0,1}, [E_{n-1,2}, X^-_{1,1}]] = 0$ since $[X^+_{0,1}, E_{n-1,2}] = [X^+_{0,1}, X^-_{1,1}] = 0$. This proves (66). The proof of the equality $[\varphi(\mathsf{X}^{+;-}_{0,1}), [\varphi(\mathsf{X}^{+;-}_{0,1}), \varphi(\mathsf{X}^+_1)]] = 0$ is analogous.

We are left to verify the relations in (53) and we do it only for the second one.

$$\begin{aligned}[\varphi(\mathsf{X}^+_0), [\varphi(\mathsf{X}^+_0), \varphi(\mathsf{X}^+_{n-1,1})]] &= \big[\varphi(\mathsf{X}^+_0), [X^+_0 - E_{n1}, X^+_{n-1,1}]\big] \\ &= \frac{1}{h}\Big(\big[X^+_0, [X^+_0, X^+_{n-1,1}]\big] - 2\big[X^+_0, [E_{n1}, X^+_{n-1,1}]\big] + \big[E_{n1}, [E_{n1}, X^+_{n-1,1}]\big]\Big)\end{aligned}$$

The first term is equal to 0. Writing $X^+_{n-1,1} = J(E_{n-1,n}) - \lambda\omega^+_{n-1}$, we find that $\big[E_{n1}, [E_{n1}, X^+_{n-1,1}]\big] = -2\lambda E_{n1}E_{n-1,1}$.

As for the term $[X^+_0, [E_{n1}, X^+_{n-1,1}]]$, which equals $[E_{n1}, [X^+_0, X^+_{n-1,1}]]$, we find that

$$\begin{aligned}\big[X^+_0, [E_{n1}, X^+_{n-1,1}]\big] &= \big[E_{n1}, [X^+_{0,1}, X^+_{n-1}] - ((\beta-\lambda)X^+_0 X^+_{n-1} - \beta X^+_{n-1}X^+_0)\big] \\ &= [E_{n-1,1}, X^+_{0,1}] + (\beta-\lambda)X^+_0 E_{n-1,1} - \beta E_{n-1,1}X^+_0 = -\lambda X^+_0 E_{n-1,1}\end{aligned}$$

Therefore,

$$[\varphi(\mathsf{X}^+_0), [\varphi(\mathsf{X}^+_0), \varphi(\mathsf{X}^+_{n-1,1})]] = \frac{2\lambda}{h}(X^+_0 - E_{n1})E_{n-1,1} = 2\lambda\varphi(\mathsf{X}^+_0)\varphi(E_{n-1,1}).$$

Proof that $\varphi$ is surjective: The kernel $\mathbf{K} \subset \widetilde{\mathbf{R}}$ is generated by $h^r\overline{C}_{i,r,s}$, by the elements $\Xi^+_r = \mathfrak{X}^+_{0,r} - [\mathfrak{X}^-_{n-1,r}, E_{n-1,1}], \Xi^-_r = \mathfrak{X}^-_{0,r} - [E_{1,n-1}, \mathfrak{X}^+_{n-1,r}]$ and by $ad(X^\pm_{i_1}) \circ ad(X^\pm_{i_2}) \circ \ldots \circ ad(X^\pm_{i_p})(\Xi^\pm_r)$ for any $0 \leq i_1, \ldots, i_p \leq n-1, p \geq 1$. Therefore, we have to show that $h^{r-1}\overline{C}_{i,r,s}$ and $\frac{\Xi^\pm_r}{h}$ (viewed as elements of $\mathbf{R}/h\mathbf{R}$) belong to the image of $\varphi$.

If $s \neq 0, r \geq 1$, we write $h^{r-1}\overline{C}_{i,r,s} = \frac{1}{2h}[\mathbf{K}_s(H_i), h^r J_r(H_i)] = \frac{1}{2h}[\mathbf{K}_s(H_i) - H_i, h^r J_r(H_i)]$. We know that $h^r J_r(H_i), \frac{\mathbf{K}_s(H_i)-H_i}{h} \in \mathrm{Image}(\varphi)$, hence $h^{r-1}\overline{C}_{i,r,s} \in \mathrm{Image}(\varphi)$. If $s = 0, r \geq 0$, then $h^{r-1}\overline{C}_{i,r,s} = \frac{1}{h}[h^r\widetilde{J}_r(H_i), \mathbf{K}_{-1}(H_i)] = \frac{1}{h}[h^r\widetilde{J}_r(H_i), \mathbf{K}_{-1}(H_i) - H_i] + [h^{r-1}\widetilde{J}_r(H_i), H_i]$. We claim that $h^r\widetilde{J}_r(H_i)$ is in the image of $\varphi$. Indeed, in $\mathbf{R}/h\mathbf{R}$, $h^r\widetilde{J}_r(E_{i+1,i}) = \frac{1}{2}[h^r J_r(H_i), \mathbf{K}_1(E_{i+1,i})]$ since the difference $\widetilde{J}_r(E_{i,i+1}) - \frac{1}{2}[J_r(H_i), \mathbf{K}_1(E_{i,i+1})] \in F_{r-1}(\widehat{\mathbf{Y}})$; so $h^r\widetilde{J}_r(H_i) = -[h^r[E_{i,i+1}, J_r(H_i)], \mathbf{K}_1(E_{i+1,i})] + \frac{1}{2}[h^r J_r(H_i), \mathbf{K}_1(H_i)] = -[h^r[E_{i,i+1}, J_r(H_i)], E_{i+1,i}] + \frac{1}{2}[h^r J_r(H_i), H_i]$ since $\mathbf{K}_1(E_{i+1,i}) = E_{i+1,i}, \mathbf{K}_1(H_i) = H_i$ in $\mathbf{R}/h\mathbf{R}$. The last two terms are in the image of $\varphi$, hence so is $h^r\widetilde{J}(H_i)$. To conclude that $h^{r-1}\overline{C}_{i,r,s}$ is in $\mathrm{Image}(\varphi)$, we have to see that $[h^{r-1}\widetilde{J}_r(H_i), H_i] \in \mathrm{Image}(\varphi)$ also. We write, in $\mathbf{R}/h\mathbf{R}$,

$$\begin{aligned}[h^{r-1}\widetilde{J}_r(H_i), H_i] &= \big[[E_{i,i+1}, h^{r-1}\widetilde{J}_r(E_{i+1,i})], H_i\big] \\ &= \frac{h^{r-1}}{2}\Big[\big[E_{i,i+1}, [J_r(H_i), E_{i+1,i}]\big], H_i\Big] + \\ &\quad \Big[\big[E_{i,i+1}, h^{r-1}(\widetilde{J}_r(E_{i+1,i}) - \frac{1}{2}[J_r(H_i), E_{i+1,i}])\big], H_i\Big] \\ &= \Big[\big[E_{i,i+1}, h^{r-1}(\widetilde{J}_r(E_{i+1,i}) - \frac{1}{2}[J_r(H_i), \mathbf{K}_1(E_{i+1,i})])\big], H_i\Big] \qquad (67)\end{aligned}$$

By definition of $\widetilde{J}_r(E_{i+1,i})$, the difference $\widetilde{J}_r(E_{i+1,i}) - \frac{1}{2}[J_r(H_i), \mathbf{K}_1(E_{i+1,i})]$ is in $F_{r-1}(\widehat{\mathbf{Y}})$. Therefore, the expression on line (67) is in the image of $\varphi$, thus so is $[h^{r-1}\widetilde{J}_r(H_i), H_i]$, which completes the proof that $h^{r-1}\overline{C}_{i,r,s} \in \mathrm{Image}(\varphi)$.

We proceed by induction to prove our claim about $\frac{\Xi_r^+}{h}$. Since $\Xi_r^+ = h^r\big(X_{0,r}^+ - [X_{n-1,r}^-, E_{n-1,1}]\big)$,

$$\begin{aligned}
[\mathfrak{H}_{n-1,1}, \frac{\Xi_r^+}{h}] &= h^r\big[H_{n-1,1}, X_{0,r}^+ - [X_{n-1,r}^-, E_{n-1,1}]\big] \\
&= -h^r\Big(X_{0,r+1}^+ + (\lambda-\beta)X_{0,r}^+ H_{n-1} + \beta H_{n-1}X_{0,r}^+ - [2X_{n-1,r+1}^-, E_{n-1,1}] \\
&\quad -\lambda[S(X_{n-1,r}^-, H_{n-1}), E_{n-1,1}] + \big[X_{n-1,r}^-, [H_{n-1,1}, E_{n-1,1}]\big]\Big) \\
&= -h^{r+1}\frac{(X_{0,r+1}^+ - [X_{n-1,r+1}^-, E_{n-1,1}])}{h} - h^r\big((\lambda-\beta)X_{0,r}^+H_{n-1} + \beta H_{n-1}X_{0,r}^+\big) \qquad (68)\\
&\quad +\lambda h^r[S(X_{n-1,r}^-, H_{n-1}), E_{n-1,1}] - h^r\big([X_{n-1,r}^-, [H_{n-1,1}, E_{n-1,1}]] - [X_{n-1,r+1}^-, E_{n-1,1}]\big) \qquad (69)
\end{aligned}$$

The second term in (68) and the first term in (69) are in the image of $\varphi$ and so is $[\mathfrak{H}_{n-1,1}, \frac{\Xi_r^+}{h}]$ by induction. The expression in parentheses on line (69) is in $F_r(Y_\lambda) \subset F_r(\widehat{\mathbf{Y}})$, so the whole expression in (69) is in $\widetilde{\mathbf{R}}$ and, consequently, belongs to the image of $\varphi$. In conclusion, $h^r(X_{0,r+1}^+ - [X_{n-1,r+1}^-, E_{n-1,1}]) = \frac{\Xi_{r+1}^+}{h}$ is in the image of $\varphi$.

Proof that $\varphi$ is injective: Let $c = \lambda, t = 2\beta - \lambda + \frac{n\lambda}{2}$. We can extend the map $\Phi_l : \widehat{\mathbf{Y}}_{\lambda,\beta} \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbf{V}^l)$ to $\widehat{\mathbf{Y}}_{\lambda,\beta} \otimes_{\mathbb{C}} \mathbb{C}[h,h^{-1}]$ by replacing $\mathbf{H}_{t,c}$ with $\mathbf{H}_{t,c} \otimes_{\mathbb{C}} \mathbb{C}[h,h^{-1}]$. The algebra map thus obtained restricts to $\mathbf{R} \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbf{A} \otimes_{\mathbb{C}[S_l]} V^{\otimes l})$, which factors through $\mathfrak{a}_l : \mathbf{R}/h\mathbf{R} \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathsf{V}^l)$ according to lemma 4.1. The composite $\mathfrak{a}_l \circ \varphi$ is exactly the map $\mathfrak{D} \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathsf{V}^l)$ coming from the $\mathfrak{D}$-module structure on $\mathsf{V}^l$ given in section 9.

Corollary 10.1 says that, given $X \in \mathfrak{D}, X \neq 0$ and $\beta \neq \frac{n\lambda}{4} + \frac{\lambda}{2}$, there exists some $l \gg 0$ such that $\mathfrak{a}_l \circ \varphi(X) \neq 0$, hence $\varphi$ is injective also if $\beta \neq \frac{n\lambda}{4} + \frac{\lambda}{2}$. It then follows that $\varphi$ must be an isomorphism for all value of $\lambda, \beta$. □

# 13 Another family of deformed double current algebras

Lemma 3.8 suggests to consider a possible different definition of deformed double current algebras. Computations involving the Schur-Weyl functor lead to the following definition.

**Definition 13.1.** *Let $\lambda, \beta \in \mathbb{C}$. We define $\mathsf{D}_{\lambda,\beta}$ to be the algebra generated by elements $z, \mathsf{K}(z), \mathsf{Q}(z), \mathsf{P}(z)$ with $z \in \mathfrak{sl}_n$, which satisfy the following relations: the elements $z_1, \mathsf{K}(z_2)\, \forall z_1, z_2 \in \mathfrak{sl}_n$, satisfy the relations for $\mathfrak{U}(\mathfrak{sl}_n[u])$ so that we have a map $\mathfrak{U}(\mathfrak{sl}_n[u]) \longrightarrow \mathsf{D}_{\lambda,\beta}$ given by $z \otimes u \mapsto \mathsf{K}(z)$, and similarly for $z_1, \mathsf{Q}(z_2)$ and $\mathfrak{U}(\mathfrak{sl}_n[v])$. The elements $z_1, \mathsf{P}(z_2)$ satisfy the relations of the Yangians of finite type $A_{n-1}$ as given in definition 3.1 with $J(z)$ replaced by $\mathsf{P}(z)$. If $a \neq b = c \neq d \neq a$ or $a \neq b \neq c \neq d = a$ (so if $[E_{ab}, E_{cd}] \neq 0$ but $E_{cd} \neq E_{ba}$ [1]), then*

$$\begin{aligned}
[\mathsf{K}(E_{ab}), \mathsf{Q}(E_{cd})] &= \mathsf{P}([E_{ab}, E_{cd}]) + \left(\beta - \frac{\lambda}{2}\right)(\delta_{bc}E_{ad} + \delta_{ad}E_{cb}) + \frac{\lambda}{4}(\delta_{ad} + \delta_{bc})S(E_{ab}, E_{cd}) \\
&\quad + \frac{\lambda}{4}\sum_{1\le i\neq j\le n} S\big([E_{ab}, E_{ij}], [E_{ji}, E_{cd}]\big) \qquad (70)
\end{aligned}$$

*If $[E_{ab}, E_{cd}] = 0$, then*

$$[\mathsf{K}(E_{ab}), \mathsf{Q}(E_{cd})] = -\lambda E_{ad}E_{cb} \qquad (71)$$

$$[\mathsf{Q}(E_{ab}), \mathsf{P}(E_{cd})] = \frac{\lambda}{2}\big(\mathsf{Q}(E_{ad})E_{cb} + \mathsf{Q}(E_{cb})E_{ad}\big) \qquad (72)$$

$$[\mathsf{K}(E_{ab}), \mathsf{P}(E_{cd})] = -\frac{\lambda}{2}\big(\mathsf{K}(E_{ad})E_{cb} + \mathsf{K}(E_{cb})E_{ad}\big) \qquad (73)$$

[1] An error here in the published version has been found by Valerio Toledano Laredo and Yaping Yang. They have also pointed out that relations (70) and (71) can be combined into one single relation by modifying slightly (70).

**Remark 13.1.** *In the case $E_{cd} \neq E_{ba}$ and $[E_{ba}, E_{cd}] = 0$, applying $[\cdot, E_{ba}]$ to (70) or (71) yields an expression for $[\mathsf{K}(H_{ab}), \mathsf{Q}(E_{cd})]$:*

$$[\mathsf{K}(H_{ab}), \mathsf{Q}(E_{cd})] = \mathsf{P}([H_{ab}, E_{cd}]) + \left(\beta - \frac{\lambda}{2}\right)(\delta_{ad}E_{ca} - \delta_{bc}E_{bd}) + \frac{\lambda}{4}\sum_{i\neq j}(\alpha_{ij}, \alpha_{ab})S(E_{ij}, [E_{ji}, E_{cd}])$$

*(72), (73) follow from this. Indeed, assuming that $[E_{ab}, E_{cd}] = 0$ and that $a, b, c, d$ are all distinct, we apply $[\mathsf{Q}(E_{ab}), \cdot]$ to the expression for $[\mathsf{K}(H_{bc}), \mathsf{Q}(E_{cd})]$ and for $[\mathsf{K}(H_{ac}), \mathsf{Q}(E_{cd})]$ to deduce (72). If $a, b, c, d$ are not all distinct, we choose $e \neq a, b, c, d$ and write $E_{cd} = [E_{ce}, E_{ed}]$ so that $[E_{ab}, E_{ce}] = 0 = [E_{ab}, E_{ed}]$. If, say, $b = d, a \neq c$, then*

$$\begin{aligned}[\mathsf{K}(E_{ab}), \mathsf{P}(E_{cd})] &= \big[\mathsf{K}(E_{ab}), [\mathsf{P}(E_{ce}), E_{ed}]\big] \\ &= -\frac{\lambda}{2}\big[\mathsf{K}(E_{ab})E_{cb} + \mathsf{K}(E_{cb})E_{ae}, E_{ed}\big] = -\frac{\lambda}{2}\big(\mathsf{K}(E_{ad})E_{cb} + \mathsf{K}(E_{cb})E_{ad}\big)\end{aligned}$$

**Proposition 13.1.** *We can define an automorphism of $\mathsf{D}_{\lambda,\beta}$ by $\mathsf{K}(z) \mapsto -\mathsf{Q}(z), \mathsf{Q}(z) \mapsto \mathsf{K}(z), \mathsf{P}(z) \mapsto -\mathsf{P}(z), z \mapsto z$, $\forall z \in \mathfrak{sl}_n$, and an anti-involution by $\mathsf{K}(z) \mapsto \mathsf{Q}(z^t), \mathsf{Q}(z) \mapsto \mathsf{K}(z^t), \mathsf{P}(z) \mapsto \mathsf{P}(z^t), z \mapsto z^t$, $\forall z \in \mathfrak{sl}_n$ where $z^t$ is the transpose of $z$.*

*Proof.* This is straightforward to verify. □

This should be compared with the involutions on $\mathsf{H}_{t,c}(S_l)$ described in [15].

The following proposition is an immediate consequence of lemma 3.8.

**Proposition 13.2.** *We have an isomorphism $\mathsf{D}_{\lambda=0,\beta=0} \xrightarrow{\sim} \mathfrak{U}(\widehat{\mathfrak{sl}}_n[u, v])$.*

**Corollary 13.1** (See proposition 7.1 in [17])**.** *The following relation holds in $\mathsf{D}_{\lambda,\beta}$ :*

$$[\mathsf{K}(E_{ab}), \mathsf{Q}(E_{bc})] + [\mathsf{Q}(E_{ab}), \mathsf{K}(E_{bc})] = 2\mathsf{P}(E_{ac}) \text{ if } a \neq b \neq c \neq a.$$

*Proof.* This follows immediately from relation (70). □

# 14 Schur-Weyl duality for $\mathsf{D}$

There exist a duality of Schur-Weyl type between $\mathsf{H}_{t,c}(S_l)$ and $\mathsf{D}_{\lambda,\beta}$. The proof of this fact below is simpler than the one given in [17] since we do not have to prove it first for affine Yangians.

**Theorem 14.1.** *Suppose that $\lambda = c, \beta = \frac{t}{2} - \frac{nc}{4} + \frac{c}{2}$. The functor $\mathsf{F} : M \mapsto M \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$ sends a right $\mathsf{H}_{t,c}(S_l)$-module to an integrable left $\mathsf{D}_{\lambda,\beta}$-module of level $l$ (as $\mathfrak{sl}_n$-module). Furthermore, if $l + 2 < n$, this functor is an equivalence.*

*Proof.* Suppose that $c = \lambda$ and $\beta = \frac{t}{2} - \frac{nc}{4} + \frac{c}{2}$. We would like to let $\mathsf{K}(z), \mathsf{Q}(z), \mathsf{P}(z)$ act on $M \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$ in the following way:

$$\mathsf{K}(z)(m \otimes \mathbf{v}) = \sum_{k=1}^{l} mx_k \otimes z^{(k)}(\mathbf{v}),\;\; \mathsf{Q}(z)(m \otimes \mathbf{v}) = \sum_{k=1}^{l} my_k \otimes z^{(k)}(\mathbf{v}),\;\; \mathsf{P}(z)(m \otimes \mathbf{v}) = \sum_{k=1}^{l} m\mathsf{Y}_k \otimes z^{(k)}(\mathbf{v}).$$

Assuming that $a \neq b = c \neq d \neq a$ or $a \neq b \neq c \neq d = a$, we find that $[\mathsf{K}(E_{ab}), \mathsf{Q}(E_{cd})](m \otimes \mathbf{v})$ is equal to

$$\begin{aligned}
= \quad & \sum_{j\neq k} m[y_j, x_k] \otimes E_{ab}^{(k)} E_{cd}^{(j)}(\mathbf{v}) + \delta_{bc} \sum_{j=1}^{l} m y_j x_j \otimes E_{ad}^{(j)}(\mathbf{v}) - \delta_{ad} \sum_{j=1}^{l} m x_j y_j \otimes E_{cb}^{(j)}(\mathbf{v}) \\
= \quad & -c \sum_{j\neq k} m s_{jk} \otimes E_{ab}^{(k)} E_{cd}^{(j)}(\mathbf{v}) + \sum_{j=1}^{l} m \mathsf{Y}_j \otimes (\delta_{bc} E_{ad} - \delta_{ad} E_{cb})^{(j)}(\mathbf{v}) \\
& -\frac{\delta_{bc}}{2} \sum_{j=1}^{l} m[x_j, y_j] \otimes E_{ad}^{(j)}(\mathbf{v}) + \frac{\delta_{ad}}{2} \sum_{j=1}^{l} m[y_j, x_j] \otimes E_{cb}^{(j)}(\mathbf{v}) \\
= \quad & \mathsf{P}\big([E_{ab}, E_{cd}]\big)(m \otimes \mathbf{v}) + \frac{t}{2}(\delta_{bc} E_{ab} + \delta_{ad} E_{cd})(m \otimes \mathbf{v}) - c \sum_{j\neq k} m s_{jk} \otimes E_{ab}^{(k)} E_{cd}^{(j)}(\mathbf{v}) \\
& + \frac{c}{2} \sum_{j\neq k} m s_{jk} \otimes (\delta_{bc} E_{ad}^{(j)} + \delta_{ad} E_{cb}^{(j)})(\mathbf{v}) \\
= \quad & \mathsf{P}\big([E_{ab}, E_{cd}]\big)(m \otimes \mathbf{v}) + \frac{t}{2}(\delta_{bc} E_{ad} + \delta_{ad} E_{cb})(m \otimes \mathbf{v}) - c \sum_{j\neq k} m \otimes E_{cb}^{(k)} E_{ad}^{(j)}(\mathbf{v}) \\
& + \frac{c}{2} \delta_{bc} \sum_{j\neq k} \sum_{e=1}^{n} m \otimes E_{ae}^{(k)} E_{ed}^{(j)}(\mathbf{v}) + \frac{c}{2} \delta_{ad} \sum_{j\neq k} \sum_{e=1}^{n} m \otimes E_{ce}^{(k)} E_{eb}^{(j)}(\mathbf{v}) \\
= \quad & \mathsf{P}\big([E_{ab}, E_{cd}]\big)(m \otimes \mathbf{v}) + \left(\frac{t}{2} - \frac{cn}{4}\right)(\delta_{bc} E_{ad} + \delta_{ad} E_{cb})(m \otimes \mathbf{v}) - c \sum_{j\neq k} m \otimes E_{cb}^{(k)} E_{ad}^{(j)}(\mathbf{v}) \\
& + \frac{c}{4} \delta_{bc} \sum_{e=1}^{n} S(E_{ae}, E_{ed})(m \otimes \mathbf{v}) + \frac{c}{4} \delta_{ad} \sum_{e=1}^{n} S(E_{ce}, E_{eb})(m \otimes \mathbf{v}) \\
= \quad & \Big(\mathsf{P}\big([E_{ab}, E_{cd}]\big) + \left(\beta - \frac{\lambda}{2}\right)(\delta_{bc} E_{ad} + \delta_{ad} E_{cb}) + \frac{\lambda}{4}(\delta_{bc} + \delta_{ad}) S(E_{ab}, E_{cd}) \\
& + \frac{\lambda}{4}(\delta_{bc} + \delta_{ad}) \sum_{1\leq i\neq j\leq n} S\big([E_{ab}, E_{ij}], [E_{ji}, E_{cd}]\big)\Big)(m \otimes \mathbf{v})
\end{aligned}$$

Now let $N$ be an integrable left module of level $l$ over $\mathsf{D}_{\lambda,\beta}$ and suppose that $l + 2 < n$. We have to show that there exists a module $M$ over $\mathsf{H}$ such that $\mathsf{F}(M) = N$. From the Schur-Weyl duality between $\mathbb{C}[w_1, \ldots, w_l] \rtimes S_l$ and $\mathfrak{U}(\mathfrak{sl}_n[u])$, we know that there exist modules $M^1, M^2$ over $\mathbb{C}[x_1, \ldots, x_l] \rtimes S_l$ and $\mathbb{C}[y_1, \ldots, y_l] \rtimes S_l$, respectively, such that $\mathsf{F}(M^1) \cong N$ (resp. $\mathsf{F}(M^2) \cong N$) as $\mathfrak{U}(\mathfrak{sl}_n[u])$-module (resp. as $\mathfrak{U}(\mathfrak{sl}_n[v])$-module). Since they must be isomorphic as $S_l$-modules, we can denote them simply by $M$. We must show that $M$ is actually a module over $\mathsf{H}_{t,c}$. We proceed exactly as in [17], so we will need the following lemma.

**Lemma 14.1.** *If $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_l}$ and $i_j \neq i_k$ for any $j \neq k$, then $m \otimes \mathbf{v} = 0 \Longrightarrow m = 0$.*

Fix $1 \leq j, k \leq l, j \neq k$. Choose $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_l}$ such that $i_k = 2, i_j = n-1$, $i_r = r + 2$ if $r < j, r \neq k$, $i_r = r + 1$ if $r > j, r \neq k$. Set $\widetilde{\mathbf{v}} = E_{n2}^{(k)} E_{1,n-1}^{(j)}(\mathbf{v})$. On one hand,

$$\big(\mathsf{Q}(E_{1,n-1})\mathsf{K}(E_{n2}) - \mathsf{K}(E_{n2})\mathsf{Q}(E_{1,n-1})\big)(m \otimes \mathbf{v}) =$$

$$\sum_{s=1}^{l} \sum_{r=1}^{l} m x_r y_s \otimes E_{1,n-1}^{(s)} E_{n2}^{(r)}(\mathbf{v}) - \sum_{s=1}^{l} \sum_{r=1}^{l} m y_s x_r \otimes E_{n2}^{(r)} E_{1,n-1}^{(s)}(\mathbf{v}) \quad = \quad m(x_k y_j - y_j x_k) \otimes \widetilde{\mathbf{v}} \tag{74}$$

Using relation (71) for $E_{ab} = E_{1,n-1}$ and $E_{cd} = E_{n2}$, we find that $[\mathsf{Q}(E_{1,n-1}), \mathsf{K}(E_{n2})] = \lambda E_{12} E_{n,n-1}$, so $[\mathsf{Q}(E_{1,n-1}), \mathsf{K}(E_{n2})](m \otimes \mathbf{v}) = \lambda m \otimes E_{12}^{(k)} E_{n,n-1}^{(j)}(\mathbf{v}) = \lambda m s_{jk} \otimes \widetilde{\mathbf{v}}$. Therefore, we conclude that $m(x_k y_j - y_j x_k - \lambda s_{jk}) \otimes \widetilde{\mathbf{v}} = 0$. From lemma 14.1 and our assumption that $\lambda = c$, we deduce that $m(x_k y_j - y_j x_k - c s_{jk}) = 0$.

We use equation (70) in the case $E_{ab}=E_{n1}, E_{cd}=E_{1,n-1}$ and vice-versa. It implies that

$$\begin{aligned}[\mathsf{K}(E_{n1}),\mathsf{Q}(E_{1,n-1})]-[\mathsf{Q}(E_{n1}),\mathsf{K}(E_{1,n-1})] &= (2\beta-\lambda)E_{n,n-1}+\frac{\lambda}{2}\sum_{d=2}^{n-2}S(E_{nd},E_{d,n-1})+\frac{\lambda}{2}S(E_{n1},E_{1,n-1}) \\ &\quad +\frac{\lambda}{2}S(H_0,E_{n,n-1})+\frac{\lambda}{2}S\big(E_{n,n-1},(E_{n-1,n-1}-E_{11})\big)\end{aligned}$$

Now fix $k$ and let $\mathbf{v}$ be determined by $i_k=n-1$, $i_j=j+1$ if $j\neq k$. Set $\widehat{\mathbf{v}}=E^{(k)}_{n,n-1}(\mathbf{v})$. Applying both sides of the previous equality to $m\otimes\mathbf{v}$, we deduce that

$$m(y_kx_k-x_ky_k)\otimes\widehat{\mathbf{v}}=(2\beta-\lambda)m\otimes E_{n,n-1}(\mathbf{v})+\frac{\lambda(n+1)}{2}m\otimes E_{n,n-1}(\mathbf{v})+\lambda\sum_{j\neq k}\sum_{d=2}^{n-2}m\otimes E^{(j)}_{nd}E^{(k)}_{d,n-1}(\mathbf{v})$$

$$\Longrightarrow m[y_k,x_k]\otimes\widehat{\mathbf{v}}=\left(2\beta-\lambda+\frac{\lambda n}{2}+\frac{\lambda}{2}\right)m\otimes\widehat{\mathbf{v}}+\lambda\sum_{j\neq k}ms_{jk}\otimes\widehat{\mathbf{v}}$$

Lemma 14.1 and the assumption $2\beta-\lambda+\frac{\lambda n}{2}+\frac{\lambda}{2}=t$ imply that $[y_k,x_k]=t+c\sum_{j\neq k}s_{jk}$. □

# 15 Equivalence of definitions 8.1 and 13.1

**Theorem 15.1.** *The algebras $\mathsf{D}_{\lambda,\beta}$ and $\mathfrak{D}_{\lambda,\beta}$ are isomorphic.*

*Proof.* We define a surjective map $f:\mathfrak{D}\longrightarrow\mathsf{D}$ by setting $f(\mathsf{X}_i^{\pm})=E_i^{\pm}, f(\mathsf{X}_{i,1}^{\pm})=\mathsf{Q}(E_i^{\pm})$ for $1\le i\le n-1$, $f(\mathsf{X}_0^{\pm})=\mathsf{K}(E_{\mp\theta}), f(\mathsf{X}_{0,1}^{+,\pm})=\mathsf{P}(E_{-\theta})-\lambda\omega_0^{+,\pm}$. We have to verify that all the relations in definition 8.1 are satisfied. We give details only in the "+" case.

The relation $[\mathsf{P}(E_{n1}),\mathsf{K}(E_{n1})]=\lambda E_{n1}\mathsf{K}(E_{n1})$ implies $[f(\mathsf{X}_{0,1}^{+,+}),f(\mathsf{X}_0^{+})]=2\lambda f(E_{n1})f(\mathsf{X}_0^{+})$; $[\mathsf{P}(E_{n1}),E_{n1}]=0$ leads to $[f(\mathsf{X}_{0,1}^{+,-}),f(E_{n1})]=\lambda f(E_{n1}^2)$. Relation (43) follows from the definition of $\omega_0^{+,\pm}$ and (44) is a consequence of (72) and the definition of $\omega_0^{+,+}$. As for (45),(46), they follow from (71), whereas (47) is a consequence of remark 11.1. The same is true for the second and fourth relations in (48), whereas the first and third relations follow from (71).

The relation $[f(\mathsf{X}_{n-1,1}^{+}),f(\mathsf{X}_0^{+})]-[f(\mathsf{X}_{n-1,0}^{+}),f(\mathsf{X}_{0,1}^{+,-})]=(\beta-\lambda)f(E_{n1})f(E_{n-1,n})-\beta f(E_{n-1,n})f(E_{n1})$ is equivalent to

$$\begin{aligned}\Leftrightarrow[\mathsf{Q}(E_{n-1,n}),\mathsf{K}(E_{n1})] &= [E_{n-1,n},\mathsf{P}(E_{n1})]-\lambda[E_{n-1,n},\omega_0^{+,-}]+(\beta-\lambda)E_{n1}E_{n-1,n}-\beta E_{n-1,n}E_{n1}\\ \Leftrightarrow[\mathsf{Q}(E_{n-1,n}),\mathsf{K}(E_{n1})] &= \mathsf{P}(E_{n-1,1})-\left(\beta-\frac{\lambda}{2}\right)E_{n-1,1}-\frac{\lambda}{4}S(E_{n-1,n},E_{n1})\\ &\quad -\frac{\lambda}{4}\sum_{1\le i\neq j\le n}S([E_{n-1,n},E_{ij}],[E_{ji},E_{n1}])\end{aligned}$$

The last equality is indeed satisfied in $\mathsf{D}$. Relation (50) can be checked similarly.

We can verify that $[f(\mathsf{X}_{0,1}^{+,-}),[f(\mathsf{X}_{0,1}^{+,-}),f(\mathsf{X}_1^{+})]]=0=[f(\mathsf{X}_1^{+}),[f(\mathsf{X}_1^{+}),f(\mathsf{X}_{0,1}^{+,-})]]$ (and similarly with $\mathsf{X}_1^{\pm}$ replaced by $\mathsf{X}_{n-1}^{\pm}$ and $\mathsf{X}_{0,1}^{\pm,-}$ par $\mathsf{X}_{0,1}^{\pm,+}$) by using remark 11.1.

The relation $[f(\mathsf{X}_{n-1,1}^{+}),[f(\mathsf{X}_{n-1,1}^{+}),f(\mathsf{X}_0^{+})]]=2\lambda f(\mathsf{X}_{n-1,1}^{+})f(E_{n-1,1})$ is equivalent to

$$\begin{aligned}\Leftrightarrow\big[\mathsf{Q}(E_{n-1,n}),[\mathsf{Q}(E_{n-1,n}),\mathsf{K}(E_{n1})]\big] &= 2\lambda\mathsf{Q}(E_{n-1,n})E_{n-1,1}\\ \Leftrightarrow[\mathsf{Q}(E_{n-1,n}),\mathsf{P}(E_{n-1,1})] &= \frac{\lambda}{2}\mathsf{Q}(E_{n-1,n})E_{n-1,1}+\frac{\lambda}{2}\mathsf{Q}(E_{n-1,1})E_{n-1,n}\end{aligned}$$

which is again true in $\mathsf{D}$. The other relation in (52) can be verified similarly.

Finally, we verify (53), so we compute that $\big[f(\mathsf{X}_0^+), [f(\mathsf{X}_0^+), f(\mathsf{X}_{n-1,1}^+)]\big]$ equals

$$\begin{aligned}
&= \big[\mathsf{K}(E_{n1}), [\mathsf{K}(E_{n1}), \mathsf{Q}(E_{n-1,n})]\big] \\
&= -\Big[\mathsf{K}(E_{n1}), \mathsf{P}(E_{n-1,1}) - \frac{\lambda}{4}S(E_{n1}, \mathsf{K}(E_{n-1,1})) + \frac{\lambda}{2}S(\mathsf{K}(E_{n1}), E_{n-1,1}) + \frac{\lambda}{4}S(E_{n-1,1}, \mathsf{K}(E_{n1}))\Big] \\
&= 2\lambda \mathsf{K}(E_{n1})E_{n-1,1} = 2\lambda f(E_{n-1,1}) f(\mathsf{X}_0^+)
\end{aligned}$$

The first equality in (53) can be verified similarly. That $f$ is an isomorphism is a consequence of corollary 10.1. $\square$

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