Central extensions of matrix Lie superalgebras over $\mathbb{Z}/2\mathbb{Z}$-graded algebras

Hongjia Chen, Nicolas Guay

Abstract

We study central extensions of the Lie superalgebra $\mathfrak{sl}_n(A)$ when A is a $\mathbb{Z}/2\mathbb{Z}$-graded superalgebra over a commutative ring K. Steinberg Lie superalgebras and their central extensions play an essential role. We use a $\mathbb{Z}/2\mathbb{Z}$-graded version of cyclic homology to study the center of the extensions in question.

1 Introduction

The article [KaLo] is one of the main references on the subject of central extensions of the Lie algebra $\mathfrak{sl}_n(A)$ when A is an associative ring. Its results have been extended to many other related algebras: for instance, when A is commutative, $\mathfrak{sl}_n$ can be replaced by a simple Lie algebra [Ka1] or by a Lie superalgebra [IoKo1]. [Ne] is a good reference for the general theory of central extensions. To the authors' best knowledge, it seems however that the central extensions of the Lie superalgebra $\mathfrak{sl}_n(A)$ when A is a $\mathbb{Z}/2\mathbb{Z}$-graded associative superalgebra over a unital commutative ring K have never been studied, so the aim of our paper is to provide general results about these extensions in the spirit of [KaLo]. The paper [IoKo1] provides general results about central extensions of Lie superalgebras of the form $\mathfrak{g} \otimes_k A$ where A is a commutative k-algebra, k a commutative ring and $\mathfrak{g}$ is a basic classical Lie superalgebra. The article [IoKo2] of the same authors computes the second homology groups of Lie superalgebras of the form $\mathfrak{g} \otimes_{\mathbb{K}} A$ where $\mathbb{K}$ is a field of characteristic zero, A is a supercommutative superalgebra and $\mathfrak{g}$ is a Lie superalgebra.

This work is an outgrowth of a section of [ChGu] where the following two cases are considered: if $K = \mathbb{C}$ and A is the Clifford algebra $\mathbb{C}\langle c\rangle/(c^2+1)$ with c an odd element of degree 1, then $\mathfrak{gl}_n(A)$ is the Lie superalgebra $\mathfrak{q}_n$ and $\mathfrak{sl}_n(A)$ is its derived Lie subsuperalgebra $\mathfrak{sq}_n$. (The quotient $\mathfrak{psq}_n$ of $\mathfrak{sq}_n$ by the subspace spanned by the identity matrix is isomorphic to the "strange" simple Lie superalgebra of type Q_n in Kac's classification [K]). If $K = \mathbb{C}$ and A is the affine Clifford algebra $\mathbb{C}\langle c, x, x^{-1}\rangle/(c^2-1, cx - x^{-1}c)$ with $\deg(x) = 0 = \deg(x^{-1}), \deg(c) = 1$, then $\mathfrak{gl}_n(A)$ is a twisted loop superalgebra of type Q and it admits a non-trivial central extension. Quantized enveloping superalgebras attached to these Lie superalgebras have received some attention lately [GJKK, GJKKK, ChGu] since they have an interesting representation theory. (See also [Na] for Yangians of type Q.) It is thus natural to try to develop a more general theory for extensions of Lie superalgebras of the form $\mathfrak{sl}_n(A)$ when A is $\mathbb{Z}/2\mathbb{Z}$-graded. We should mention that [IoKo2] provides results about central extensions of Lie superalgebras of the form $\mathfrak{psq}_n \otimes_{\mathbb{K}} A$ with A a supercommutative superalgebra over the field $\mathbb{K}$. In particular, proposition 5.9 in [IoKo2] states that $\mathfrak{sq}_n$ is the universal central extension of $\mathfrak{psq}_n$. This is in agreement with the following corollary of the results of our paper: when A is $\mathbb{C}\langle c\rangle/(c^2+1)$ as above, then $\mathfrak{sl}_n(A)$ is centrally closed since $HC_1(A) = 0$ - see [ChGu]. Another paper on this subject is [MiPi2].

A central role is played by Steinberg Lie superalgebras, whose definition is a natural super version of the Steinberg Lie algebras denoted $\mathfrak{st}_n$ in [KaLo]. When $n \geq 5$ and for arbitrary base ring K, they are the universal central extensions of $\mathfrak{sl}_n(A)$; moreover, the kernel of this extension is isomorphic to $HC_1(A)$ where HC_1 is the first $\mathbb{Z}/2\mathbb{Z}$-graded cyclic homology group of A. (The $\mathbb{Z}/2\mathbb{Z}$-graded version of cyclic homology that we use was introduced in [Ka2].) When $n = 3$ or $n = 4$, $\mathfrak{st}_n(A)$ is not the universal central extension of $\mathfrak{sl}_n(A)$: to obtain the latter, we need to construct the universal central extension of $\mathfrak{st}_n(A)$. The Steinberg Lie superalgebras that we consider are different from those studied in [CGS, MiPi1]: in those two articles, they provide central extensions of the Lie superalgebras $\mathfrak{sl}_{m|n}(A)$ (with A viewed as an ungraded algebra). Many of the arguments in our paper are similar to those used in [CGS] and in [GaSh].

2 Acknowledgements

The second author was partially supported by an NSERC Discovery Grant and a start-up grant from the University of Alberta. The authors thank Erhard Neher for pointing out an error in an earlier version.

3 Extensions of Lie superalgebras

Let K be a unital commutative ring. We will use the definition of Lie superalgebra over K given in section 1.2 of [Ne].

Let $L = L_0 \oplus L_1$ be a Lie superalgebra over K. The pair $(\widetilde{L}, \varphi)$, where $\widetilde{L} = \widetilde{L}_0 \oplus \widetilde{L}_1$ is a Lie superalgebra and $\varphi : \widetilde{L} \to L$ an epimorphism, is called a central extension of L if $[\mathrm{Ker}(\varphi), \widetilde{L}] = 0$. A central extension $(\widetilde{L}, \varphi)$ of L is called universal if for any central extension (L', ψ) of L there exists a unique homomorphism $h : \widetilde{L} \to L'$ such that $\psi \circ h = \varphi$.

Let $A = A_0 \oplus A_1$ be a unital, $\mathbb{Z}/2\mathbb{Z}$-graded, associative K-superalgebra. Elements of K have degree 0 and we always assume that A has a homogeneous K-basis $\{a_\beta\}_{\beta \in \mathcal{B}}$ ($\mathcal{B}$ is an index set), which contains the identity element 1 of A. Let $M_n(A)$ be the $n \times n$ matrix superalgebra with coefficients in A and $\deg(E_{ij}(a)) = \deg(a) = |a|$, for any homogeneous element $a \in A$. Under the Lie superbracket $[A \otimes a, B \otimes b] = AB \otimes ab - (-1)^{|a||b|} BA \otimes ba$, where $A, B \in M_n(K)$ and $a, b \in A$, $M_n(A)$ becomes a Lie superalgebra denoted $\mathfrak{gl}_n(A)$. (A more general construction in the context of Leibniz superalgebras and super dialgebras is considered in [HuLi].) For $n \geq 3$, the Lie superalgebra $\mathfrak{sl}_n(A)$ is the subsuperalgebra of $\mathfrak{gl}_n(A)$ generated by the elements $E_{ij}(a)$, $1 \leq i \neq j \leq n$, $a \in A$. One can show that $\mathfrak{sl}_n(A)$ can be equivalently defined as $\mathfrak{sl}_n(A) = [\mathfrak{gl}_n(A), \mathfrak{gl}_n(A)]$, the derived subsuperalgebra of $\mathfrak{gl}_n(A)$, or as the set of matrices $X \in \mathfrak{gl}_n(A)$ such that $\mathrm{Tr}(X) \in [A, A]$.

Definition 3.1. *For $n \geq 3$, the Steinberg Lie superalgebra $\mathfrak{st}_n(A)$ is defined to be the Lie superalgebra over K generated by the homogeneous elements $F_{ij}(a)$, $a \in A$ homogeneous, $1 \leq i \neq j \leq n$ and $\deg F_{ij}(a) = |a|$, subject to the following relations for $a, b \in A$:*

$$a \mapsto F_{ij}(a) \text{ is a } K\text{-linear map,} \tag{1}$$

$$[F_{ij}(a), F_{jk}(b)] = F_{ik}(ab), \text{ for distinct } i, j, k, \tag{2}$$

$$[F_{ij}(a), F_{kl}(b)] = 0, \text{ for } i \neq j \neq k \neq l \neq i. \tag{3}$$

$\mathfrak{st}_n(A)$ is a central extension of $\mathfrak{sl}_n(A)$ which is centrally closed most of the time: this is explained below.

4 Central extensions of $\mathfrak{st}_n(A)$ constructed from super 2-cocycles

Definition 4.1. *Let $L = L_0 \oplus L_1$ be a Lie superalgebra and $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded free module over K. A K-bilinear map $\psi : L \times L \to \mathcal{C}$ is called a super 2-cocycle if it is super skew-symmetric and*

$$(-1)^{deg(x)deg(z)} \psi([x, y], z) + (-1)^{deg(x)deg(y)} \psi([y, z], x) + (-1)^{deg(y)deg(z)} \psi([z, x], y) = 0$$

for homogenous elements $x, y, z \in L$ and $\psi(w, w) = 0$ for $w \in L_0$.

A central extension of the Lie superalgebra L can be constructed from a super 2-cocycle in the standard way. (See [Ne] for instance.) The following definition provides a priori a slightly different construction.

Definition 4.2. *Let ξ be a super 2-cocycle on $\mathfrak{st}_n(A)$ with value in $\mathcal{C}$ as above. Let $\mathfrak{st}_n(A)^\sharp$ be the Lie superalgebra generated by elements $F_{ij}^\sharp(a), a \in A, 1 \leq i \neq j \leq n$, and by elements of $\mathcal{C}$ with* $\deg(F_{ij}^\sharp(a)) = |a|$

and satisfying the relations:

$$[\mathcal{C},\mathcal{C}]=[F_{ij}^{\sharp}(a),\mathcal{C}]=0,\ \textit{for distinct } i,j,\ \ a\mapsto F_{ij}{}^{\sharp}(a)\ \textit{is a } K\textit{-linear map,} \tag{4}$$

$$[F_{ij}^{\sharp}(a),F_{jk}^{\sharp}(b)]=F_{ik}^{\sharp}(ab)+\xi(F_{ij}(a),F_{jk}(b)),\ \textit{for distinct } i,j,k, \tag{5}$$

$$[F_{ij}^{\sharp}(a),F_{kl}^{\sharp}(b)]=\xi(F_{ij}(a),F_{kl}(b)),\ \textit{for distinct } i\neq j\neq k\neq l\neq i \tag{6}$$

where $a,b\in A, 1\le i,j,k,l\le n$*.*

The following lemma shows that this construction of a central extension of $\mathfrak{st}_n(A)$ is actually the same as the classical construction if ξ is surjective.

Lemma 4.1. *If* $0\longrightarrow\mathcal{C}\longrightarrow\mathfrak{st}'_n(A)\xrightarrow{\pi}\mathfrak{st}_n(A)\longrightarrow 0$ *is a central extension of* $\mathfrak{st}_n(A)$ *constructed from a surjective super* 2*-cocycle* ξ *(so that* $\mathfrak{st}'_n(A)\cong\mathfrak{st}_n(A)\oplus\mathcal{C}$ *as* K*-modules), then there is an isomorphism* $\rho:\mathfrak{st}_n(A)^{\sharp}\to\mathfrak{st}'_n(A)$ *with* $\rho(F_{ij}^{\sharp}(a))=F_{ij}(a)$ *and* $\rho(c)=c$ *for* $a\in A, c\in\mathcal{C}$*.*

Proof. Since $\mathcal{C}$ and $F_{ij}(a)$ in $\mathfrak{st}'_n(A)$ satisfy (4)-(6), there is a Lie superalgebra epimomorphism $\rho:\mathfrak{st}_n(A)^{\sharp}\to\mathfrak{st}'_n(A)$ with $\rho(F_{ij}^{\sharp}(a))=F_{ij}(a)$ and $\rho(c)=c$, for $a\in A, c\in\mathcal{C}$. Also, since $\mathcal{C}$ is an ideal of $\mathfrak{st}_n(A)^{\sharp}$ and since (4)-(6) for $\mathfrak{st}_n(A)^{\sharp}$ reduce mod $\mathcal{C}$ to (1)-(3) for $\mathfrak{st}_n(A)$, there exists a homomorphism $\theta:\mathfrak{st}_n(A)\to\mathfrak{st}_n(A)^{\sharp}/\mathcal{C}$ with $\theta(F_{ij}(a))=\overline{F_{ij}^{\sharp}(a)}$. Moreover, $\mathcal{C}\subset\ker(\pi\circ\rho)$, so there is an induced homomorphism $\overline{\pi\circ\rho}:\mathfrak{st}_n(A)^{\sharp}/\mathcal{C}\to\mathfrak{st}_n(A)$ with $(\overline{\pi\circ\rho})(\overline{F_{ij}^{\sharp}(a)})=F_{ij}(a)$. Since $\{\overline{F_{ij}^{\sharp}(a)}|1\le i\neq j\le n, a\in A\}$ is a set of generators of $\mathfrak{st}_n(A)^{\sharp}/\mathcal{C}$, we see that $\overline{\pi\circ\rho}$ is an isomorphism and $\overline{\pi\circ\rho}=\theta^{-1}$. Thus, $\mathcal{C}=\ker(\pi\circ\rho)$, $\ker(\rho)\subset\mathcal{C}$. But $\rho|_{\mathcal{C}}=id$ yields $\ker(\rho)=0$. Therefore ρ is injective and it is thus an isomorphism. □

5 Universal Central Extension of $\mathfrak{st}_n(A)$

For $n\ge 3$, both Lie superalgebras $\mathfrak{sl}_n(A)$ and $\mathfrak{st}_n(A)$ are perfect. Let φ be the epimorphism of Lie superalgebras $\varphi:\mathfrak{st}_n(A)\longrightarrow\mathfrak{sl}_n(A)$ given by $\varphi(F_{ij}(a))=E_{ij}(a)$ for $1\le i\neq j\le n$. Let $H_{ij}(a,b)=[F_{ij}(a),F_{ji}(b)]$. Let $\mathfrak{N}^+$ and $\mathfrak{N}^-$ be the K-submodules of $\mathfrak{st}_n(A)$ generated by $F_{ij}(a)$ for $1\le i<j\le n$ and $F_{ij}(a)$ for $1\le j<i\le n$, respectively. Let H be the K-submodule of $\mathfrak{st}_n(A)$ generated by $H_{ij}(a,b)$ for $1\le i\neq j\le n$.

Before stating and proving the main theorem of this section (theorem 5.1), we need a few lemmas.

Lemma 5.1. *For* $n\ge 3$*, we have a triangular decomposition* $\mathfrak{st}_n(A)=\mathfrak{N}^+\oplus H\oplus\mathfrak{N}^-$*. Moreover, we have* $\mathfrak{st}_n(A)=H\bigoplus\left(\sum_{1\le i\neq j\le n}F_{ij}(A)\right)$*.*

Proof. The proof is the same as for lemma 1.11 in [KaLo]. □

Lemma 5.2. *For* $n\ge 3$*, we have* $\mathrm{Ker}\varphi\subseteq H$ *and* $(\mathfrak{st}_n(A),\varphi)$ *is a central extension of* $\mathfrak{sl}_n(A)$*, i.e.,* $[\mathrm{Ker}\varphi,\mathfrak{st}_n(A)]=0$*.*

Proof. The proof is the same as for proposition 1.12 in [KaLo]. □

Cyclic homology of $\mathbb{Z}/2\mathbb{Z}$-graded algebras was studied in [Ka2] - see also [IoKo2]. We define the chain complex of K-modules $C_*(A)$ where $C_0(A)=A$ and for $n\ge 1$ the module $C_n(A)$ is the quotient of the K-module $A^{\otimes(n+1)}$ by the K-submodule I_n generated by the elements

$$a_0\otimes a_1\otimes\cdots\otimes a_n-(-1)^{n+|a_n|\sum_{i=0}^{n-1}|a_i|}a_n\otimes a_0\otimes a_1\otimes\cdots\otimes a_{n-1},$$

with homogeneous elements $a_i \in A$, for $0 \le i \le n$. The homomorphism $\widetilde{d_n} : A^{\otimes(n+1)} \to A^{\otimes n}$ is given by

$$\widetilde{d_n}(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^{n+|a_n| \sum_{i=0}^{n-1} |a_i|} a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}. \quad (7)$$

One can check $\widetilde{d_n}(I_n) \subseteq I_{n-1}$, hence it induces a homomorphism $d_n : C_n(A) \to C_{n-1}(A)$ and we have $d_{n-1} d_n = 0$. This is a $\mathbb{Z}/2\mathbb{Z}$-graded version of the Connes complex. The n^{th} $\mathbb{Z}/2\mathbb{Z}$-graded cyclic homology group $HC_n(A)$ of the superalgebra A is defined by $HC_n(A) = \mathrm{Ker}(d_n)/\mathrm{Image}(d_{n+1})$.

The following is the $\mathbb{Z}/2\mathbb{Z}$-graded analog of one of the main results in [KaLo].

Theorem 5.1. *If $n \ge 3$, the kernel of the central extension $(\mathfrak{st}_n(A), \varphi)$ of the Lie superalgebra $\mathfrak{sl}_n(A)$ is isomorphic to $HC_1(A)$.*

To prove this theorem below, we will need a few more lemmas.

Lemma 5.3. *For $H_{ij}(a,b) = [F_{ij}(a), F_{ji}(b)], 1 \le i \ne j \le n$ and homogeneous elements $a, b, c \in A$ we have*

1. $(-1)^{|a||c|} H_{ij}(ab, c) = (-1)^{|a||c|} H_{ik}(a, bc) + (-1)^{|a||b|} H_{kj}(b, ca)$ *for pairwise distinct i, j, k.*
2. $H_{1j}(a,b) - (-1)^{|a||b|} H_{1j}(1, ba) = H_{1k}(a,b) - (-1)^{|a||b|} H_{1k}(1, ba)$ *for any $j, k \ne 1$.*

From 2 in the previous lemma, we see that it is possible to define, for $a, b \in A$ homogeneous, the elements $h(a,b) = H_{1j}(a,b) - (-1)^{|a||b|} H_{1j}(1, ba)$ and this definition does not depend on j, $j \ne 1$.

Lemma 5.4. *The elements $h(a,b)$ satisfy the relation*

$$(-1)^{|a||c|} h(ab, c) = (-1)^{|a||c|} h(a, bc) + (-1)^{|a||b|} h(b, ca).$$

Proof. $(-1)^{|a||c|} h(ab, c)$ equals

$$\begin{aligned}
&(-1)^{|a||c|} H_{1j}(ab, c) - (-1)^{|a||c|} (-1)^{|c|(|a|+|b|)} H_{1j}(1, cab) \\
&= (-1)^{|a||c|} H_{1k}(a, bc) + (-1)^{|a||b|} H_{kj}(b, ca)) - (-1)^{|c||b|} \big(H_{1k}(1, cab) + H_{kj}(1, cab)\big) \\
&= (-1)^{|a||c|} h(a, bc) + (-1)^{|a||b|} H_{kj}(b, ca) - (-1)^{|c||b|} H_{kj}(1, cab) \\
&+ (-1)^{|a||b|} H_{1k}(1, bca) - (-1)^{|c||b|} H_{1k}(1, cab) \\
&= (-1)^{|a||c|} h(a, bc) + (-1)^{|a||b|} \big(H_{k1}(1, bca) + H_{1j}(b, ca)\big) \\
&- (-1)^{|c||b|} \big(H_{k1}(1, cab) + H_{1j}(1, cab)\big) + (-1)^{|a||b|} H_{1k}(1, bca) - (-1)^{|c||b|} H_{1k}(1, cab) \\
&= (-1)^{|a||c|} h(a, bc) + (-1)^{|b||a|} h(b, ca)
\end{aligned}$$

∎

Lemma 5.5. *Every element $x \in H$ can be written as $x = \sum_{i \in I_x} h(a_i, b_i) + \sum_{j=2}^{n} H_{1j}(1, c_j)$ where $a_i, b_i, c_j \in A$ and I_x is a finite indexing set.*

We now collect some formulas which may be useful to prove some of the results below. For i, j, k all distinct,

$$[H_{ij}(a,b), F_{ik}(c)] = F_{ik}(abc), \quad [H_{ij}(a,b), F_{ki}(c)] = -(-1)^{|c|(|a|+|b|)} F_{ki}(cab) \quad (8)$$

$$[H_{ij}(a,b), F_{ij}(c)] = F_{ij}\big(abc + (-1)^{|a||b|+|b||c|+|a||c|} cba\big), \quad [h(a,b), F_{jk}(c)] = 0 \text{ for } j,k \ge 2 \quad (9)$$

$$[h(a,b), F_{1i}(c)] = F_{1i}\big((ab - (-1)^{|a||b|} ba)c\big), \ [h(a,b), F_{i1}(c)] = -(-1)^{|c|(|a|+|b|)} F_{i1}\big(c(ab - (-1)^{|a||b|} ba)\big) \quad (10)$$

$$[H_{1k}(a,1), H_{1m}(b,1)] = h(a,b), \quad [H_{1k}(a,1), H_{1k}(b,1)] = 2h(a,b) - H_{1k}(1, [a,b]), \ 2 \le k \ne m \le n \quad (11)$$

$$[h(a,b), H_{1k}(c,1)] = h([a,b], c), \ 2 \le k \le n, \quad [h(a,b), h(c,d)] = h([a,b],[c,d]) \quad (12)$$

Note that, as a consequence of these formulas, if a and b supercommute, then $h(a,b)$ is in the center of $\mathfrak{st}_n(A)$.

Proof of theorem 5.1. Let $\langle A,A\rangle$ be the quotient of $A\otimes A$ by the two-sided K-submodule I $(=I_1+\mathrm{Image}(\widetilde{d_2}))$ generated by $a_1\otimes a_2+(-1)^{|a_1||a_2|}a_2\otimes a_1$ and $(-1)^{|a_2||a_1|}a_2a_3\otimes a_1+(-1)^{|a_3||a_2|}a_3a_1\otimes a_2+(-1)^{|a_3||a_1|}a_1a_2\otimes a_3$. Hence $\langle A,A\rangle\cong C_1(A)/\mathrm{Image}(d_2)$. The first cyclic homology group $HC_1(A)$ is, by definition, the kernel of the map $d:\langle A,A\rangle\twoheadrightarrow[A,A]$, $a_1\otimes a_2\mapsto[a_1,a_2]=a_1a_2-(-1)^{|a_1||a_2|}a_2a_1$ (d is induced by d_1).

The proof follows the same ideas as the proof of theorem 1.7 in [KaLo]. The main steps of the proof in *loc. cit.* are to show that the kernel of the map $\mathfrak{st}_n(A)\twoheadrightarrow\mathfrak{sl}_n(A)$ is contained in the submodule H, that this kernel is central in $\mathfrak{st}_n(A)$ and, finally, isomorphic to $HC_1(A)$. The first two steps are lemma 5.2, and the last one reduces to showing that the map $\eta:\langle A,A\rangle\longrightarrow\mathfrak{st}_n(A)$ given by $\eta(a\otimes b)=h(a,b)$ identifies $HC_1(A)$ with the kernel of the projection $\mathfrak{st}_n(A)\twoheadrightarrow\mathfrak{sl}_n(A)$. (Note that lemma 5.4 implies that η is well-defined.) $\square$

Corollary 5.1. *Suppose $\mathbb{Q}\subseteq K$. For $n\geq 3$, $\mathfrak{st}_n(A)\cong\mathfrak{sl}_n(A)\oplus HC_1(A)$ as super vector spaces and we have a Lie superalgebra isomorphism $\mathfrak{st}_n(A)\cong(\mathfrak{sl}_n(K)\otimes_K A)\oplus\langle A,A\rangle$ where the Lie superbracket with respect to this second decomposition is given as follows:*

$$[x\otimes a,y\otimes b]=\frac{1}{n}(x,y)\langle a,b\rangle+\frac{1}{2}[x,y]\otimes[a,b]_++\frac{1}{2}[x,y]_+\otimes[a,b],\tag{13}$$

$$[\langle a_1,a_2\rangle,\langle b_1,b_2\rangle]=\langle[a_1,a_2],[b_1,b_2]\rangle,\ \ [\langle a_1,a_2\rangle,y\otimes b]=y\otimes\big[[a_1,a_2],b\big].\tag{14}$$

where

$$[x,y]_+=xy+yx-\frac{2}{n}(x,y)I,\ \ [a,b]_+=ab+(-1)^{|a||b|}ba$$

and $(\cdot,\cdot)$ is the Killing form on $\mathfrak{sl}_n(K)$.

Proof. We have a Lie superalgebra homomorphism $\mathfrak{st}_n(A)\longrightarrow(\mathfrak{sl}_n(K)\otimes_K A)\oplus\langle A,A\rangle$ given by $F_{ij}(a)\mapsto E_{ij}\otimes a$ for $1\leq i\neq j\leq n$ and one can check, using relations (8)-(12), that its inverse is given by $E_{ij}\otimes a\mapsto F_{ij}(a)$ for $1\leq i\neq j\leq n$, $(E_{ii}-E_{jj})\otimes a\mapsto H_{ij}(a,1)$ and $\langle A,A\rangle\ni\langle a,b\rangle\mapsto nh(a,b)-\sum_{k=2}^n H_{1k}([a,b],1)$. Detailed computations are available in the appendix. $\square$

Since $\mathfrak{sl}_n(A)$ and $\mathfrak{st}_n(A)$ are perfect Lie superalgebras and $(\mathfrak{st}_n(A),\varphi)$ is a central extension of $\mathfrak{sl}_n(A)$ for $n\geq 3$, the universal central extension $\widehat{\mathfrak{sl}}_n(A)$ of the Lie superalgebra $\mathfrak{sl}_n(A)$ is also the universal central extension of $\mathfrak{st}_n(A)$, which is denoted $\widehat{\mathfrak{st}}_n(A)$. Indeed, by the universal property of $\widehat{\mathfrak{sl}}_n(A)$, we have a homomorphism $f_1:\widehat{\mathfrak{sl}}_n(A)\longrightarrow\mathfrak{st}_n(A)$ with $\mathrm{Ker}(f_1)$ central in $\widehat{\mathfrak{sl}}_n(A)$. Moreover, f_1 is surjective since $\mathfrak{st}_n(A)$ is generated by any choice of preimages of $E_{ij}\otimes a\in\mathfrak{sl}_n(A)$. Therefore, $\widehat{\mathfrak{sl}}_n(A)$ is a central extension of $\mathfrak{st}_n(A)$, so we have a homomorphism $f_2:\widehat{\mathfrak{st}}_n(A)\longrightarrow\widehat{\mathfrak{sl}}_n(A)$ whose kernel is central in $\widehat{\mathfrak{st}}_n(A)$. Moreover, f_2 is onto because $\widehat{\mathfrak{sl}}_n(A)$ is perfect and thus generated by a set of preimages of the elements in a K-spanning set of $\mathfrak{sl}_n(A)$. (Note that the composite $\widehat{\mathfrak{st}}_n(A)\twoheadrightarrow\mathfrak{st}_n(A)\twoheadrightarrow\mathfrak{sl}_n(A)$ is onto, hence $\mathrm{Image}(f_2)$ contains a set of generators of $\widehat{\mathfrak{sl}}_n(A)$.) Since f_2 is an epimorphism and $\widehat{\mathfrak{sl}}_n(A)$ is its own universal central extension, f_2 must admit a splitting $f_3:\widehat{\mathfrak{sl}}_n(A)\longrightarrow\widehat{\mathfrak{st}}_n(A)$: this can happen only if f_2 is an isomorphism, in which case $f_3=f_2^{-1}$. In conclusion, $\widehat{\mathfrak{sl}}_n(A)$ is isomorphic to $\widehat{\mathfrak{st}}_n(A)$.

Our purpose now is to calculate $\widehat{\mathfrak{st}}_n(A)$ for any ring K and $n\geq 3$.

6 Universal central extension of $\mathfrak{st}_n(A)$, $n\geq 5$.

When $n\geq 5$, we have the following super analog of one of the main results in [KaLo].

Theorem 6.1. *Let $n \geq 5$ and let $(\mathfrak{e}, \psi)$ be a central extension of the Lie superalgebra $\mathfrak{sl}_n(A)$. Then there exists a unique homomorphism $\eta : \mathfrak{st}_n(A) \to \mathfrak{e}$ such that $\varphi = \psi \circ \eta$. Therefore $\mathfrak{st}_n(A)$ is centrally closed and is the universal central extension of $\mathfrak{sl}_n(A)$.*

Since ψ is surjective, for any $E_{ij} \otimes a \in \mathfrak{sl}_n(A)$ we can choose some $\widetilde{F}_{ij}(a) \in \psi^{-1}(E_{ij} \otimes a)$. We need the following lemmas.

Lemma 6.1. *The commutator $[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)]$ doesn't depend on the choice of representatives in $\psi^{-1}(E_{ij} \otimes a)$ and $\psi^{-1}(E_{kl} \otimes b)$.*

Proof. This follows from the fact that, if $\check{F}_{ij}(a) \in \psi^{-1}(E_{ij} \otimes a)$, then $\check{F}_{ij}(a) - \widetilde{F}_{ij}(a) \in \mathrm{Ker}(\psi)$ and $\mathrm{Ker}(\psi)$ is central in $\mathfrak{e}$. $\square$

Lemma 6.2. *Let $1 \leq i \neq j \leq n$, and suppose that $1 \leq k, l \leq n$, and k, l are different from i and j. Then we have $[\widetilde{F}_{ik}(a), \widetilde{F}_{kj}(b)] = [\widetilde{F}_{il}(a), \widetilde{F}_{lj}(b)]$.*

Proof. If $k \neq l$, then

$$\begin{aligned} [\widetilde{F}_{ik}(a), \widetilde{F}_{kj}(b)] &= \big[\widetilde{F}_{ik}(a), [\widetilde{F}_{kl}(1), \widetilde{F}_{lj}(b)] + c_1\big] = \big[[\widetilde{F}_{ik}(a), \widetilde{F}_{kl}(1)], \widetilde{F}_{lj}(b)\big] + \big[\widetilde{F}_{kl}(1), [\widetilde{F}_{ik}(a), \widetilde{F}_{lj}(b)]\big] \\ &= [\widetilde{F}_{il}(a) + c_2, \widetilde{F}_{lj}(b)] + [\widetilde{F}_{kl}(1), c_3] = [\widetilde{F}_{il}(a), \widetilde{F}_{lj}(b)] \end{aligned}$$

where c_1, c_2, c_3 are central elements in $\mathfrak{e}$. $\square$

Proof of Theorem 6.1. For $1 \leq i \neq j \leq n$, set $w_{ij}(a) = [\widetilde{F}_{il}(a), \widetilde{F}_{lj}(1)]$ for some $l \neq i, j$; by lemma 6.2, this does not depend on the choice of l.

We would like to define a homomorphism $\eta : \mathfrak{st}_n(A) \to \mathfrak{e}$ is by $\eta(F_{ij}(a)) = w_{ij}(a)$. To see that this makes sense, we have to prove the following relations in $\mathfrak{e}$.

$$w_{ij}(xa + yb) = x w_{ij}(a) + y w_{ij}(b), \quad \text{for all } a, b \in A,\ x, y \in K. \tag{15}$$
$$[w_{ij}(a), w_{jk}(b)] = w_{ik}(ab), \ \text{for distinct } i, j, k, \tag{16}$$
$$[w_{ij}(a), w_{kl}(b)] = 0, \ \text{for } j \neq k, i \neq l. \tag{17}$$

Equality (15) follows from the fact that $\widetilde{F}_{il}(xa+yb) = x\widetilde{F}_{il}(a) + y\widetilde{F}_{il}(b) + c$ with $c \in \mathrm{Ker}(\psi)$. As for (16), choose $l \neq i, j, k$; then

$$\begin{aligned} [w_{ij}(a), w_{jk}(b)] &= \big[w_{ij}(a), [\widetilde{F}_{jl}(b), \widetilde{F}_{lk}(1)]\big] \\ &= \big[[w_{ij}(a), \widetilde{F}_{jl}(b)], \widetilde{F}_{lk}(1)\big] + (-1)^{|a||b|}\big[\widetilde{F}_{jl}(b), [w_{ij}(a), \widetilde{F}_{lk}(1)]\big] \\ &= [\widetilde{F}_{il}(ab) + c_1, \widetilde{F}_{lk}(1)] + [\widetilde{F}_{jl}(b), c_2] = w_{ik}(ab) \end{aligned}$$

where $c_1, c_2 \in \mathrm{Ker}(\psi)$.

We need our assumption that $n \geq 5$ to prove that equality (17) holds. Choose $m \neq i, j, k, l$. Then

$$\begin{aligned} [w_{ij}(a), w_{kl}(b)] &= \big[w_{ij}(a), [\widetilde{F}_{km}(b), \widetilde{F}_{ml}(1)]\big] \\ &= \big[[w_{ij}(a), \widetilde{F}_{km}(b)], \widetilde{F}_{ml}(1)\big] + (-1)^{|a||b|}\big[\widetilde{F}_{km}(b), [w_{ij}(a), \widetilde{F}_{ml}(1)]\big] \\ &= [c_1, \widetilde{F}_{ml}(1)] + [\widetilde{F}_{km}(b), c_2] = 0 \end{aligned}$$

where again $c_1, c_2 \in \mathrm{Ker}(\psi)$.

We have thus established relations (15)-(17), which proves that η is a well-defined homomorphism. The uniqueness of η follows from the fact that, since $F_{ij}(a) = [F_{ik}(a), F_{kj}(1)]$ for any distinct i, j, k, we must have $\eta(F_{ij}(a)) = [\eta(F_{ik}(a)), \eta(F_{kj}(1))]$ and $\eta(F_{ik}(a)) - \widetilde{F}_{ik}(a), \eta(F_{kj}(1)) - \widetilde{F}_{kj}(1) \in \mathrm{Ker}(\psi)$. $\square$

7 Central Extension of $\mathfrak{st}_4(A)$

In this section, we define a super 2-cocycle on $\mathfrak{st}_4(A)$ and construct the Lie superalgebra $\widehat{\mathfrak{st}}_4(A)$ as a covering (eventually a universal covering) of $\mathfrak{st}_4(A)$.

For any positive integer m, let $\mathcal{I}_m$ be the 2-sided $\mathbb{Z}/2\mathbb{Z}$-graded ideal of A generated by the elements: ma and $ab-(-1)^{|a||b|}ba$, for homogeneous elements $a, b \in A$.

Lemma 7.1. *$\mathcal{I}_m = mA + A[A,A]$ and $[A,A]A = A[A,A]$.*

Proof. Since ma and $ab-(-1)^{|a||b|}ba$ generate $\mathcal{I}_m$, $\mathcal{I}_m = mA + A[A,A]A$. Moreover, $[A,A]A \subseteq A[A,A] + [[A,A],A] \subseteq A[A,A] + [A,A] \subseteq A[A,A]$, and similarly we have $A[A,A] \subseteq [A,A]A$, so $[A,A]A = A[A,A]$ and the lemma is proved. $\square$

Let $A_m := A/\mathcal{I}_m$ be the quotient superalgebra over K; it is super commutative. Write $\bar{a} = a + \mathcal{I}_m$ for $a \in A$. If $m = 2$ and 2 is invertible in A, then $A_2 = 0$.

For $\{i,j,k,l\} = \{1,2,3,4\}$, let $\epsilon_{ijkl}(A_2)$ denote a copy of A_2 and identify $\epsilon_{ijkl}(\bar{r})$, $\epsilon_{ilkj}(\bar{r})$, $\epsilon_{kjil}(\bar{r})$ and $\epsilon_{klij}(\bar{r})$ for $\bar{r} \in A_2$. Thus, we have six distinct copies of A_2 whose direct sum is denoted by $\mathcal{W}$. Using the decomposition in lemma 5.1 of $\mathfrak{st}_4(A)$, we define a K-bilinear map $\psi : \mathfrak{st}_4(A) \times \mathfrak{st}_4(A) \to \mathcal{W}$ by $\psi(F_{ij}(a), F_{kl}(b)) = \epsilon_{ijkl}(\overline{ab})$ for $\{i,j,k,l\} = \{1,2,3,4\}$ and $a, b \in A$, and by $\psi(x,y) = 0$ for all other pairs of elements from the summands of lemma 5.1. Note that, if $m = 2$, then $\bar{a} = -\bar{a}$ in A_2.

The following lemma is central to the construction of $\widehat{\mathfrak{st}}_4(A)$.

Lemma 7.2. *The bilinear map ψ is a super 2-cocycle.*

Proof. Since there is a Lie superalgebra homomorphism $\alpha : \mathfrak{st}_4(A) \to \mathfrak{st}_4(A_2)$ with $\alpha(F_{ij}(a)) = F_{ij}(\bar{a})$ and $\psi(\alpha(x), \alpha(y)) = \psi(x,y)$ is well-defined for $x, y \in \mathfrak{st}_4(A)$, it suffices to verify the lemma for A_2, i.e., we can assume that $A = A_2$ and the proof is similar to the corresponding one in [GaSh]. In this case, it is clear that ψ is super skew-symmetric and $\psi(x,x) = 0$ for all x in $\mathfrak{st}_4(A)_0$. Now let $J(x,y,z) = (-1)^{|x||z|}\psi([x,y],z) + (-1)^{|x||y|}\psi([y,z],x) + (-1)^{|y||z|}\psi([z,x],y)$. for homogeneous elements $x, y, z \in \mathfrak{st}_4(A_2)$.

We will show $J(x,y,z) = 0$ by taking homogenous elements x, y, z in summands of lemma 5.1. If a term of $J(x,y,z)$ is not 0, we can reorder to assume that $z = F_{kl}(d)$ and $0 \neq [x,y] \in F_{ij}(A)$ with $\{i,j,k,l\} = \{1,2,3,4\}$.

Case 1: If x or y is in H, we can assume without loss of generality $x = H_{pq}(a,b)$ and $y = F_{ij}(c)$. Since $A(= A_2)$ is (super) commutative with $2A = 0$, $[x,y] \neq 0$ forces precisely one of p or q to be in $\{i,j\}$ (so the other to be in $\{k,l\}$ by (8),(9)). Moreover, in this case, $[x,y] = F_{ij}(abc)$, $[y,z] = 0$ and $[z,x] = F_{kl}(abd)$, so

$$\begin{aligned} J(x,y,z) &= \psi(F_{ij}(abc), F_{kl}(d)) + \psi(0, H_{pq}(a,b)) + \psi(F_{kl}(abd), F_{ij}(c)) \\ &= \epsilon_{ijkl}(abcd) + 0 + \epsilon_{klij}(abdc) = 0. \end{aligned}$$

Case 2: If neither x nor y is in H, we can assume that $x = F_{ip}(a)$ and $y = F_{pj}(b)$ with $p \in \{k,l\}$, so $[x,y] = F_{ij}(ab)$. For $p = k$, we have $[y,z] = 0$ and $[z,x] = F_{il}(ad)$, so

$$\begin{aligned} J(x,y,z) &= \psi(F_{ij}(ab), F_{kl}(d)) + \psi(0, F_{ik}(a)) + \psi(F_{il}(ad), F_{kj}(b)) \\ &= \epsilon_{ijkl}(abd) + 0 + \epsilon_{ilkj}(adb) = 0. \end{aligned}$$

For $p = l$, we have $[y,z] = F_{kj}(db)$ and $[z,x] = 0$, so

$$\begin{aligned} J(x,y,z) &= \psi(F_{ij}(ab), F_{kl}(d)) + \psi(F_{kj}(db), F_{il}(a)) + \psi(0, F_{lj}(b)) \\ &= \epsilon_{ijkl}(abd) + \epsilon_{kjil}(dba) + 0 = 0. \end{aligned}$$

$\square$

We therefore obtain a central extension of the Lie superalgebra $\mathfrak{st}_4(A)$, $0 \to \mathcal{W} \to \widehat{\mathfrak{st}}_4(A) \xrightarrow{\pi} \mathfrak{st}_4(A) \to 0$, i.e. $\widehat{\mathfrak{st}}_4(A) = \mathfrak{st}_4(A) \oplus \mathcal{W}$, with Lie superbracket $[(x,c),(y,c')] = ([x,y], \psi(x,y))$ for all $x, y \in \mathfrak{st}_4(A)$ and $c, c' \in \mathcal{W}$. π is the projection on the first summand $\pi : \mathfrak{st}_4(A) \oplus \mathcal{W} \to \mathfrak{st}_4(A)$.

We can now apply definition 4.2 with $\mathcal{C} = \mathcal{W}$ and $\xi = \psi$ to obtain the Lie superalgebra $\mathfrak{st}_4(A)^\sharp$. Since A is a unital algebra, $\mathfrak{st}_4(A)^\sharp$ is perfect. By lemma 4.1, there exists a unique Lie superalgebra isomorphism $\rho : \mathfrak{st}_4(A)^\sharp \to \widehat{\mathfrak{st}}_4(A)$ such that $\rho(F^\sharp_{ij}(a)) = F_{ij}(a)$ and $\rho|_{\mathcal{W}} = id$.

8 Central Extension of $\mathfrak{st}_3(A)$

In this section, we shall handle $\mathfrak{st}_3(A)$. Recall that $\mathcal{I}_3 = 3A + A[A,A]$ and $A_3 = A/\mathcal{I}_3$ is an associative super commutative K-algebra.

For $\{i,j,k\} = \{1,2,3\}$, let $\epsilon_{ijpq}(A_3)$ for $(p,q) = (i,k)$ or (k,j) denote a copy of A_3 and identify $\epsilon_{ijpq}(\bar{r})$ with $\epsilon_{pqij}(-\bar{r})$. Thus, we have six distinct copies of A_3 whose direct sum is denoted $\mathcal{U}$. Using the decomposition in lemma 5.1 of $\mathfrak{st}_3(A)$, we define a K-bilinear map $\psi : \mathfrak{st}_3(A) \times \mathfrak{st}_3(A) \to \mathcal{U}$ by $\psi(F_{ij}(r), F_{pq}(s)) = \epsilon_{ijpq}(\overline{rs})$ for $(p,q) = (i,k)$ or (k,j) with $\{i,j,k\} = \{1,2,3\}$ and $r, s \in A$, and by $\psi(x,y) = 0$ for all other pairs of elements from the summands of lemma 5.1.

As for $\mathfrak{st}_4(A)$, we have the following lemma.

Lemma 8.1. *The bilinear map ψ is a super 2-cocycle.*

Proof. As in the proof of Lemma 7.2, we can assume $A = A_3$, i.e., A is super commutative and $3A = 0$. By definition, ψ is super skew-symmetric and $\psi(x,x) = 0$ for all x in $\mathfrak{st}_3(A)_0$.

Similarly to the proof of lemma 7.2, we show that $J(x,y,z) = 0$ by taking homogeneous elements x, y, z in summands of lemma 5.1. If a term of $J(x,y,z)$ is not 0, we can assume that $z = F_{pq}(d)$ and $0 \neq [x,y] \in F_{st}(A)$ with $(p,q) = (s,u)$ or (u,t) and $\{s,t,u\} = \{1,2,3\}$.

Case 1: If x or y is in H, we can assume $x = H_{ij}(a,b)$ and $y = F_{st}(c)$. By (8),(9) for $A = A_3$, we have $J(x,y,z) = \epsilon_{stpq}(\theta abcd)$ with $\theta = 0, 3$ or -3, thus $J(x,y,z) = 0$.

Case 2: If neither x nor y is in H, we can assume that $x = F_{su}(a)$ and $y = F_{ut}(b)$. We have, if $(p,q) = (s,u)$,

$$\begin{aligned} J(x,y,z) &= (-1)^{|a||d|}\psi(F_{st}(ab), F_{su}(d)) + (-1)^{|a||b|}\psi(-(-1)^{|b||d|}F_{st}(db), F_{su}(a)) \\ &= (-1)^{|a||d|}\epsilon_{stsu}(abd) - (-1)^{|a||b|+|b||d|}\epsilon_{stsu}(dba) = 0, \end{aligned}$$

while, if $(p,q) = (u,t)$,

$$\begin{aligned} J(x,y,z) &= (-1)^{|a||d|}\psi(F_{st}(ab), F_{ut}(d)) + (-1)^{|b||d|}\psi(-(-1)^{|a||d|}F_{st}(ad), F_{ut}(b)) \\ &= (-1)^{|a||d|}\epsilon_{stut}(abd) - (-1)^{|a||d|+|b||d|}\epsilon_{stut}(adb) = 0. \end{aligned}$$

□

As in the $\mathfrak{st}_4(A)$ case, we have a central extension of $\mathfrak{st}_3(A)$, $0 \to \mathcal{U} \to \widehat{\mathfrak{st}}_3(A) \xrightarrow{\pi} \mathfrak{st}_3(A) \to 0$ i.e. $\widehat{\mathfrak{st}}_3(A) = \mathfrak{st}_3(A) \oplus \mathcal{U}$, and we can apply definition 4.2 with $\mathcal{C} = \mathcal{U}$ and $\xi = \psi$ to obtain the Lie superalgebra $\mathfrak{st}_3(A)^\sharp$. $\mathfrak{st}_3(A)^\sharp$ is perfect and, by lemma 4.1, there exists a unique Lie superalgebra isomorphism $\rho : \mathfrak{st}_3(A)^\sharp \to \widehat{\mathfrak{st}}_3(A)$.

9 Proof of the main theorem when $n = 3, 4$.

In last two sections, we constructed the Lie superalgebra $\widehat{\mathfrak{st}}_n(A)$ as a covering of $\mathfrak{st}_n(A)$ for $n = 3, 4$. Now we can prove the main theorem when $n = 3, 4$. We follow ideas used in [CGS, GaSh].

Theorem 9.1. *For $n = 3$ or 4, the universal central extension of $\mathfrak{st}_n(A)$ is $(\widehat{\mathfrak{st}}_n(A), \pi)$.*

Proof. Suppose that

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathfrak{g} \xrightarrow{\chi} \mathfrak{st}_n(A) \longrightarrow 0$$

is a central extension of $\mathfrak{st}_n(A)$. We will show that we can choose a preimage $\widetilde{F}_{ij}(a)$ of $F_{ij}(a)$ under χ for $1 \le i \neq j \le n, a \in A$ and a linear map $\mu : \mathcal{C} \to \mathcal{V}$ which satisfy the relations (4)-(6) for $\mathfrak{st}_n(A)^\sharp$ with $\xi = \mu \circ \psi$ and $\mathcal{C} = \mathcal{U}, \mathcal{W}$ for $n = 3, 4$ respectively. Thus, we will have a homomorphism $\theta : \mathfrak{st}_n(A)^\sharp \to \mathfrak{g}$ with $\theta(F^\sharp_{ij}(a)) = \widetilde{F}_{ij}(a)$ so $\chi \circ \theta = \pi \circ \rho$ as in lemma 4.1 (with θ, χ playing the roles of ρ, π). This will give us the homomorphism $\theta \circ \rho^{-1} : \widehat{\mathfrak{st}}_n(A) \to \mathfrak{g}$ satisfying $\chi \circ (\theta \circ \rho^{-1}) = \pi$ and we will be able to conclude that $(\widehat{\mathfrak{st}}_n(A), \pi)$ is the universal covering of $\mathfrak{st}_n(A)$.

We begin by choosing any preimage $\widetilde{F}_{ij}(a)$ under χ of $F_{ij}(a)$ for a in a K-basis of A and extend linearly to all elements in A. We observe as before that, since $\mathfrak{g}$ is a central extension, $\widetilde{H}_{ij}(a, b) := [\widetilde{F}_{ij}(a), \widetilde{F}_{ji}(b)]$ is independent of the choice of $\widetilde{F}_{ij}(a), \widetilde{F}_{ji}(b)$. Moreover, if $[H_{pq}(a, b), F_{ij}(c)] = F_{ij}(d)$, then $[\widetilde{H}_{pq}(a, b), \widetilde{F}_{ij}(c)] \in \widetilde{F}_{ij}(d) + \mathcal{V}$. Also, if $[H_{pq}(1, 1), F_{ij}(c)] = F_{ij}(mc)$, where $m \in \{0, \pm 1, \pm 2\}$, then we always have $[H_{pq}(1, 1), F_{ji}(d)] = F_{ji}(-md)$ for any $d \in A$. Therefore,

$$[\widetilde{H}_{pq}(1, 1), \widetilde{H}_{ij}(a, b)] \in [\widetilde{F}_{ij}(ma) + \mathcal{V}, \widetilde{F}_{ji}(b)] + [\widetilde{F}_{ij}(a), \widetilde{F}_{ji}(-mb) + \mathcal{V}] = \{0\} \tag{18}$$

for any $1 \le i, j, p, q \le n$, $i \neq j, p \neq q$.

Now fix some $k \neq i, j$ and replace $\widetilde{F}_{ij}(a)$ by $\check{F}_{ij}(a)$ with $\check{F}_{ij}(a) = [\widetilde{H}_{ik}(1, 1), \widetilde{F}_{ij}(a)]$. It then follows that $[\widetilde{H}_{ik}(1, 1), \check{F}_{ij}(a)] = [\widetilde{H}_{ik}(1, 1), [\widetilde{H}_{ik}(1, 1), \widetilde{F}_{ij}(a)]] = [\widetilde{H}_{ik}(1, 1), \widetilde{F}_{ij}(a) + \nu] = \check{F}_{ij}(a)$ for some $\nu \in \mathcal{V}$. $\check{F}_{ij}(a)$ is another element in the preimage of $F_{ij}(a)$ under χ, which we will rename $\widetilde{F}_{ij}(a)$. In other words, we can assume, without loss of generality, that $[\widetilde{H}_{ik}(1, 1), \widetilde{F}_{ij}(a)] = \widetilde{F}_{ij}(a)$ and this holds for any a in a K-basis of A.

Using (18), we see that

$$[\widetilde{H}_{pq}(a, b), \widetilde{F}_{ij}(c)] = \big[\widetilde{H}_{pq}(a, b), [\widetilde{H}_{ik}(1, 1), \widetilde{F}_{ij}(c)]\big] = \big[\widetilde{H}_{ik}(1, 1), [\widetilde{H}_{pq}(a, b), \widetilde{F}_{ij}(c)]\big] = \widetilde{F}_{ij}(d)$$

if $[H_{pq}(a, b), F_{ij}(c)] = F_{ij}(d)$. In particular, taking $(p, q) = (i, l)$, we observe that $\widetilde{F}_{ij}(a)$ does not depend on the choice of $k \neq i, j$. Applying $ad(\widetilde{H}_{ij}(1, 1))$ to $[\widetilde{F}_{ij}(a), \widetilde{F}_{jk}(b)] \in \widetilde{F}_{ik}(ab) + \mathcal{V}$ (with i, j, k all distinct) yields $[\widetilde{F}_{ij}(2a) + \nu_1, \widetilde{F}_{jk}(b)] + [\widetilde{F}_{ij}(a), \widetilde{F}_{jk}(-b) + \nu_2] = \widetilde{F}_{ik}(ab)$ with $\nu_1, \nu_2 \in \mathcal{V}$, hence $[\widetilde{F}_{ij}(a), \widetilde{F}_{jk}(b)] = \widetilde{F}_{ik}(ab)$. Thus, $\widetilde{F}_{ij}(a), \mathcal{V}$ satisfy relations (4)-(5). (Recall that $\psi(F_{ij}(a), F_{jk}(b)) = 0$.)

We now focus on relation (6). Applying $ad(\widetilde{F}_{ik}(1))$ to $[\widetilde{F}_{kj}(a), \widetilde{F}_{ij}(b)] \in \mathcal{V}$ for i, j, k all distinct gives $[\widetilde{F}_{ij}(a), \widetilde{F}_{ij}(b)] = 0$. When $n = 4$, picking $l \neq i, j, k$ and applying $ad(\widetilde{H}_{kl}(1, 1))$ to $[\widetilde{F}_{kj}(a), \widetilde{F}_{ij}(b)]$ yields $[\widetilde{F}_{kj}(a), \widetilde{F}_{ij}(b)] = 0$; similarly, if $n = 4$, $[\widetilde{F}_{ij}(a), \widetilde{F}_{ik}(b)] = 0$.

Therefore, when $n = 3$ or 4, all cases in relation (6) are satisfied, except perhaps

$$[\widetilde{F}_{ij}(a), \widetilde{F}_{pq}(b)] = \mu(\epsilon_{ijpq}(\overline{ab})) \quad \text{with } (p, q) = (i, k) \text{ or } (p, q) = (k, j), \text{ if } \ n = 3 \tag{19}$$

$$[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)] = \mu(\epsilon_{ijkl}(\overline{ab})) \ \text{ if } \ n = 4, \text{ and } i, j, k, l \text{ are all distinct.} \tag{20}$$

for some map μ which remains to be defined.

Let us establish (19). Define a K-linear map $\mu_{ijpq} : A \longrightarrow \mathcal{V}$ by setting $\mu_{ijpq}(c) = [\widetilde{F}_{ij}(c), \widetilde{F}_{pq}(1)]$ for $(p,q) = (i,k)$ or (k,j). Applying $ad(\widetilde{H}_{ij}(a,b))$ to this central element gives

$$\mu_{ijpq}(abc + (-1)^{|a||b|+|b||c|+|c||a|}cba) + (-1)^{|c||a|+|b||c|}[\widetilde{F}_{ij}(c), \widetilde{F}_{pq}(d)] = 0 \tag{21}$$

where $d = ab$ if $(p,q) = (i,k)$ and $d = (-1)^{|a||b|}ba$ if $(p,q) = (k,j)$. Setting $a = b = 1$ implies $\mu_{ijpq}(3c) = 0$ for any $c \in A$, while setting $c = 1$ gives $\mu_{ijpq}(ab + (-1)^{|a||b|}ba) + [\widetilde{F}_{ij}(1), \widetilde{F}_{pq}(d)] = 0$, so that $\mu_{pqij}(d) = \mu_{ijpq}(ab + (-1)^{|a||b|}ba)$. In particular, $b = 1$ gives $\mu_{pqij}(a) = \mu_{ijpq}(2a) = -\mu_{ijpq}(a)$ for any $a \in A$, so $\mu_{ijpq}(2d) = \mu_{ijpq}(-d) = \mu_{pqij}(d) = \mu_{ijpq}(ab + (-1)^{|a||b|}ba)$, which implies $\mu_{ijpq}(ab) = (-1)^{|a||b|}\mu_{ijpq}(ba)$ whether $d = ab$ or $(-1)^{|a||b|}ba$. Using this and letting $b = 1$ in (21) yields $(-1)^{|a||c|}[\widetilde{F}_{ij}(c), \widetilde{F}_{pq}(a)] = -\mu_{ijpq}(ac + (-1)^{|c||a|}ca) = -\mu_{ijpq}(ac + ac) = \mu_{ijpq}(ac)$. We can now deduce that (21) is equivalent to $\mu_{ijpq}(abc + (-1)^{|b||c|}acb) + \mu_{ijpq}(dc) = 0$, hence to $\mu_{ijpq}(abc - (-1)^{|b||c|}acb) = 0$. Therefore, $\mu_{ijpq}(\mathcal{I}_3) = 0$ since $\mathcal{I}_3$ is linearly spanned by the elements $3a$ and $a[b,c]$ for $a,b,c \in A$ by lemma 7.1. We can now define $\mu : \mathcal{U} \to \mathcal{V}$ by $\mu(\epsilon_{ijpq}(\bar{a})) = \mu_{ijpq}(a)$ such that (19) holds.

To establish (20), we set $\mu_{ijkl}(c) = [\widetilde{F}_{ij}(c), \widetilde{F}_{kl}(1)] \in \mathcal{V}$ for $\{i,j,k,l\} = \{1,2,3,4\}$. Applying $ad(\widetilde{H}_{ij}(a,b))$ gives $\mu_{ijkl}(abc + (-1)^{|a||b|+|b|c|+|c||a|}cba) = 0$. If we set $b = c = 1$, we get $\mu_{ijkl}(2a) = 0$; using this and setting $c = 1$, we obtain $\mu_{ijkl}(ab) = -(-1)^{|a||b|}\mu_{ijkl}(ba) = \mu_{ijkl}(ba)$. Moreover,

$$\begin{aligned}\mu_{ijkl}(a[b,c]) &= \mu_{ijkl}(abc) - (-1)^{|b|c|}\mu_{ijkl}(acb) = \mu_{ijkl}(abc) - (-1)^{|b|c|}\mu_{ijkl}(-(-1)^{|a||b|+|c||a|}cba) \\ &= \mu_{ijkl}(abc + (-1)^{|a||b|+|b|c|+|c||a|}cba) = 0\end{aligned}$$

Therefore, $\mu_{ijkl}(\mathcal{I}_2) = 0$. We now have to verify the invariance property of μ_{ijpq} upon certain permutation of its indices. Note that $[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)] = -[\widetilde{F}_{il}(ab), \widetilde{F}_{kj}(1)] = -\mu_{ilkj}(ab)$; in particular, $b = 1$ gives $\mu_{ijkl}(a) = -\mu_{ilkj}(a)$, hence $\mu_{ijkl}(a) = \mu_{ilkj}(a)$ for any $a \in A$ because $\mu_{ijkl}(a) = -\mu_{ijkl}(a)$. It now follows that $[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)] = \mu_{ijkl}(ab)$.

Furthermore, $\mu_{klij}(a) = [\widetilde{F}_{kl}(1), \widetilde{F}_{ij}(a)] = -[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)] = -\mu_{ijkl}(a) = \mu_{ijkl}(a)$. In conclusion, $\mu_{kjil}(a) = \mu_{ilkj}(a) = \mu_{ijkl}(a)$. This implies that we can now define $\mu : \mathcal{W} \to \mathcal{V}$ by $\mu(\epsilon_{ijkl}(\bar{a})) = \mu_{ijkl}(a)$ so (20) holds.

We have now established that (19) and (20) hold and this completes the proof of theorem 9.1. $\square$

10 Appendix

In this appendix, we present computations which establish the relations (11),(12) and are useful in the proof of theorem 5.1 and corollary 5.1. We set $F_{ab} = F_{ab}(1)$.

If $1 \leq k \neq m \leq n$:

$$\begin{aligned}
[H_{1k}(a,1), H_{1m}(b,1)] &= \big[[F_{1k}(a), F_{k1}], [F_{1m}(b), F_{m1}]\big] \\
&= \Big[\big[F_{1k}(a), [F_{1m}(b), F_{m1}]\big], F_{k1}\Big] + \Big[F_{1k}(a), \big[F_{k1}, [F_{1m}(b), F_{m1}]\big]\Big] \\
&= -(-1)^{|a||b|}[F_{1k}(ba), F_{k1}] + [F_{1k}(a), F_{k1}(b)] = H_{1k}(a,b) - (-1)^{|a||b|}H_{1k}(1, ba) \\
&= h(a,b)
\end{aligned}$$

$$\begin{aligned}
[H_{1k}(a,1), H_{1k}(b,1)] &= \big[H_{1k}(a,1), [F_{1k}(b), F_{k1}]\big] \\
&= \big[[H_{1k}(a,1), F_{1k}(b)], F_{k1}\big] + (-1)^{|a||b|}\big[F_{1k}(b), [H_{1k}(a,1), F_{k1}]\big] \\
&= [F_{1k}(ab + (-1)^{|a||b|}ba), F_{k1}] - 2(-1)^{|a||b|}[F_{1k}(b), F_{k1}(a)] \\
&= H_{1k}(ab,1) + (-1)^{|a||b|}H_{1k}(1,ba) - 2(-1)^{|a||b|}H_{1k}(b,a) \\
&= H_{1k}(ab,1) + (-1)^{|a||b|}H_{1k}(1,ba) - 2(-1)^{|a||b|}h(b,a) - 2H_{1k}(1,ab) \\
&= 2h(a,b) - H_{1k}(1,[a,b])
\end{aligned}$$

We have proved the two relations in (11). If $2 \leq k \leq n$,

$$\begin{aligned}
[h(a,b), H_{1k}(c,1)] &= \big[h(a,b), [F_{1k}(c), F_{k1}]\big] = \big[[h(a,b), F_{1k}(c)], F_{k1}\big] + (-1)^{|c|(|a|+|b|)}\big[F_{1k}(c), [h(a,b), F_{k1}]\big] \\
&= [F_{1k}([a,b]c), F_{k1}] - (-1)^{|c|(|a|+|b|)}[F_{1k}(c), F_{k1}([a,b])] \\
&= H_{1k}([a,b]c, 1) - (-1)^{|c|(|a|+|b|)}H_{1k}(c, [a,b]) \\
&= -(-1)^{|c|(|a|+|b|)}h(c, [a,b]) = h([a,b], c)
\end{aligned}$$

This establishes the first relation in (12). As for the second one,

$$\begin{aligned}
[h(a,b), h(c,d)] &= [h(a,b), H_{1j}(c,d) - (-1)^{|c||d|}H_{1j}(1,dc)] \\
&= H_{1j}([a,b]c,d) - (-1)^{(|c|+|d|)(|a|+|b|)}H_{1j}(c, d[a,b]) - (-1)^{|c||d|}H_{1j}([a,b],dc) \\
&\quad +(-1)^{|c||d|+(|c|+|d|)(|a|+|b|)}H_{1j}(1, dc[a,b]) \\
&= h([a,b]c,d) + (-1)^{|d|(|c|+|a|+|b|)}H_{1j}(1, d[a,b]c) - (-1)^{(|c|+|d|)(|a|+|b|)}h(c, d[a,b]) \\
&\quad -(-1)^{|d|(|a|+|b|+|c|)}H_{1j}(1, d[a,b]c) \\
&\quad -(-1)^{|c||d|}h([a,b],dc) - (-1)^{|c||d|+(|c|+|d|)(|a|+|b|)}H_{1j}(1, dc[a,b]) \\
&\quad +(-1)^{|c||d|+(|c|+|d|)(|a|+|b|)}H_{1j}(1, dc[a,b]) \\
&= h([a,b],[c,d])
\end{aligned}$$

We now give all the computations relevant for the proof of corollary 5.1. In order to verify that the natural homomorphism $\mathfrak{st}_n(A) \longrightarrow (\mathfrak{sl}_n(K) \otimes_K A) \oplus \langle A, a \rangle$ admits an inverse given by the formula in the proof of corollary 5.1, we have to replace, in relations (13),(14), the elements $E_{ij} \otimes a, \langle a, b \rangle$ by their images as specified in that proof and see if the relations that we obtain are satisfied in $\mathfrak{st}_n(A)$. Relation (13) can be checked quickly if $(x,y) = 0$, so we verify only the case $x = F_{ij}, y = F_{ji}$ with $1 \leq i \neq j \leq n$. The right-hand side becomes

$$\begin{aligned}
& h(a,b) - \frac{1}{n}\sum_{k=2}^{n} H_{1k}\big([a,b],1\big) + \frac{1}{2}H_{ij}(ab+(-1)^{|a||b|}ba,1) + \frac{1}{2n}\sum_{\substack{k=1\\k\neq i}}^{n} H_{ik}\big([a,b],1\big) + \frac{1}{2n}\sum_{\substack{k=1\\k\neq j}}^{n} H_{jk}\big([a,b],1\big) \\
= \; & h(a,b) - \frac{1}{n}\sum_{k=2}^{n} H_{1k}([a,b],1) + \frac{1}{2}H_{ij}(ab+(-1)^{|a||b|}ba,1) + \frac{1}{2}H_{i1}\big([a,b],1\big) \\
& + \frac{1}{2}H_{j1}\big([a,b],1\big) + \frac{1}{n}\sum_{k=2}^{n} H_{1k}\big([a,b],1\big) \\
= \; & h(a,b) + \frac{1}{2}H_{ij}(ab+(-1)^{|a||b|}ba,1) + \frac{1}{2}H_{i1}([a,b],1) + \frac{1}{2}H_{j1}([a,b],1) \\
= \; & h(a,b) + \frac{1}{2}H_{i1}(ab+(-1)^{|a||b|}ba,1) + \frac{1}{2}H_{1j}(ab+(-1)^{|a||b|}ba,1) + \frac{1}{2}H_{i1}([a,b],1) + \frac{1}{2}H_{j1}([a,b],1) \\
= \; & h(a,b) + H_{i1}(ab,1) + (-1)^{|a||b|}H_{1j}(ba,1) = H_{ij}(a,b)
\end{aligned}$$

The first relation in (14) is very similar to the second one in (12) but it does not follow automatically from it. It left-hand side equals

$$\begin{aligned}
& \left[nh(a_1,a_2) - \sum_{k=2}^{n} H_{1k}([a_1,a_2],1), nh(b_1,b_2) - \sum_{m=2}^{n} H_{1m}([b_1,b_2],1)\right] \\
= \; & n^2 h\big([a_1,a_2],[b_1,b_2]\big) - n\sum_{m=2}^{n} h\big([a_1,a_2],[b_1,b_2]\big) + n(-1)^{(|a_1|+|a_2|)(|b_1|+|b_2|)}\sum_{k=2}^{n} h\big([b_1,b_2],[a_1,a_2]\big) \\
& + \sum_{\substack{k,m=2\\k\neq m}}^{n} h\big([a_1,a_2],[b_1,b_2]\big) + \sum_{k=2}^{n}\Big(2h\big([a_1,a_2],[b_1,b_2]\big) - H_{1k}\big(1,[[a_1,a_2],[b_1,b_2]]\big)\Big) \\
= \; & nh\big([a_1,a_2],[b_1,b_2]\big) - \sum_{k=2}^{n} H_{1k}\Big(\big[[a_1,a_2],[b_1,b_2]\big],1\Big)
\end{aligned}$$

As for the second relation in (14), it follows from (8), (9) when $i \neq 1$ and $j \neq 1$; when $i = 1, j \neq 1$, its left-hand side equals

$$\begin{aligned}
& nF_{1j}\big([a_1,a_2]b\big) - \sum_{k=2,k\neq j}^{n} \big[H_{1k}([a_1,a_2]),1), F_{1j}(b)\big] - \big[H_{1j}\big([a_1,a_2],1\big), F_{1j}(b)\big] \\
= \; & nF_{1j}\big([a_1,a_2]b\big) - \sum_{k=2,k\neq j}^{n} F_{1j}\big([a_1,a_2]b\big) - F_{1j}\big([a_1,a_2]b + (-1)^{|b|(|a_1|+|a_2|)}b[a_1,a_2]\big) \\
= \; & F_{ij}([[a_1,a_2],b])
\end{aligned}$$

References

[CGS] H. Chen, Y. Gao, S. Shang, *Central extensions of Steinberg Lie superalgebras of small rank*, Comm. Algebra **35** (2007), no. 12, 4225–4244.

[ChGu] H. Chen, N. Guay *Twisted affine Lie superalgebra of type Q and quantization of its enveloping superalgebra*, to appear in Mathematische Zeitschrift.

[FSS] L. Frappat, A. Sciarrino, P. Sorba, *Dictionary on Lie algebras and superalgebras*, Academic Press, Inc., San Diego, CA, 2000. xxii+410 pp.

[GaSh] Y. Gao, S. Shang, *Universal coverings of Steinberg Lie algebras of small characteristic*, J. Algebra **311** (2007), no. 1, 216-230.

[GJKK] D. Grantcharov, J.H. Jung. S.-J. Kang, M. Kim, *Highest weight modules over quantum queer superalgebra $U_q(\mathfrak{q}(n))$*, Comm. Math. Phys. **296** (2010), no. 3, 827–860.

[GJKKK] D. Grantcharov, J.H. Jung. S.-J. Kang, M. Kashiwara, M. Kim, *Quantum queer superalgebra and crystal bases*, Proc. Japan Acad. Ser. A Math. Sci **86** (2010), no. 10, 177–182.

[HuLi] D. Liu, N. Hu, *Leibniz superalgebras and central extensions*, J. Algebra Appl. **5** (2006), no. 6, 765-780.

[IoKo1] K. Iohara, Y. Koga, *Central extensions of Lie superalgebras*, Comment. Math. Helv. **76** (2001), no. 1, 110–154.

[IoKo2] K. Iohara, Y. Koga, *Second homology of Lie superalgebras*, Math. Nachr. **278** (2005), no. 9, 1041-1053.

[K] V.G. Kac, *Lie superalgebras*, Advances in Math. **26** (1977), no. 1, 8–96.

[Ka1] C. Kassel, *Kähler differentials and coverings of complex simple Lie algebras extended over a commutative algebra*, Proceedings of the Luminy conference on algebraic **K**-theory (Luminy, 1983), J. Pure Appl. Algebra **34** (1984), no. 2-3, 265–275.

[Ka2] C. Kassel, *A Künneth formula for the cyclic cohomology of $\mathbb{Z}/2$-graded algebras*, Math. Ann. 275 (1986), no. 4, 683–699.

[KaLo] C. Kassel, J.-L. Loday, *Extensions centrales d'algèbres de Lie*, Ann. Inst. Fourier (Grenoble) **32** (1982), no. 4, 119–142 (1983).

[MiPi1] A.V. Mikhalev and I.A. Pinchuk, *Universal central extension of the matrix Lie superalgebras $sl(m,n,A)$*, Contemp. Math. 264 (2000), 111–125.

[MiPi2] A.V. Mikhalev and I.A. Pinchuk, *The universal central extension* $\mathrm{Q}(n,A)$ *of Lie superalgebras*, (Russian) Chebyshevskii Sb. **6** (2005), no. 4(16), 149-153.

[Na] M. Nazarov, *Yangian of the queer Lie superalgebra*, Comm. Math. Phys. **208** (1999), no. 1, 195–223.

[Ne] E. Neher, *An introduction to universal central extensions of Lie superalgebras*, Groups, rings, Lie and Hopf algebras, St. John's, NF, 2001, 141–166 Math. Appl., **555**, Kluwer Acad. Publ., Dordrecht, 2003.

Department of Mathematical and Statistical Sciences
University of Alberta
CAB 632
Edmonton, AB T6G 2G1
Canada

nguay@math.ualberta.ca, hongchen@math.ualberta.ca