Highest Weight Modules Over The Quantum Periplectic Superalgebra of Type P

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Abstract

In this paper, we begin the study of highest weight representations of the quantized enveloping superalgebra $\mathfrak{U}_q\mathfrak{p}_n$ of type P. We introduce a Drinfeld-Jimbo representation and establish a triangular-decomposition of $\mathfrak{U}_q\mathfrak{p}_n$. We explain how to relate modules over $\mathfrak{U}_q\mathfrak{p}_n$ to modules over $\mathfrak{p}_n$, the Lie superalgebra of type P, and we prove that the category of tensor modules over $\mathfrak{U}_q\mathfrak{p}_n$ is not semisimple.

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Introduction

The classification of finite-dimensional simple Lie superalgebras and the foundations of their representation theory was established by V. Kac in [K1] and [K2]. The representation theory of Lie superalgebras has been known, since its inception, to be more complicated than that of Lie algebras. The Lie superalgebras of types P and Q are especially interesting due to the algebraic, geometric, and combinatorial properties of their representations. The study of the representations of type P Lie superalgebras, which are also called periplectic in the literature, has attracted considerable attention in the last several years. Interesting results on the category $\mathcal{O}$, the associated periplectic Brauer algebras, and related theories have been established in [AGG], [BDEA$^+$1], [BDEA$^+$2], [CP], [Co], [CE1], [CE2], [DHIN], [EAS1], [EAS2], [HIR], [IN], [IRS], [KT], [Ser], among others.

In this paper we initiate the study of highest weight representations of the quantum superalgebra $\mathfrak{U}_q\mathfrak{p}_n$. In [AGG] we constructed a flat deformation of the universal enveloping algebra $\mathfrak{U}\mathfrak{p}_n$ which is a quantum enveloping superalgebra in the sense of Drinfeld ([Dr], §7). The idea was to apply a suitable modification of the procedure used by Faddeev, Reshetikhin, and Takhtajan in [FRT] using an element S in $\mathrm{End}(\mathbb{C}_q(n|n)^{\otimes 2})$ that satisfies the quantum Yang-Baxter equation.

In the present paper, based on the definition of $\mathfrak{U}_q\mathfrak{p}_n$ in [AGG], we give a presentation of $\mathfrak{U}_q\mathfrak{p}_n$ in terms of Drinfeld-Jimbo generators and relations. These relations are quantum deformations of those obtained in [DKM]. Using this new presentation, we find a natural triangular decomposition of $\mathfrak{U}_q\mathfrak{p}_n$, and then introduce the notion of highest weight module. This matches the corresponding result of Moon in [M] for $\mathfrak{U}\mathfrak{p}_n$. We also obtain the explicit decomposition of the second and the third tensor power of the natural representation of $\mathfrak{U}_q\mathfrak{p}_n$. These decompositions, in particular, imply that the category of tensor representations is not semisimple, which is expected.

The structure of the paper is as follows. We give the notation and basic definitions related to the classical periplectic Lie superalgebra in Section 1. In Section 2, we present a Drinfeld-Jimbo representation of $\mathfrak{U}_q\mathfrak{p}_n$ and prove its triangular decomposition. We introduce standard notation, definitions, and results related to highest weight $\mathfrak{U}_q\mathfrak{p}_n$-modules in Section 3. In Section 4 we discuss the classical limit and how the highest weight representations of $\mathfrak{U}_q\mathfrak{p}_n$ relate to those of $\mathfrak{p}_n$ (cf. Theorem 4.12). In the last section, we discuss tensor representations of $\mathfrak{U}_q\mathfrak{p}_n$ and use particular modules to prove that not every tensor representation of $\mathfrak{U}_q\mathfrak{p}_n$ is completely reducible (cf. Theorem 5.14).

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1 The Lie superalgebra $\mathfrak{p}_n$ and its representations

By $\mathbb{Z}_2 = \{0, 1\}$ we denote the group $\mathbb{Z}/2\mathbb{Z}$. All Lie superalgebras and homomorphisms are over $\mathbb{C}$ unless otherwise stated.

We will use the same setting as in [AGG]. We will denote by $\mathbb{C}(n|n)$ as the vector superspace $\mathbb{C}^n \oplus \mathbb{C}^n$ spanned by the standard basis vectors $u_{-n}, \dots, u_{-1}, u_1, \dots, u_n$. We say that u_i is odd if $i < 0$ and even if $i > 0$. Denote the elementary matrices in $M_{n|n}(\mathbb{C})$, the vector superspace consisting of square $(2n)\times(2n)$-matrices with entries in $\mathbb{C}$ by E_{ij}, with $i, j \in \{\pm 1, \pm 2, \dots, \pm n\}$. Set the parity function $p : \{\pm 1, \pm 2, \dots, \pm n\} \longrightarrow \mathbb{Z}_2$ to be $p(i) = 0$ if $i > 0$ and $p(i) = 1$ if $i < 0$. We set $\mathsf{E}_{ij} = E_{ij} - (-1)^{p(i)(p(j)+1)} E_{-j,-i}$ and observe that $\mathsf{E}_{ij} = -(-1)^{p(i)(p(j)+1)} \mathsf{E}_{-j,-i}$ for all $i, j \in \{\pm 1, \dots, \pm n\}$. Therefore, $\mathsf{E}_{i,-i} = 0$ when $1 \le i \le n$.

The Lie superalgebra $\mathfrak{p}_n$ of type P is the subsuperalgebra of $\mathfrak{gl}(n|n)$ that consists of matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A, B, C, D \in \mathfrak{gl}(n)$, $D = -A^t$, $B = B^t$, and $C = -C^t$. A basis of $\mathfrak{p}_n$ is provided by all the matrices E_{ij} with indices i and j respecting one of the following series of inequalities:

$$1 \le |j| < |i| \le n \text{ or } 1 \le i = j \le n \text{ or } -n \le i = -j \le -1.$$

The superbracket on $\mathfrak{p}_n$ is given by

$$\begin{aligned} [\mathsf{E}_{ji}, \mathsf{E}_{lk}] &= \delta_{il} \mathsf{E}_{jk} - (-1)^{(p(i)+p(j))(p(k)+p(l))} \delta_{jk} \mathsf{E}_{li} \\ &\quad - \delta_{i,-k} (-1)^{p(l)(p(k)+1)} \mathsf{E}_{j,-l} - \delta_{-j,l} (-1)^{p(j)(p(i)+1)} \mathsf{E}_{-i,k}. \end{aligned} \tag{1}$$

Throughout the paper $\mathfrak{h}$ will be the Lie subsuperalgebra of $\mathfrak{p}_n$ with basis $\{k_1, \dots, k_n\}$, where $k_i := \mathsf{E}_{ii}$ for $1 \le i \le n$. Note that $\mathfrak{h}$ is purely even, and is also a self-normalizing nilpotent subsuperalgebra of $\mathfrak{p}_n$, hence a Cartan subsuperalgebra of $\mathfrak{p}_n$. By $\{\epsilon_1, \dots, \epsilon_n\}$ we denote the basis of $\mathfrak{h}^*$ dual to $\{k_1, \dots, k_n\}$.

Set $I := \{1, \dots, n-1\}$. The root system Δ of $\mathfrak{p}_n$ relative to $\mathfrak{h}$ consists of the roots $\epsilon_i - \epsilon_j$ (for $i \ne j$), $\epsilon_i + \epsilon_j$ (for $i < j$), and $-\epsilon_i - \epsilon_j$ (for $i \le j$). Let $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $\beta_i = 2\epsilon_i$, and $\gamma_i = \epsilon_i + \epsilon_{i+1}$. Set

$$e_i := \mathsf{E}_{-i-1,-i}, \qquad e_{\bar{i}} := \mathsf{E}_{i+1,-i}, \qquad F_{\bar{j}} := \mathsf{E}_{-j,j}.$$
$$f_i := \mathsf{E}_{i+1,i}, \qquad f_{\bar{i}} := \mathsf{E}_{-i-1,i},$$

for $i \in I$ and $j \in I \cup \{n\}$. The root spaces of α_i, $-\alpha_i$, γ, $-\gamma_i$, and $-\beta_i$ are spanned, respectively, by $e_i, f_i, e_{\bar{i}}, f_{\bar{i}}$, and $F_{\bar{i}}$. Note that $\beta_i \notin \Delta$.

Using the root space decomposition $\mathfrak{p}_n = \mathfrak{h} \oplus \left(\bigoplus_{\mu \in \Delta} (\mathfrak{p}_n)_\mu \right)$ we define the triangular decomposition $\mathfrak{p}_n = \mathfrak{p}_n^- \oplus \mathfrak{h} \oplus \mathfrak{p}_n^+$ as follows: $\mathfrak{p}_n^-$ is spanned by $\{f_i, f_{\bar{i}}, F_{\bar{j}} \mid i \in I, j \in I \cup \{n\}\}$ and $\mathfrak{p}_n^+$ is spanned by $\{e_i, e_{\bar{i}} \mid i \in I\}$. Alternatively, $\Delta = \Delta_+ \sqcup \Delta_-$, where

$$\Delta_+ = \Delta(\mathfrak{p}_n^+) = \{\alpha_i, \gamma_i \mid i \in I\}, \quad \Delta_- = \Delta(\mathfrak{p}_n^-) = \{-\alpha_i, -\gamma_i, -\beta_j \mid i \in I, j \in I \cup \{n\}\}.$$

In this paper, all highest weight modules of $\mathfrak{p}_n$ will be relative to the Borel subalgebra $\mathfrak{b}_n = \mathfrak{h} \oplus \mathfrak{p}_n^+$.

The cone of positive roots will be denoted by $Q_+ := \sum_{i=1}^{n-1} \mathbb{Z}_{\ge 0} \alpha_i + \sum_{i=1}^{n-1} \mathbb{Z}_{\ge 0} \gamma_i$ and $Q_- := -\sum_{i=1}^{n-1} \mathbb{Z}_{\ge 0} \alpha_i - \sum_{i=1}^{n-1} \mathbb{Z}_{\ge 0} \gamma_i - \sum_{i=1}^{n} \mathbb{Z}_{\ge 0} \beta_i$ denotes the cone of negative roots. Set $Q = Q_+ + Q_-$.

We will also denote $P \coloneqq \bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i$ to be the weight lattice of $\mathfrak{p}_n$, and denote $P^\vee \coloneqq \bigoplus_{i=1}^{n} \mathbb{Z}k_i$ to be the coweight lattice.

We next give a presentation of $\mathfrak{p}_n$ (hence of $\mathfrak{U}\mathfrak{p}_n$) in terms of generators and relations. This presentation will be used to define $\mathfrak{U}_q\mathfrak{p}_n$ in terms of Drinfeld-Jimbo generators and relations.

Proposition 1.1 ([DKM]). *The complex Lie superalgebra $\mathfrak{p}_n$ is generated by the elements e_i, $e_{\bar{i}}$, f_i, $f_{\bar{i}}$ ($i \in I$), $\mathfrak{h}$ and $F_{\bar{j}}$ ($j \in I \cup \{n\}$) subject to following defining relations (for $h \in \mathfrak{h}$):*

$$\begin{aligned}
&[\mathfrak{h},\mathfrak{h}] = 0 \\
&[h, e_i] = \alpha_i(h)e_i \\
&[h, f_i] = -\alpha_i(h)f_i \\
&[h, e_{\bar{i}}] = \gamma_i(h)e_{\bar{i}} \\
&[h, f_{\bar{i}}] = -\gamma_i(h)f_{\bar{i}} \\
&[h, F_{\bar{i}}] = -\beta_i(h)F_{\bar{i}} \\
&[e_i, e_j] = [f_i, f_j] = 0 \textit{ for } |i-j| \neq 1 \\
&[e_i, f_j] = -\delta_{ij}(k_i - k_{i+1}) \\
&[e_{\bar{i}}, f_{\bar{i}}] = -(k_i - k_{i+1}) \\
&[f_{\bar{i}}, e_{\bar{j}}] = 0 \textit{ if } |i-j| > 1 \\
&[f_{\overline{i+1}}, e_{\bar{i}}] = [e_{i+1}, e_i] \\
&[f_{\bar{i}}, e_{\overline{i+1}}] = [f_{i+1}, f_i] \\
&[e_{\overline{i+1}}, e_i] = [f_{i+1}, e_{\bar{i}}] \\
&[f_{\overline{i+1}}, f_i] = [e_{i+1}, f_{\bar{i}}]
\end{aligned}
\qquad
\begin{aligned}
&[f_{\bar{i}}, f_i] = F_{\bar{i}} \\
&[e_i, f_{\bar{i}}] = F_{\overline{i+1}} \\
&[e_{\bar{i}}, e_{\bar{j}}] = [f_{\bar{i}}, f_{\bar{j}}] = 0 \textit{ for } i, j \in I \\
&[f_i, e_{\bar{j}}] = 0 \textit{ if } i \neq j+1 \\
&[e_{\bar{i}}, e_j] = 0 \textit{ if } i \neq j+1 \\
&[e_i, f_{\bar{j}}] = 0 \textit{ if } i \neq j, j+1 \\
&[f_{\bar{i}}, f_j] = 0 \textit{ if } i \neq j, j+1 \\
&[F_{\bar{j}}, e_i] = -\beta_i(k_j)f_{\bar{i}} \\
&[F_{\bar{j}}, f_i] = \beta_{i+1}(k_j)f_{\bar{i}} \\
&[e_i, [e_i, e_{i\pm1}]] = 0 \\
&[f_i, [f_i, f_{i\pm1}]] = 0 \\
&[e_{\overline{i+1}}, [e_{i+1}, e_i]] = e_{\bar{i}}
\end{aligned}$$

Remark 1.2. We note that we use a slightly different presentation of $\mathfrak{p}_n$ in terms of generators and relations than the one used in Definition 3.1.1 in [DKM]. To define an isomorphism between the two presentations we proceed as follows. A homomorphism from the presentation in Proposition 1.1 to the one in Definition 3.1.1 in [DKM] can be defined by the following maps:

$$k_i \longmapsto -H_i, \qquad e_i \longmapsto -F_i, \qquad f_i \longmapsto E_i,$$
$$f_{\bar{i}} \longmapsto B_i, \qquad e_{\bar{i}} \longmapsto -C_i, \qquad F_{\bar{1}} \longmapsto -\frac{1}{2}B_{1,1}.$$

These maps indeed define a homomorphism because $F_{\bar{j}} = [e_{j-1}, f_{\overline{j-1}}] = [f_{\bar{j}}, f_j]$ and all relations listed in Proposition 1.1 follow from the relations in Definition 3.1.1 (see for example Lemma 3.2.1 in [DKM]). The details are left to the reader. To define a reverse homomorphism is easier. We note that neither of the sets of generators is minimal, but the larger set of generators used in this paper will serve better our purpose.

The following are relations of $\mathfrak{p}_n$ that can be obtained from the relations in Proposition 1.1.

Lemma 1.3. *The following relations hold in $\mathfrak{p}_n$:*

(a) $[F_{\bar{j}}, e_{\bar{i}}] = \begin{cases} 2f_i & \textit{if } j = i \\ 2e_i & \textit{if } j = i+1 \\ 0 & \textit{otherwise} \end{cases},$

(b) $[F_{\bar{j}}, f_{\bar{i}}] = 0,$

(c) $[e_i, [e_i, e_{\overline{i\pm1}}]] = 0,$

(d) $[f_i, [f_i, f_{\overline{i\pm1}}]] = 0,$

(e) $[F_{\bar{i}}, F_{\bar{j}}] = 0$ *for* $i, j \in I \cup \{n\}$.

Proof. We will prove (a) and (c). The remaining parts can be deduced similarly.

First, we prove (a) for $j = n$. For every i we have

$$[F_{\bar{n}}, e_{\bar{i}}] = [[e_{n-1}, f_{\overline{n-1}}], e_{\bar{i}}] = [e_{\bar{i}}, [e_{n-1}, f_{\overline{n-1}}]] = [e_{n-1}, [f_{\overline{n-1}}, e_{\bar{i}}]] + [f_{\overline{n-1}}, [e_{\bar{i}}, e_{n-1}]] = [e_{n-1}, [f_{\overline{n-1}}, e_{\bar{i}}]].$$

If $i = n-2$, then we have that $[e_{n-1}, [f_{\overline{n-1}}, e_{\overline{n-2}}]] = [e_{n-1}, [e_{n-1}, e_{n-2}]] = 0$. If $i = n-1$, then we have that

$$[e_{n-1}, [f_{\overline{n-1}}, e_{\overline{n-1}}]] = [e_{n-1}, -k_{n-1} + k_n] = -[e_{n-1}, k_{n-1}] + [e_{n-1}, k_n] = 2e_{n-1}.$$

Otherwise, we have that $[F_{\bar{n}}, e_{\bar{i}}] = 0$.

Next we prove (a) for $j < n$. Using the relations in Proposition 1.1, we have that:

$$\begin{aligned}
[F_{\bar{j}}, e_{\bar{i}}] &= [[f_{\bar{j}}, f_j], e_{\bar{i}}] \\
&= [e_{\bar{i}}, [f_{\bar{j}}, f_j]] \\
&= [f_{\bar{j}}, [f_j, e_{\bar{i}}]] - [f_j, [e_{\bar{i}}, f_{\bar{j}}]].
\end{aligned}$$

Note that $[F_{\bar{j}}, e_{\bar{i}}] = 0$ from above, unless $|i-j| \le 2$. So, we need to check the three subcases $i - j = 0, 1, -1$.

If $j = i$, then

$$\begin{aligned}
[F_{\bar{i}}, e_{\bar{i}}] &= [f_{\bar{i}}, [f_i, e_{\bar{i}}]] - [f_i, [e_{\bar{i}}, f_{\bar{i}}]] \\
&= -[f_i, -k_i + k_{i+1}] \\
&= 2f_i.
\end{aligned}$$

If $i = j+1$, then

$$\begin{aligned}
[F_{\bar{j}}, e_{\overline{j+1}}] &= [f_{\bar{j}}, [f_j, e_{\overline{j+1}}]] - [f_j, [e_{\overline{j+1}}, f_{\bar{j}}]] \\
&= -[f_j, [f_{j+1}, f_j]] \\
&= [f_j, [f_j, f_{j+1}]] + [f_j, [f_{j+1}, f_j]] \\
&= 0.
\end{aligned}$$

If $j = i+1$, then

$$\begin{aligned}
[F_{\overline{i+1}}, e_{\bar{i}}] &= [f_{\overline{i+1}}, [f_{i+1}, e_{\bar{i}}]] - [f_{i+1}, [e_{\bar{i}}, f_{\overline{i+1}}]] \\
&= [f_{\overline{i+1}}, [e_{\overline{i+1}}, e_i]] - [f_{i+1}, [e_{i+1}, e_i]] \\
&= [e_{\overline{i+1}}, [e_i, f_{\overline{i+1}}]] - [e_i, [f_{\overline{i+1}}, e_{\overline{i+1}}]] + [e_{i+1}, [e_i, f_{i+1}]] + [e_i, [f_{i+1}, e_{i+1}]] \\
&= -[e_i, -k_{i+1} + k_{i+2}] - [e_i, -k_{i+1} + k_{i+2}] \\
&= 2e_i.
\end{aligned}$$

Now, we prove (c). Note that $[e_i, e_{\overline{i-1}}] = 0$ for all $2 \le i \le n$, so $[e_i, [e_i, e_{\overline{i-1}}]] = 0$. Also,

$$\begin{aligned}
[e_i, [e_i, e_{\overline{i+1}}]] &= [e_i, [e_{\bar{i}}, f_{i+1}]] \\
&= [e_{\bar{i}}, [f_{i+1}, e_i]] + [f_{i+1}, [e_i, e_{\bar{i}}]] \\
&= 0.
\end{aligned}$$

□

2 Quantized enveloping superalgebra $\mathfrak{U}_q\mathfrak{p}_n$

Let $\mathbb{C}(q)$ be the field of rational functions in the variable q, and let $\mathbb{C}_q(n|n) = \mathbb{C}(q) \otimes_{\mathbb{C}} \mathbb{C}(n|n)$. Definition 3.6 from [AGG] gives that $\mathfrak{U}_q\mathfrak{p}_n$ is defined to be the associative superalgebra over $\mathbb{C}(q)$ generated by elements t_{ij}, t_{ii}^{-1} with $1 \leq |i| \leq |j| \leq n$ and $i, j \in \{\pm 1, \ldots, \pm n\}$, such that $t_{ii} = t_{-i,-i}$, $t_{-i,i} = 0$ if $i \geq 0$, $t_{ij} = 0$ if $|i| > |j|$, and the following relation is satisfied:

$$\begin{aligned}
&(-1)^{(p(i)+p(j))(p(k)+p(l))} t_{ij}t_{kl} - t_{kl}t_{ij} + \theta(i,j,k)\big(\delta_{|j|<|l|} - \delta_{|k|<|i|}\big)\epsilon t_{il}t_{kj} \\
&\quad + (-1)^{(p(i)+p(j))(p(k)+p(l))}\big(\delta_{j>0}(q-1) + \delta_{j<0}(q^{-1}-1)\big)\big(\delta_{jl} + \delta_{j,-l}\big)t_{ij}t_{kl} \\
&\qquad - \big(\delta_{i>0}(q-1) + \delta_{i<0}(q^{-1}-1)\big)\big(\delta_{ik} + \delta_{i,-k}\big)t_{kl}t_{ij} \\
&\qquad\quad + \theta(i,j,k)\delta_{j>0}\delta_{j,-l}\epsilon t_{i,-j}t_{k,-l} - (-1)^{p(j)}\delta_{i<0}\delta_{i,-k}\epsilon t_{-k,l}t_{-i,j} \\
&\quad + (-1)^{p(j)(p(i)+1)}\epsilon \sum_{-n\leq a\leq n} \big((-1)^{p(i)p(a)}\theta(i,j,k)\delta_{j,-l}\delta_{|a|<|l|}t_{i,-a}t_{ka} + (-1)^{p(-j)p(a)}\delta_{i,-k}\delta_{|k|<|a|}t_{al}t_{-a,j}\big) \\
&\quad = 0,
\end{aligned} \tag{2}$$

where $\epsilon = q - q^{-1}$ and $\theta(i,j,k) = \mathrm{sgn}(\mathrm{sgn}(i) + \mathrm{sgn}(j) + \mathrm{sgn}(k))$.

Now, let

$$\begin{aligned}
q^{k_i} &:= t_{ii}, \qquad e_i := \frac{-1}{q-q^{-1}}t_{-i,-i-1}, \qquad f_{\bar{i}} := \frac{-1}{q-q^{-1}}t_{i,-i-1}, \\
f_i &:= \frac{1}{q-q^{-1}}t_{i,i+1}, \qquad e_{\bar{i}} := \frac{1}{q-q^{-1}}t_{-i,i+1}, \qquad F_{\bar{i}} := \frac{-2}{q-q^{-1}}t_{i,-i}.
\end{aligned} \tag{3}$$

Using (3), we have the following relations between the two sets of generators of $\mathfrak{U}_q\mathfrak{p}_n$

$$\begin{aligned}
t_{-i,-i-j} &= -(q-q^{-1})q^{-\sum_{h=1}^{j-1}k_{i+h}} \prod_{h=1}^{j-1} \mathrm{ad}\ e_{i+h}(e_i), \\
t_{-i,i+j} &= (q-q^{-1})q^{-\sum_{h=1}^{j-1}k_{i+h}} \prod_{h=1}^{j-1} \mathrm{ad}\ f_{i+h}(e_{\bar{i}}), \\
t_{i,-i-j} &= -(q-q^{-1})q^{-\sum_{h=1}^{j-1}k_{i+h}} \prod_{h=1}^{j-1} \mathrm{ad}\ e_{i+h}(f_{\bar{i}}), \\
t_{i,i+j} &= (q-q^{-1})q^{-\sum_{h=1}^{j-1}k_{i+h}} \prod_{h=1}^{j-1} \mathrm{ad}\ f_{i+h}(f_i),
\end{aligned} \tag{4}$$

where $\mathrm{ad}\ a_i(a_j) := [a_i, a_j]$, $\prod_{h=1}^{j} \mathrm{ad}\ a_{i+h}(a_i) := \mathrm{ad}\ a_{i+j}\mathrm{ad}\ a_{i+j-1}\ldots\mathrm{ad}\ a_{i+1}(a_i)$, and $\prod_{h=1}^{0} \mathrm{ad}\ a_{i+h}(a_i) := a_i$, for $a_i = e_i, e_{\bar{i}}, f_i, f_{\bar{i}}$. From (4), one can obtain the following relations

$$\begin{aligned}
t_{i,i+j} &= q^{-k_{i+j-1}}(f_{i+j-1}t_{i,i+j-1} - t_{i,i+j-1}f_{i+j-1}), \\
t_{-i,i+j} &= q^{-k_{i+j-1}}(f_{i+j-1}t_{-i,i+j-1} - t_{-i,i+j-1}f_{i+j-1}), \\
t_{i,-i-j} &= q^{-k_{i+j-1}}(e_{i+j-1}t_{i,-i-j+1} - t_{i,-i-j+1}e_{i+j-1}), \\
t_{-i,-i-j} &= q^{-k_{i+j-1}}(e_{i+j-1}t_{-i,-i-j+1} - t_{-i,-i-j+1}e_{i+j-1}).
\end{aligned} \tag{5}$$

Equivalently, the above relations can be written as follows

$$\begin{aligned}
t_{ij} &= -q^{-k_{i+1}}(f_i t_{i+1,j} - t_{i+1,j}f_i), \\
t_{-i,j} &= q^{-k_{i+1}}(e_i t_{i+1,j} - t_{i+1,j}e_i),
\end{aligned} \tag{6}$$

where $i > 0$.

The relations (5) (respectively, (6)) can serve as an alternative way to define the generators of $\mathfrak{U}_q\mathfrak{p}_n$ inductively. These relations also allows us to obtain the following relations.

Lemma 2.1. *The following relations hold in $\mathfrak{U}_q\mathfrak{p}_n$ for all $i \in I$.*

(a) $e_if_i - f_ie_i = e_{\bar{i}}f_{\bar{i}} + f_{\bar{i}}e_{\bar{i}}$,

(b) $\dfrac{2}{1+q^2}f_{\overline{i+1}}f_{i+1}f_i - f_{\overline{i+1}}f_if_{i+1} - f_{i+1}f_if_{\overline{i+1}} + q^2f_if_{i+1}f_{\overline{i+1}} = q^2q^{2k_{i+1}}f_{\bar{i}} - \dfrac{1-q^2}{1+q^2}f_{i+1}f_{\overline{i+1}}f_i$,

(c) $f_ie_i = e_if_i + q^2\dfrac{q^{2k_i} - q^{k_{i+1}}}{q^2-1} + (q^2-1)e_{\bar{i}}f_{\bar{i}}$.

Our first main result is the following presentation of $\mathfrak{U}_q\mathfrak{p}_n$.

Proposition 2.2. *The quantum superalgebra $\mathfrak{U}_q\mathfrak{p}_n$ is isomorphic to the unital associative superalgebra over $\mathbb{C}(q)$ generated by the even elements q^h for $h \in P^\vee$, e_i, f_i for $i \in I$, and the odd elements $e_{\bar{i}}, f_{\bar{i}}$, for $i \in I$, $F_{\bar{i}}$ for $i \in I \cup \{n\}$, that satisfy the following relations*

$$
\begin{aligned}
&q^0 = 1,\ \ q^{h_1+h_2} = q^{h_1}q^{h_2} \quad \textit{for } h_1, h_2 \in P^\vee,\\
&q^he_i = q^{\alpha_i(h)}e_iq^h,\ \ q^hf_i = q^{-\alpha_i(h)}f_iq^h \quad \textit{for } h \in P^\vee,\\
&q^he_{\bar{i}} = q^{\gamma_i(h)}e_{\bar{i}}q^h,\ \ q^hf_{\bar{i}} = q^{-\gamma_i(h)},f_{\bar{i}}q^h,\ \ q^hF_{\bar{i}} = q^{-\beta_i(h)}F_{\bar{i}}q^h \quad \textit{for } h \in P^\vee,
\end{aligned}
$$

$$
\begin{aligned}
&e_ie_j - e_je_i = 0,\ \ f_if_j - f_jf_i = 0,\ \ f_{\bar{i}}f_{\bar{j}} + f_{\bar{j}}f_{\bar{i}} = 0 \quad \textit{if } |i-j| > 1,\\
&e_{\bar{i}}e_{\bar{j}} + e_{\bar{j}}e_{\bar{i}} = 0,\ \ F_{\bar{i}}F_{\bar{j}} + F_{\bar{j}}F_{\bar{i}} = 0 \quad \textit{if } |i-j| > 0,\\
&e_if_j - f_je_i = 0 \quad \textit{if } j \neq i, i+1,\\
&e_if_{\bar{j}} - f_{\bar{j}}e_i = 0,\ \ f_if_{\bar{j}} - f_{\bar{j}}f_i = 0,\ \ e_{\bar{i}}f_{\bar{j}} + f_{\bar{j}}e_{\bar{i}} = 0 \quad \textit{if } |i-j| > 1,\\
&e_ie_{\bar{j}} - e_{\bar{j}}e_i = 0,\ \ f_je_{\bar{i}} - e_{\bar{i}}f_j = 0 \quad \textit{if } j \neq i+1,
\end{aligned}
$$

$$
\begin{aligned}
&F_{\bar{i}}e_j - e_jF_{\bar{i}} = 0,\ \ F_{\bar{i}}f_j - f_jF_{\bar{i}} = 0 \quad \textit{if } i \neq j, j+1,\\
&F_{\bar{i}}e_{\bar{j}} + e_{\bar{j}}F_{\bar{i}} = 0,\ \ F_{\bar{i}}f_{\bar{j}} + f_{\bar{j}}F_{\bar{i}} = 0 \quad \textit{if } i \neq j, j+1,\\
&e_{\bar{i}}^2 = 0,\ \ f_{\bar{i}}^2 = 0,\ \ F_{\bar{i}}^2 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&e_{i+1}e_i - e_ie_{i+1} = e_{\bar{i}}f_{\overline{i+1}} + f_{\overline{i+1}}e_{\bar{i}},\ \ f_{i+1}f_i - f_if_{i+1} = f_{\bar{i}}e_{\overline{i+1}} + e_{\overline{i+1}}f_{\bar{i}},\\
&e_{\overline{i+1}}e_i - e_ie_{\overline{i+1}} = f_{i+1}e_{\bar{i}} - e_{\bar{i}}f_{i+1},\ \ f_{\overline{i+1}}f_i - f_if_{\overline{i+1}} = e_{i+1}f_{\bar{i}} - f_{\bar{i}}e_{i+1},\\
&e_if_i - f_ie_i = -\frac{q^{2k_i} - q^{2k_{i+1}}}{q^2-1} + \frac{q^2-1}{q^2}f_{\bar{i}}e_{\bar{i}},\\
&f_{\bar{i}}e_{\bar{i}} + q^2e_{\bar{i}}f_{\bar{i}} = -\frac{q^2}{q^2-1}(q^{2k_i} - q^{2k_{i+1}}),\\
&qe_if_{\bar{i}} - q^{-1}f_{\bar{i}}e_i = \frac{(1+q^2)}{2}q^{k_{i+1}}F_{\overline{i+1}} = q^{-1}f_{\overline{i+1}}f_{i+1} - qf_{i+1}f_{\overline{i+1}},
\end{aligned}
$$

$$
\begin{aligned}
&qF_{\overline{i+1}}e_i - e_iF_{\overline{i+1}} = 0,\ \ qF_{\bar{i}}f_i - f_iF_{\bar{i}} = 0,\\
&F_{\bar{i}}e_i - qe_iF_{\bar{i}} = -2f_{\bar{i}}q^{k_i},\ \ q^{-1}F_{\overline{i+1}}f_i - f_iF_{\overline{i+1}} = 2q^{k_{i+1}}f_{\bar{i}},
\end{aligned}
$$

$$F_{\bar{i}}e_{\bar{i}} + qe_{\bar{i}}F_{\bar{i}} = 2f_i q^{k_i}, \quad F_{\bar{i}}f_{\bar{i}} + q^{-1}f_{\bar{i}}F_{\bar{i}} = 0,$$
$$F_{\overline{i+1}}e_{\bar{i}} + qe_{\bar{i}}F_{\overline{i+1}} = 2e_i q^{k_{i+1}}, \quad F_{\overline{i+1}}f_{\bar{i}} + q^{-1}f_{\bar{i}}F_{\overline{i+1}} = 0,$$

$$q^{-1}e_i^2 e_{i+1} - (q+q^{-1})e_i e_{i+1} e_i + q e_{i+1} e_i^2 = 0,$$
$$q e_{i+2}^2 e_i - (q+q^{-1})e_{i+1} e_i e_{i+1} + q^{-1} e_i e_{i+1}^2 = 0,$$
$$q f_i^2 f_{i+1} - (q+q^{-1})f_i f_{i+1} f_i + q^{-1} f_{i+1} f_i^2 = 0,$$
$$q^{-1} f_{i+1}^2 f_i - (q+q^{-1})f_{i+1} f_i f_{i+1} + q f_i f_{i+1}^2 = 0,$$
$$q^{-1}e_i^2 e_{\overline{i+1}} - (q+q^{-1})e_i e_{\overline{i+1}} e_i + q e_{\overline{i+1}} e_i^2 = 0,$$
$$q f_i^2 f_{\overline{i+1}} - (q+q^{-1})f_i f_{\overline{i+1}} f_i + q^{-1} f_{\overline{i+1}} f_i^2 = 0,$$
$$e_{i+1}e_i e_{\overline{i+1}} - e_i e_{i+1} e_{\overline{i+1}} - q^2 e_{\overline{i+1}} e_{i+1} e_i + q^2 e_{\overline{i+1}} e_i e_{i+1} = q^{2k_{i+1}} e_{\bar{i}},$$

$$2qq^{k_{i+1}}(f_{i+1}f_{\bar{i}} - f_{\bar{i}}f_{i+1}) = (1-q^{-2})F_{\overline{i+1}}(f_{i+1}f_i - f_i f_{i+1}),$$
$$-2qq^{k_{i+1}}(f_{\overline{i+1}}e_i - e_i f_{\overline{i+1}}) = (1-q^{-2})F_{\overline{i+1}}(e_{i+1}e_i - e_i e_{i+1}),$$
$$-2qq^{k_{i+1}}(f_{\overline{i+1}}f_{\bar{i}} + f_{\bar{i}}f_{\overline{i+1}}) = (1-q^{-2})F_{\overline{i+1}}(f_{\overline{i+1}}f_i - f_i f_{\overline{i+1}}),$$
$$2qq^{k_{i+1}}(f_{i+1}e_i - e_i f_{i+1}) = (1-q^{-2})F_{\overline{i+1}}(e_{\overline{i+1}}e_i - e_i e_{\overline{i+1}}),$$

Proof. Let U be the unital associative superalgebra over $\mathbb{C}(q)$ generated by the elements $e_i, f_i, e_{\bar{i}}, f_{\bar{i}}$ for $i \in I$, $F_{\bar{i}}$ for $i \in I \cup \{n\}$, and q^h for $h \in P^\vee$ with defining relations given in the statement of the proposition above.

We first note that using (3) and particular choices for i, j, k, l in (2) one can establish on a case-by-case basis all relations in the proposition. Hence, we have an associative superalgebra homomorphism $\psi : U \to \mathfrak{U}_q\mathfrak{p}_n$. The relations in (4) immediately show that ψ is surjective.

It remains to show that ψ is injective. For this, we prove that (2) is obtained from the relations in the statement of the proposition by considering the following 26 cases:

1. $|i| = |j| < |k| < |l|$
2. $|k| < |i| = |j| < |l|$
3. $|k| < |l| < |i| = |j|$
4. $|k| = |l| < |i| < |j|$
5. $|i| < |k| = |l| < |j|$
6. $|i| < |j| < |k| = |l|$
7. $|i| = |k| < |j| < |l|$
8. $|i| = |k| < |l| < |j|$
9. $|i| < |k| < |l| = |j|$
10. $|k| < |i| < |l| = |j|$
11. $|k| < |l| = |i| < |j|$
12. $|i| < |j| = |k| < |l|$
13. $|i| < |j| < |k| < |l|$
14. $|i| < |k| < |j| < |l|$
15. $|i| < |k| < |l| < |j|$
16. $|k| < |i| < |j| < |l|$
17. $|k| < |i| < |l| < |j|$
18. $|k| < |l| < |i| < |j|$
19. $|k| = |l| < |i| = |j|$
20. $|i| = |j| < |k| = |l|$
21. $|i| = |k| < |j| = |l|$
22. $|i| = |j| = |k| < |l|$
23. $|k| < |i| = |j| = |l|$
24. $|i| = |k| = |l| < |j|$
25. $|i| < |j| = |k| = |l|$
26. $|i| = |j| = |k| = |l|$

The verification in each case uses the relations (5) and (6) and appropriate induction. In fact, for some cases, we apply useful identities that follow from (5) and (6), see Lemma 2.1 below. For example, in Case 21 we use Lemma 2.1(c). For reader's convenience, we write detailed proofs for cases 2 and 25. The remaining cases are established using analogous reasoning.

Case 2. Suppose that $|k| < |i| = |j| < |l|$ in (2). We prove that

$$(-1)^{(p(i)+p(j))(p(k)+p(\ell))}t_{ij}t_{k\ell} - t_{k\ell}t_{ij} = 0$$

is obtained from some of the relations in the proposition applying induction on $|\ell| - |i|$ first, and then induction on $|i| - |k|$. We consider only the case of when $i = -j > 0$ and $k, \ell > 0$ as the other cases follow similarly.

We start with the base case of the first induction, i.e., $\ell = i+1$. For the base case of the second induction, we have $i = k+1$. Then

$$\begin{aligned}
t_{k+1,-k-1}t_{k,k+2} - t_{k,k+2}t_{k+1,-k-1} &= -\frac{(q-q^{-1})^2}{2}\left(F_{\overline{k+1}}q^{-k_{k+1}}(f_{k+1}f_k - f_kf_{k+1}) - q^{-k_{k+1}}(f_{k+1}f_k - f_kf_{k+1})F_{\overline{k+1}}\right)\\
&= -\frac{(q-q^{-1})^2}{2}q^{-k_{k+1}}[q^{-2}F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1}) - (f_{k+1}f_k - f_kf_{k+1})F_{\overline{k+1}}]\\
&= -\frac{(q-q^{-1})^2}{2}q^{-k_{k+1}}[F_{\overline{k+1}}f_{k+1}f_k - F_{\overline{k+1}}f_kf_{k+1} - f_{k+1}f_kF_{\overline{k+1}} + f_kf_{k+1}F_{\overline{k+1}}\\
&+ (q^{-2}-1)F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1})]\\
&= -\frac{(q-q^{-1})^2}{2}q^{-k_{k+1}}[q^{-1}f_{k+1}F_{\overline{k+1}}f_k - F_{\overline{k+1}}f_kf_{k+1} - f_{k+1}f_kF_{\overline{k+1}} + qf_kF_{\overline{k+1}}f_{k+1}\\
&+ (q^{-2}-1)F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1})]\\
&= -\frac{(q-q^{-1})^2}{2}q^{-k_{k+1}}[f_{k+1}(q^{-1}F_{\overline{k+1}}f_k - f_kF_{\overline{k+1}}) - q(q^{-1}F_{\overline{k+1}}f_k - f_kF_{\overline{k+1}})f_{k+1}\\
&+ (q^{-2}-1)F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1})]\\
&= -\frac{(q-q^{-1})^2}{2}q^{-k_{k+1}}[2f_{k+1}q^{k_{k+1}}f_{\overline{k}} - 2qq^{k_{k+1}}f_{\overline{k}}f_{k+1} + (q^{-2}-1)F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1})]\\
&= -\frac{(q-q^{-1})^2}{2}q^{-k_{k+1}}[2qq^{k_{k+1}}(f_{k+1}f_{\overline{k}} - f_{\overline{k}}f_{k+1}) + (q^{-2}-1)F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1})]\\
&= -\frac{(q-q^{-1})^2}{2}q^{-k_{k+1}}[(1-q^{-2})F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1}) + (q^{-2}-1)F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1})]\\
&= 0.
\end{aligned}$$

The induction step for the second induction ($i - k \geq 2$) is established as follows:

$$\begin{aligned}
t_{i,-i}t_{k,i+1} - t_{k,i+1}t_{i,-i} &= -\frac{q-q^{-1}}{2}[F_{\bar{i}}q^{-k_{k+1}}(f_kt_{k+1,i+1} - t_{k+1,i+1}f_k) - t_{k,i+1}F_{\bar{i}}]\\
&= -\frac{q-q^{-1}}{2}[q^{-k_{k+1}}F_{\bar{i}}(f_kt_{k+1,i+1} - t_{k+1,i+1}f_k) - t_{k,i+1}F_{\bar{i}}]\\
&= -\frac{q-q^{-1}}{2}[q^{-k_{k+1}}(f_kt_{k+1,i+1} - t_{k+1,i+1}f_k)F_{\bar{i}} - t_{k,i+1}F_{\bar{i}}]\\
&= -\frac{q-q^{-1}}{2}[t_{k,i+1}F_{\bar{i}} - t_{k,i+1}F_{\bar{i}}]\\
&= 0.
\end{aligned}$$

For the induction step of the first induction ($\ell - i \geq 2$) we proceed as follows:

$$\begin{aligned}
t_{i,-i}t_{k\ell} - t_{k\ell}t_{i,-i} &= -\frac{q-q^{-1}}{2}[F_{\bar{i}}q^{-k_{\ell-1}}(f_{\ell-1}t_{k,\ell-1} - t_{k,\ell-1}f_{\ell-1}) - q^{-k_{\ell-1}}(f_{\ell-1}t_{k,\ell-1} - t_{k,\ell-1}f_{\ell-1})F_{\bar{i}}]\\
&= -\frac{q-q^{-1}}{2}[q^{-k_{\ell-1}}F_{\bar{i}}(f_{\ell-1}t_{k,\ell-1} - t_{k,\ell-1}f_{\ell-1}) - q^{-k_{\ell-1}}(f_{\ell-1}t_{k,\ell-1} - t_{k,\ell-1}f_{\ell-1})F_{\bar{i}}]\\
&= -\frac{q-q^{-1}}{2}[q^{-k_{\ell-1}}(f_{\ell-1}t_{k,\ell-1} - t_{k,\ell-1}f_{\ell-1})F_{\bar{i}} - q^{-k_{\ell-1}}(f_{\ell-1}t_{k,\ell-1} - t_{k,\ell-1}f_{\ell-1})F_{\bar{i}}]\\
&= 0
\end{aligned}$$

Case 25. Suppose that $|i| < |j| = |k| = |l|$ in (2). We prove that

$$0 = (-1)^{(p(i)+p(j))(p(j)+p(k))}q^{\mathrm{sgn}(j)}t_{ij}t_{kj} - t_{kj}t_{ij}$$

for $j=\ell$, and

$$0=(-1)^{(p(i)+p(j))(p(j)+p(k))}q^{\mathrm{sgn}(j)}t_{ij}t_{k,-j}-t_{k,-j}t_{ij}+(-1)^{p(i)}\delta_{j>0}(q-q^{-1})t_{i,-j}t_{kj}$$

for $j=-\ell$, using some of the relations in the proposition. We proceed by induction on $|j|-|i|$ and consider only the case $i>0$ and $j=k>0$ as the other cases follow similarly.

For the base case $j=i+1$, the relations

$$qt_{i,i+1}t_{i+1,i+1}=t_{i+1,i+1}t_{i,i+1}$$

when $j=\ell$, and

$$qt_{i,i+1}t_{i+1,-i-1}-t_{i+1,-i-1}t_{i,i+1}=-(q-q^{-1})t_{i,-i-1}t_{i+1,i+1}$$

when $j=-\ell$, follow from the relations $q^{k_{i+1}}f_i=qf_iq^{k_{i+1}}$ and $f_iF_{\overline{i+1}}-q^{-1}F_{\overline{i+1}}f_i=-2q^{k_{i+1}}f_{\bar{i}}$. For the induction step ($j-i\geq 2$) we have:

$$\begin{aligned}
qt_{ij}t_{jj}&=qq^{-k_{i+1}}(f_it_{i+1,j}-t_{i+1,j}f_i)q^{k_j}\\
&=q^{k_j}q^{-k_{i+1}}(f_it_{i+1,j}-t_{i+1,j}f_i)\\
&=t_{jj}t_{ij}
\end{aligned}$$

for $j=\ell$, and

$$\begin{aligned}
qt_{ij}t_{j,-j}&=-q\frac{q-q^{-1}}{2}q^{-k_{i+1}}(f_it_{i+1,j}-t_{i+1,j}f_i)F_{\bar{j}}\\
&=-q\frac{q-q^{-1}}{2}q^{-k_{i+1}}f_it_{i+1,j}F_{\bar{j}}+q\frac{q-q^{-1}}{2}q^{-k_{i+1}}t_{i+1,j}f_iF_{\bar{j}}\\
&=-\frac{q-q^{-1}}{2}q^{-k_{i+1}}f_i(F_{\bar{j}}t_{i+1,j}+2t_{i+1,-j}q^{k_j})+\frac{q-q^{-1}}{2}q^{-k_{i+1}}(F_{\bar{j}}t_{i+1,j}+2t_{i+1,-j}q^{k_j})f_i\\
&=\frac{q-q^{-1}}{2}q^{-k_{i+1}}F_{\bar{j}}(f_it_{i+1,j}-t_{i+1,j}f_i)-(q-q^{-1})q^{-k_{i+1}}(f_it_{i+1,-j}-t_{i+1,-j}f_i)q^{k_j}\\
&=\frac{q-q^{-1}}{2}F_{\bar{j}}q^{-k_{i+1}}(f_it_{i+1,j}-t_{i+1,j}f_i)-(q-q^{-1})q^{-k_{i+1}}(f_it_{i+1,-j}-t_{i+1,-j}f_i)q^{k_j}\\
&=t_{j,-j}t_{ij}-(q-q^{-1})t_{i,-j}t_{jj}
\end{aligned}$$

for $j=-\ell$. $\square$

We define a standard grading onto $\mathfrak{U}_q\mathfrak{p}_n$, namely we let $\deg e_i=\alpha_i$, $\deg f_i=-\alpha_i$, $\deg q^h=0$, $\deg e_{\bar{i}}=\gamma_i$, $\deg f_{\bar{i}}=-\gamma_i$, and $\deg F_{\bar{i}}=-\beta_i$. With this grading, all of the defining relations of the quantum superalgebra $\mathfrak{U}_q\mathfrak{p}_n$ are homogeneous. Hence, we say that $\mathfrak{U}_q\mathfrak{p}_n$ have a Q-grading

$$\mathfrak{U}_q\mathfrak{p}_n=\bigoplus_{\alpha\in Q}(\mathfrak{U}_q)_\alpha,$$

where $(\mathfrak{U}_q)_\alpha=\left\{v\in\mathfrak{U}_q\mathfrak{p}_n\mid q^hvq^{-h}=q^{\alpha(h)}v \text{ for all } h\in P^\vee\right\}$. In what follows we write $\deg u=\alpha$ whenever $u\in(\mathfrak{U}_q)_\alpha$.

The comultiplication Δ of $\mathfrak{U}_q\mathfrak{p}_n$ is given in [AGG] by the formula

$$\Delta(t_{ij})=\sum_{k=-n}^{n}(-1)^{(p(i)+p(k))(p(k)+p(j))}t_{ik}\otimes t_{kj}.$$

Through direct computations, we can express the comultiplication Δ in terms of the new generators in Proposition 2.2. The details are left to the reader.

Lemma 2.3. *In terms of the generators $e_i, f_i, e_{\bar{i}}, f_{\bar{i}}$ for $i \in I$, $F_{\bar{i}}$ for $i \in I \cup \{n\}$, and q^h for $h \in P^\vee$, the following hold in $\mathfrak{U}_q\mathfrak{p}_n$:*

$$\begin{aligned}
\Delta(q^h) &= q^h \otimes q^h, \\
\Delta(e_i) &= q^{k_i} \otimes e_i + e_i \otimes q^{k_{i+1}} - \frac{q-q^{-1}}{2} e_{\bar{i}} \otimes F_{\overline{i+1}}, \\
\Delta(f_i) &= q^{k_i} \otimes f_i + f_i \otimes q^{k_{i+1}} + \frac{q-q^{-1}}{2} F_{\bar{i}} \otimes e_{\overline{i+1}}, \\
\Delta(e_{\bar{i}}) &= q^{k_i} \otimes e_{\bar{i}} + e_{\bar{i}} \otimes q^{k_{i+1}}, \\
\Delta(f_{\bar{i}}) &= q^{k_i} \otimes f_{\bar{i}} + f_{\bar{i}} \otimes q^{k_{i+1}} - \frac{q-q^{-1}}{2} F_{\bar{i}} \otimes e_i + \frac{q-q^{-1}}{2} f_i \otimes F_{\overline{i+1}}, \\
\Delta(F_{\bar{i}}) &= q^{k_i} \otimes F_{\bar{i}} + F_{\bar{i}} \otimes q^{k_i}.
\end{aligned}$$

Let $\mathfrak{U}_q^+$ (respectively $\mathfrak{U}_q^-$) be the subsuperalgebra of $\mathfrak{U}_q\mathfrak{p}_n$ generated by the elements e_i and $e_{\bar{i}}$ (respectively f_i, $f_{\bar{i}}$, and $F_{\bar{j}}$) for $i \in I$ (and $j \in I \cup \{n\}$). Also, let $\mathfrak{U}_q^0$ be the subsuperalgebra of $\mathfrak{U}_q\mathfrak{p}_n$ generated by q^h for $h \in P^\vee$. In order to establish the triangular decomposition of $\mathfrak{U}_q\mathfrak{p}_n$ we need the following lemma.

Lemma 2.4. *Let $\mathfrak{U}_q^{\geq 0}$ (respectively, $\mathfrak{U}_q^{\leq 0}$) be generated by $\mathfrak{U}_q^0$ and $\mathfrak{U}_q^+$ (respectively, $\mathfrak{U}_q^0$ and $\mathfrak{U}_q^-$). Then the following $\mathbb{C}(q)$-linear isomorphisms hold.*

$$\mathfrak{U}_q^{\geq 0} \cong \mathfrak{U}_q^0 \otimes \mathfrak{U}_q^+, \qquad \mathfrak{U}_q^{\leq 0} \cong \mathfrak{U}_q^- \otimes \mathfrak{U}_q^0.$$

Proof. We will prove the second isomorphism. Let $\{f_\zeta \mid \zeta \in \Omega\}$ be a basis of $\mathfrak{U}_q^-$ consisting of monomials in f_i's, $f_{\bar{i}}$'s, and $F_{\bar{j}}$'s ($1 \leq i \leq n-1$, $1 \leq j \leq n$), with Ω being an index set. Consider the map $\varphi : \mathfrak{U}_q^- \otimes \mathfrak{U}_q^0 \to \mathfrak{U}_q^{\leq 0}$ defined by $\varphi(f_\zeta \otimes q^h) = f_\zeta q^h$. The defining relations of $\mathfrak{U}_q\mathfrak{p}_n$ imply that $f_\zeta q^h$ span $\mathfrak{U}_q^{\leq 0}$, so φ is surjective. It remains to show that the set $\{f_\zeta q^h \mid \zeta \in \Omega, h \in P^\vee\}$ is linearly independent over $\mathbb{C}(q)$.

Suppose

$$\sum_{\substack{\zeta \in \Omega \\ h \in P^\vee}} C_{\zeta,h} f_\zeta q^h = 0,$$

for some $C_{\zeta,h} \in \mathbb{C}(q)$. Then

$$\sum_{\alpha \in Q_-} \left(\sum_{\substack{\deg f_\zeta = \alpha \\ h \in P^\vee}} C_{\zeta,h} f_\zeta q^h \right) = 0.$$

Write $\alpha = -\sum_{i=1}^{n-1}(m_i\alpha_i + n_i\gamma_i) - \sum_{i=1}^{n} r_i\beta_i$, for $m_i, n_i, r_i \in \mathbb{Z}_{\geq 0}$, and let $h_\alpha = \sum_{i=1}^{n-1}(m_i + n_i)k_{i+1} + \sum_{i=1}^{n} r_i k_i$ and $h'_\alpha = \sum_{i=1}^{n-1}(m_i + n_i)k_i + r_i k_i$. From $\mathfrak{U}_q\mathfrak{p}_n = \bigoplus_{\alpha \in Q} (\mathfrak{U}_q)_\alpha$, we have that, for each $\alpha \in Q_-$,

$$\sum_{\substack{\deg f_\zeta = \alpha \\ h \in P^\vee}} C_{\zeta,h} f_\zeta q^h = 0. \tag{7}$$

Since f_ζ is a monomial in f_i's, $f_{\bar{i}}$'s, and $F_{\bar{i}}$'s, we have

$$\Delta(f_\zeta) = f_\zeta \otimes q^{h_\alpha} + \ldots + q^{h'_\alpha} \otimes f_\zeta.$$

Hence, the degree $(\alpha, 0)$ term in the decomposition of $\Delta(f_\zeta)$ equals $f_\zeta \otimes q^{h_\alpha}$. Applying the comultiplication to (7) gives

$$\sum_{\substack{\deg f_\zeta = \alpha \\ h \in P^\vee}} C_{\zeta,h}(f_\zeta q^h \otimes q^{h+h_\alpha} + \ldots + q^{h+h'_\alpha} \otimes f_\zeta q^h) = 0.$$

Collecting the terms of degree $(\alpha, 0)$ gives that

$$\sum_{\substack{\deg f_\zeta = \alpha \\ h \in P^\vee}} C_{\zeta,h}(f_\zeta q^h \otimes q^{h+h_\alpha}) = 0.$$

Since for every α, the set $\{q^{h+h_\alpha} \mid h \in P^\vee\}$ is linearly independent, we have that, for all $h \in P^\vee$:

$$\sum_{\deg f_\zeta = \alpha} C_{\zeta,h} f_\zeta q^h = 0.$$

Due to the linear independence of f_ζ, we conclude that $C_{\zeta,h} = 0$ for all ζ, h, as desired. $\square$

Theorem 2.5. *There is a $\mathbb{C}(q)$-linear isomorphism*

$$\mathfrak{U}_q(\mathfrak{p}_n) \cong \mathfrak{U}_q^- \otimes \mathfrak{U}_q^0 \otimes \mathfrak{U}_q^+.$$

Proof. Let $\{f_\zeta \mid \zeta \in \Omega\}$, $\{q^h \mid h \in P^\vee\}$, and $\{e_\tau \mid \tau \in \Omega'\}$ be monomial bases of $\mathfrak{U}_q^-$, $\mathfrak{U}_q^0$, and $\mathfrak{U}_q^+$, respectively, where Ω is the index set as in the proof of Lemma 2.4, and Ω' is another index set. Using the defining relations of $\mathfrak{U}_q\mathfrak{p}_n$ in Proposition 2.2, we can express every monomial in $\mathfrak{U}_q(\mathfrak{p}_n)$ as a linear combination of monomials each of which has e_i and $e_{\bar{i}}$ on the right. By Lemma 2.4, the monomials $f_\zeta q^h e_\tau$ span $\mathfrak{U}_q\mathfrak{p}_n$. Hence, it remains to show that $f_\zeta q^h e_\tau$ are linearly independent over $\mathbb{C}(q)$.

Suppose

$$\sum_{\substack{\zeta \in \Omega, \tau \in \Omega' \\ h \in P^\vee}} C_{\zeta,h,\tau} f_\zeta q^h e_\tau = 0,$$

where $C_{\zeta,h,\tau}$ is some nonzero constant in $\mathbb{C}(q)$. Due to the Q-grading of $\mathfrak{U}_q\mathfrak{p}_n$, we have that, for all $\alpha \in Q$:

$$\sum_{\substack{\deg f_\zeta + \deg e_\tau = \alpha \\ h \in P^\vee}} C_{\zeta,h,\tau} f_\zeta q^h e_\tau = 0. \tag{8}$$

Define a partial ordering on $\mathfrak{h}^*$ by $\lambda \leq \mu$ if and only if $\lambda - \mu \in Q_-$ for $\lambda, \mu \in \mathfrak{h}^*$. We then choose $\gamma = \deg f_\zeta$ and $\beta = \deg e_\tau$, which are minimal and maximal, respectively, among those for which $\gamma + \beta = \alpha$

and $C_{\zeta,h,\tau} \neq 0$. If $\gamma = -\sum_{i=1}^{n-1}(m_i\alpha_i + n_i\gamma_i) - \sum_{i=1}^{n} r_i\beta_i$, set $h_\gamma = \sum_{i=1}^{n-1}(m_i + n_i)k_{i+1} + \sum_{i=1}^{n} r_i k_i$, and if $\beta = \sum_{i=1}^{n-1}(m_i'\alpha_i + n_i'\gamma_i)$, set $h_\beta = \sum_{i=1}^{n-1}(m_i'k_i + n_i'k_i)$, for $m_i, m_i', n_i, n_i', r_i \in \mathbb{Z}_{\geq 0}$.

The term of degree $(0, \beta)$ in $\Delta(e_\tau)$ is $q^{h_\beta} \otimes e_\tau$ and the term of degree $(\gamma, 0)$ in $\Delta(f_\zeta)$ is $f_\zeta \otimes q^{h_\gamma}$. Applying the comultiplication to the sum in (8), and looking at the terms of degree (γ, β), we have that

$$\sum_{\substack{\deg f_\zeta = \gamma \\ \deg e_\tau = \beta \\ h \in P^\vee}} C_{\zeta,h,\tau}(f_\zeta q^{h+h_\beta} \otimes q^{h+h_\gamma} e_\tau) = 0.$$

By Lemma 2.4, the elements $f_\zeta q^h$ are linearly independent for $\zeta \in \Omega, h \in P^\vee$. Thus, for all $h \in P^\vee$, we have that

$$\sum_{\deg e_\tau = \beta} C_{\zeta,h,\tau} q^{h+h_\gamma} e_\tau = 0.$$

Due to the linear independence of e_τ, we conclude that $C_{\zeta,h,\tau} = 0$, leading to contradiction. Therefore all coefficients in (8) are zero. □

3 Highest weight representation theory of $\mathfrak{U}_q\mathfrak{p}_n$

A $\mathfrak{U}_q\mathfrak{p}_n$-module V^q is called a *weight module* if it admits a weight space decomposition

$$V^q = \bigoplus_{\mu \in P} V^q_\mu$$

where $V^q_\mu = \{v \in V^q \mid q^h v = q^{\mu(h)} v \text{ for all } h \in P^\vee\}$ is the *μ-weight space*. We call $\mu \in P$ a *weight* of V^q if $V^q_\mu \neq 0$. A nonzero vector $v \in V^q_\mu$ is called a *weight vector* of weight μ. If $v \in V^q$ is a nonzero vector such that $\mathfrak{U}_q^+ v = 0$, then v is called a *maximal vector*.

The dimension of each weight space $\dim V^q_\mu$ is called the *weight multiplicity* of μ. If $\dim V^q_\mu < \infty$ for all $\mu \in P$, the *character* of V^q is defined by

$$\operatorname{ch} V^q = \sum_\mu \dim_{\mathbb{F}} V^q_\mu e^\mu$$

where $\mathbb{F} = \mathbb{C}(q)$ and e^μ are formal basis elements of the group algebra $\mathbb{F}[P]$ with multiplication defined by $e^\alpha e^\beta = e^{\alpha+\beta}$.

The above definitions and notions can be introduced in the same way for $\mathfrak{U}\mathfrak{p}_n$-modules over $\mathbb{F} = \mathbb{C}$.

Definition 3.1. *A weight $\mathfrak{U}_q\mathfrak{p}_n$-module V^q is called a* highest weight module *with highest weight $\lambda \in P$ if the following holds for some nonzero $v \in V^q$:*

(a) v is a maximal vector of V^q,

(b) $v \in V^q_\lambda$, and

(c) $V^q = \mathfrak{U}_q\mathfrak{p}_n v$.

This vector v*, which is unique up to a constant multiple, is called a* highest weight vector *of* V^q*.*

This definition, along with Theorem 2.5, shows that $V^q = \mathfrak{U}_q^- v$ for any highest weight module with highest weight vector v and highest weight λ.

Fix $\lambda \in P$ and define $J^q(\lambda)$ to be the left ideal of $\mathfrak{U}_q\mathfrak{p}_n$ generated by e_i, $e_{\bar{i}}$, and $q^h - q^{\lambda(h)}1$, for $i \in I$ and $h \in P^\vee$. Then $M^q(\lambda) = \mathfrak{U}_q\mathfrak{p}_n/J^q(\lambda)$ is the Verma module, which is a $\mathfrak{U}_q\mathfrak{p}_n$-module by left multiplication. Set $v = 1 + J^q(\lambda)$. Then $M^q(\lambda)$ is a highest weight module with highest weight λ and highest weight vector v. The proof of the following proposition is standard. See, for example, the proof of Proposition 3.2.2 in [HK], which uses the same arguments.

Proposition 3.2.

(a) $M^q(\lambda)$ *is a free* $\mathfrak{U}_q^-$*-module of rank 1, generated by the highest weight vector* $v = 1 + J^q(\lambda)$*.*

(b) Every highest weight $\mathfrak{U}_q\mathfrak{p}_n$*-module with highest weight* λ *is a homomorphic image of* $M^q(\lambda)$*.*

(c) The Verma module $M^q(\lambda)$ *has a unique maximal submodule.*

Let $N_q(\lambda)$ denote the unique maximal submodule of the Verma module $M^q(\lambda)$ from Proposition 3.2(c). Then the unique irreducible quotient

$$V^q(\lambda) = M^q(\lambda)/N_q(\lambda)$$

is the irreducible highest weight module over $\mathfrak{U}_q\mathfrak{p}_n$ with highest weight λ.

We note again that the definitions of highest weight module can be introduced in the same way for $\mathfrak{U}\mathfrak{p}_n$-modules over $\mathbb{F} = \mathbb{C}$. In the latter case we will use the notation $M(\lambda)$ and $V(\lambda)$ for the Verma module and its irreducible quotient, respectively. We denote by Λ^+ the set of $\mathfrak{p}_n$-dominant integral weights:

$$\Lambda^+ := \{\lambda_1\epsilon_1 + \ldots + \lambda_n\epsilon_n \in \mathfrak{h}^* \mid \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0},\ \forall i \in I\}.$$

The following proposition and theorem will be used to prove an important result concerning highest weight modules over $\mathfrak{U}_q\mathfrak{p}_n$.

Proposition 3.3. *Let* V *be a highest weight* $\mathfrak{p}_n$*-module with highest weight* $\lambda \in \Lambda^+$ *and highest weight vector* v *such that* $f_i^{\lambda(k_i)-\lambda(k_{i+1})+1}v = 0$ *for all* $i \in I$*. Then* V *is finite dimensional.*

Proof. The proof is of this is similar to that of Proposition 1.9 in [GJKK]. The main idea of the proof is that since $e_{\bar{i}}^2 = 0$ and $f_{\bar{i}}^2 = 0$, then, with the aid of the Poincaré-Birkhoff-Witt theorem, we show that $\mathfrak{U}_q(\mathfrak{p}_n)_0 v$, where $(\mathfrak{p}_n)_0$ is generated by $\{e_i, f_i, k_j \mid i \in I, j \in J\}$, is finite generated, using the fact that $f_i^{\lambda(k_i)-\lambda(k_{i+1})+1}v = 0$ and v is a highest weight vector. For details, see the proof of Proposition 1.9 in [GJKK]. ☐

Theorem 3.4 ([K2])**.** *For any weight* $\lambda \in \mathfrak{h}^*$*,* $V(\lambda)$ *is finite dimensional if and only if* $\lambda \in \Lambda^+$*.*

We recall some standard definitions from q-calculus. We set

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}.$$

We also define $[0]_q! := 1$, and $[n]_q! = [n]_q \cdot [n-1]_q \cdot \ldots \cdot [1]_q$. The *divided powers* of e_i and f_i are:

$$e_i^{(m)} := \frac{e_i^m}{[m]_q!}, \qquad f_i^{(m)} := \frac{f_i^m}{[m]_q!}$$

Lemma 3.5. *For all $i \in I$ and $m \in \mathbb{Z}_{\geq 0}$, we have*

(a)

$$e_i f_i^{(m)} = f_i^{(m)} e_i - f_i^{(m-1)} \frac{q^{-m+1} q^{2k_i} - q^{m-1} q^{2k_{i+1}}}{q^2 - 1} + (1 - q^{-2}) \left(q^{m-1} f_i^{(m-1)} f_{\bar{i}} + \frac{q^{2m-2}}{2} q^{k_i} F_{\bar{i}} f_i^{(m-2)} \right) e_{\bar{i}},$$

(b)

$$f_i e_i^{(m)} = e_i^{(m)} f_i + q^2 e_i^{(m-1)} \frac{q^{m-1} q^{2k_i} - q^{-m+1} q^{2k_{i+1}}}{q^2 - 1} + (q^2 - 1) \left(q^2 e_i^{(m-1)} e_{\bar{i}} f_{\bar{i}} - \frac{1}{2} e_i^{(m-2)} q^{k_{i+1}} e_{\bar{i}} F_{\overline{i+1}} \right)$$

(in the case $m = 1$ we assume that the terms involving $f_i^{(m-2)}$ and $e_i^{(m-2)}$ are zero).

Proof. We prove (a) by induction on m. The base case $m = 1$:

$$e_i f_i = f_i e_i - \frac{q^{2k_i} - q^{2k_{i+1}}}{q^2 - 1} + \frac{q^2 - 1}{q^2} f_{\bar{i}} e_{\bar{i}}$$

follows from Proposition 2.2. For the base case $m = 2$ we multiply the above relation on the right by f_i and obtain:

$$\begin{aligned}
e_i f_i^2 &= f_i e_i f_i - \frac{q^{2k_i} - q^{2k_{i+1}}}{q^2 - 1} f_i + \frac{q^2 - 1}{q^2} f_{\bar{i}} e_{\bar{i}} f_i \\
&= f_i^2 e_i - f_i \frac{q^{2k_i} - q^{2k_{i+1}}}{q^2 - 1} + (1 - q^{-2}) f_i f_{\bar{i}} e_{\bar{i}} - f_i \frac{q^{-2} q^{2k_i} - q^2 q^{2k_{i+1}}}{q^2 - 1} + (1 - q^{-2}) f_{\bar{i}} e_{\bar{i}} f_i \\
&= f_i^2 e_i - f_i \frac{(1 + q^{-2}) q^{2k_i} - (1 + q^2) q^{2k_{i+1}}}{q^2 - 1} + (1 - q^{-2}) \left(f_i f_{\bar{i}} + f_{\bar{i}} f_i \right) e_{\bar{i}} \\
&= f_i^2 e_i - f_i \frac{(1 + q^{-2}) q^{2k_i} - (1 + q^2) q^{2k_{i+1}}}{q^2 - 1} + (1 - q^{-2}) \left((1 + q^2) f_i f_{\bar{i}} + \frac{1 + q^2}{2} q q^{k_i} F_{\bar{i}} \right) e_{\bar{i}}.
\end{aligned}$$

Dividing both sides by $[2]_q$ leads to the desired result for $m = 2$.

Now, suppose that (a) is true for some m. Then we have that

$$\begin{aligned}
e_i f_i^{(m)} f_i &= f_i^{(m)} e_i f_i - f_i^{(m-1)} \frac{q^{-m+1} q^{2k_i} - q^{m-1} q^{2k_{i+1}}}{q^2 - 1} f_i + (1 - q^{-2}) \left(q^{m-1} f_i^{(m-1)} f_{\bar{i}} + \frac{q^{2m-2}}{2} q^{k_i} F_{\bar{i}} f_i^{(m-2)} \right) e_{\bar{i}} f_i \\
&= f_i^{(m)} f_i e_i - f_i^{(m)} \frac{q^{2k_i} - q^{2k_{i+1}}}{q^2 - 1} + \frac{q^2 - 1}{q^2} f_i^{(m)} f_{\bar{i}} e_{\bar{i}} - f_i^{(m-1)} \frac{q^{-m+1} q^{2k_i} - q^{m-1} q^{2k_{i+1}}}{q^2 - 1} f_i \\
&+ (1 - q^{-2}) \left(q^{m-1} f_i^{(m-1)} f_{\bar{i}} f_i + \frac{q^{2m-2}}{2} q^{k_i} F_{\bar{i}} f_i^{(m-2)} f_i \right) e_{\bar{i}} \\
&= f_i^{(m)} f_i e_i - f_i^{(m)} \frac{q^{2k_i} - q^{2k_{i+1}}}{q^2 - 1} - f_i^{(m-1)} f_i \frac{q^{-m-1} q^{2k_i} - q^{m+1} q^{2k_{i+1}}}{q^2 - 1} \\
&+ (1 - q^{-2}) \left(f_i^{(m)} f_{\bar{i}} + q^{m-1} f_i^{(m-1)} f_{\bar{i}} f_i + \frac{q^{2m-2}}{2} q^{k_i} F_{\bar{i}} f_i^{(m-2)} f_i \right) e_{\bar{i}} \\
&= f_i^{(m)} f_i e_i - f_i^{(m)} \frac{q^{2k_i} - q^{2k_{i+1}}}{q^2 - 1} - f_i^{(m-1)} f_i \frac{q^{-m-1} q^{2k_i} - q^{m+1} q^{2k_{i+1}}}{q^2 - 1} \\
&+ (1 - q^{-2}) \left(f_i^{(m)} f_{\bar{i}} + q^{m+1} f_i^{(m-1)} f_i f_{\bar{i}} + q^m \frac{1 + q^2}{2} f_i^{(m-1)} q^{k_i} F_{\bar{i}} + \frac{q^{2m-2}}{2} q^{k_i} F_{\bar{i}} f_i^{(m-2)} f_i \right) e_{\bar{i}} \\
&= [m+1]_q f_i^{(m+1)} e_i - f_i^{(m)} \frac{(1 + q^{-m-1} [m]_q) q^{2k_i} - (1 + q^{m+1} [m]_q) q^{2k_{i+1}}}{q^2 - 1}
\end{aligned}$$

$$+ (1-q^{-2})\left((1+q^{m-1}[m]_q)f_i^{(m)}f_{\bar{i}} + q^{3m-2}\frac{1+q^2}{2}q^{k_i}F_{\bar{i}}f_i^{(m-1)} + \frac{q^{2m-2}[m-1]_q}{2}q^{k_i}F_{\bar{i}}f_i^{(m-1)}\right)e_{\bar{i}}.$$

Dividing both sides by $[m+1]_q$ completes the proof of (a).

The proof of (b) follows in a similar way, with the only main difference that we use the relation from Lemma 2.1(c). □

Proposition 3.6. *Let $\lambda \in \Lambda^+$ and $V^q(\lambda)$ be generated by a highest weight vector v. Then $f_i^{\lambda(k_i)-\lambda(k_{i+1})+1}v = 0$ for all $i \in I$.*

Proof. If $f_i^{\lambda(k_i)-\lambda(k_{i+1})+1}v \neq 0$ and $\mathfrak{U}_q^+ f_i^{\lambda(k_i)-\lambda(k_{i+1})+1}v = 0$, then $f_i^{\lambda(k_i)-\lambda(k_{i+1})+1}v$ generates a nontrivial proper submodule of $V^q(\lambda)$, which would be a contradiction to the fact that $V^q(\lambda)$ is irreducible. Therefore, it is enough to show that $\mathfrak{U}_q^+ f_i^{\lambda(k_i)-\lambda(k_{i+1})+1}v = 0$.

Note that for $j \neq i-1, i$, we have that

$$e_j f_i^{\lambda(k_i)-\lambda(k_{i+1})+1}v = f_i^{\lambda(k_i)-\lambda(k_{i+1})+1}e_j v = 0.$$

We next prove that $e_j f_i^{\lambda(k_i)-\lambda(k_{i+1})+1}v = 0$ for $j = i, i-1$. In the case $j = i$, we use Lemma 3.5(a) for $m = \lambda(k_i) - \lambda(k_{i+1}) + 1$:

$$e_i f_i^{(m)}v = \frac{q^{m-1+2\lambda(k_{i+1})} - q^{-m+1+2\lambda(k_i)}}{q^2-1}f_i^{(m-1)}v = 0.$$

For $j = i-1$ we apply induction on m. The base case $m = 1$ follows from

$$e_{i-1}f_i v = f_i e_{i-1}v - \frac{1-q^{-2}}{2q}q^{-k_i}F_{\bar{i}}(e_{\bar{i}}e_{i-1} - e_{i-1}e_{\bar{i}})v = 0.$$

Assume that $e_{i-1}f_i^m v = 0$. Then we have

$$\begin{aligned} e_{i-1}f_i^{m+1}v &= f_i e_{i-1}f_i^m v - \frac{1-q^{-2}}{2q}q^{-k_i}F_{\bar{i}}(e_{\bar{i}}e_{i-1} - e_{i-1}e_{\bar{i}})f_i^m v \\ &= f_i e_{i-1}f_i^m v - \frac{1-q^{-2}}{2q}q^{-k_i}F_{\bar{i}}e_{\bar{i}}e_{i-1}f_i^m v + \frac{1-q^{-2}}{2q}q^{-k_i}F_{\bar{i}}e_{i-1}f_i^m e_{\bar{i}}v \\ &= \left(f_i - \frac{1-q^{-2}}{2q}q^{-k_i}F_{\bar{i}}e_{\bar{i}}\right)e_{i-1}f_i^m v = 0. \end{aligned}$$

Therefore, $e_j f_i^{\lambda(k_i)-\lambda(k_{i+1})+1}v = 0$ for all $j \in I$.

Lastly, we prove that $e_{\bar{j}}f_i^{\lambda(k_i)-\lambda(k_{i+1})+1}v = 0$ for all $j \in I$. Note that for $j \neq i-1$,

$$e_{\bar{j}}f_i^{\lambda(k_i)-\lambda(k_{i+1})+1}v = f_i^{\lambda(k_i)-\lambda(k_{i+1})+1}e_{\bar{j}}v = 0.$$

For $j = i-1$, using that $e_{i-1}f_i^m v = 0$, we have

$$e_{\overline{i-1}}f_i^{m+1}v = f_i e_{\overline{i-1}}f_i^m v - e_{\bar{i}}e_{i-1}f_i^m v + e_{i-1}e_{\bar{i}}f_i^m v = f_i e_{\overline{i-1}}f_i^m v.$$

Thus we obtain $e_{\overline{i-1}}f_i^m v = 0$ by induction on m. Hence, $\mathfrak{U}_q^+ f_i^{\lambda(k_i)-\lambda(k_{i+1})+1}v = 0$ and the proof is complete. □

4 Classical Limits

Let $\mathbf{A}_1$ be the localization of $\mathbb{C}[q]$ at the ideal generated by $q-1$. Namely,

$$\mathbf{A}_1 = \{f(q) \in \mathbb{C}(q) \mid f \text{ is regular at } q=1\}.$$

For an integer $n \in \mathbb{Z}$, we define

$$[x;n]_y := \frac{xy^n - x^{-1}y^{-n}}{y-y^{-1}}, \qquad (x;n)_y := \frac{xy^n-1}{y-1}.$$

In particular, $[q^h;0]_q, (q^h;0)_q \in \mathfrak{U}_q^0$.

Definition 4.1. *The $\mathbf{A}_1$-form of $\mathfrak{U}_q\mathfrak{p}_n$, denoted by $\mathfrak{U}_{\mathbf{A}_1}$, is the $\mathbf{A}_1$-subalgebra of $\mathfrak{U}_q\mathfrak{p}_n$ generated by the elements $e_i, f_i, e_{\bar{i}}, f_{\bar{i}}$ for $i \in I$, $F_{\bar{j}}$ for $j \in I \cup \{n\}$, and q^h, $(q^h;0)_q$ for $h \in P^\vee$.*

Let $\mathfrak{U}_{\mathbf{A}_1}^+$ (respectively, $\mathfrak{U}_{\mathbf{A}_1}^-$), be the $\mathbf{A}_1$-subalgebra of $\mathfrak{U}_{\mathbf{A}_1}$ generated by the elements e_i and $e_{\bar{i}}$ for $i \in I$ (respectively, f_i, $f_{\bar{i}}$, and $F_{\bar{j}}$ for $i \in I$ and $j \in I \cup \{n\}$). Let $\mathfrak{U}_{\mathbf{A}_1}^0$ be the $\mathbf{A}_1$-subalgebra of $\mathfrak{U}_{\mathbf{A}_1}$ generated by q^h and $(q^h;0)_q$ for $h \in P^\vee$.

We will show that the triangular decomposition of $\mathfrak{U}_q\mathfrak{p}_n$ carries over to its $\mathbf{A}_1$-form. For this we first use the following lemma, whose proof is identical to the one of Lemma 5.2 in [GJKK].

Lemma 4.2.

(a) $(q^h;n)_q \in \mathfrak{U}_{\mathbf{A}_1}^0$ for all $n \in \mathbb{Z}$ and $h \in P^\vee$.

(b) $[q^h;0]_q \in \mathfrak{U}_{\mathbf{A}_1}^0$ for all $h \in P^\vee$.

Proposition 4.3. *The triangular decomposition of $\mathfrak{U}_q\mathfrak{p}_n$ in Theorem 2.5 induces an isomorphism of $\mathbf{A}_1$-modules*

$$\mathfrak{U}_{\mathbf{A}_1} \cong \mathfrak{U}_{\mathbf{A}_1}^- \otimes \mathfrak{U}_{\mathbf{A}_1}^0 \otimes \mathfrak{U}_{\mathbf{A}_1}^+.$$

Proof. Consider the isomorphism $\varphi : \mathfrak{U}_q\mathfrak{p}_n \longrightarrow \mathfrak{U}_q^- \otimes \mathfrak{U}_q^0 \otimes \mathfrak{U}_q^+$ from Theorem 2.5. Note that the following relations hold:

$$e_i(q_h;0)_q = (q^h;-\alpha_i(h))_q e_i \qquad e_{\bar{i}}(q_h;0)_q = (q^h;-\gamma_i(h))_q e_{\bar{i}}$$

$$(q_h;0)_q f_i = f_i(q^h;-\alpha_i(h))_q \qquad (q_h;0)_q f_{\bar{i}} = f_{\bar{i}}(q^h;-\gamma_i(h))_q$$

$$(q_h;0)_q F_{\bar{i}} = F_i(q^h;-\beta(h))_q$$

$$e_i f_i = f_i e_i - q^{-1}q^{k_i+k_{i+1}}\left[q^{k_i-k_{i+1}};0\right]_q + \frac{q^2-1}{q^2} f_{\bar{i}} e_{\bar{i}}$$

$$e_i f_{i+1} = f_{i+1}e_i - \frac{q-q^{-1}}{2} F_{\overline{i+1}} q^{-k_{i+1}}(e_{\overline{i+1}}e_i - e_i e_{\overline{i+1}}) \qquad e_i f_j = f_j e_i \text{ for } j \neq i, i+1$$

$$e_i f_{\bar{i}} = q^{-1} f_{\bar{i}} e_i + \frac{q^2+1}{2q^3} F_{\overline{i+1}} q^{k_{i+1}} \qquad e_{i+1} f_{\bar{i}} = f_{\bar{i}} e_{i+1} + f_{\overline{i+1}} f_i - f_i f_{\overline{i+1}}$$

$$e_i f_{\overline{i+1}} = f_{\overline{i+1}} e_i + \frac{q-q^{-1}}{2} F_{\overline{i+1}} q^{-k_{i+1}}(e_{i+1}e_i - e_i e_{i+1}) \qquad e_i f_{\bar{j}} = f_{\bar{j}} e_i \text{ for } |i-j|>1$$

$$e_{\bar{i}} f_{i+1} = f_{i+1} e_{\bar{i}} - e_{\overline{i+1}} e_i + e_i e_{\overline{i+1}} \qquad e_{\bar{i}} f_j = f_j e_{\bar{i}} \text{ for } j \neq i+1$$

$$e_{\bar{i}} f_{\bar{i}} = -q^{-2} f_{\bar{i}} e_{\bar{i}} - q^{-1} q^{k_i+k_{i+1}}\left[q^{k_i-k_{i+1}};0\right]_q \qquad e_{\bar{i}} f_{\overline{i+1}} = -f_{\overline{i+1}} e_{\bar{i}} + e_{i+1}e_i - e_i e_{i+1}$$

$$e_{\overline{i+1}} f_{\bar{i}} = -f_{\bar{i}} e_{\overline{i+1}} + f_{i+1} f_i - f_i f_{i+1} \qquad e_{\bar{i}} f_{\bar{j}} = f_{\bar{j}} e_{\bar{i}} \text{ for } |i-j|>1$$

$$e_i F_{\bar{i}} = q^{-1} F_{\bar{i}} e_i + 2q^{-1} f_{\bar{i}} q^{k_i} \qquad e_i F_{\overline{i+1}} = q F_{\overline{i+1}} e_i$$
$$e_{\bar{i}} F_{\bar{i}} = -q^{-1} F_{\bar{i}} e_{\bar{i}} + 2q^{-1} f_i q^{k_i} \qquad e_{\bar{i}} F_{\overline{i+1}} = -q^{-1} F_{\overline{i+1}} e_{\bar{i}} + 2q^{k_{i+1}} e_i$$
$$e_i F_{\bar{j}} = F_{\bar{j}} e_i \text{ for } j \neq i, i+1 \qquad e_{\bar{i}} F_{\bar{j}} = F_{\bar{j}} e_{\bar{i}} \text{ for } j \neq i, i+1$$

These relations, together with Lemma 4.2, imply that the image of the restriction $\overline{\varphi}$ of φ to $\mathfrak{U}_{\mathbf{A}_1}$ is a subset of $\mathfrak{U}^-_{\mathbf{A}_1} \otimes \mathfrak{U}^0_{\mathbf{A}_1} \otimes \mathfrak{U}^+_{\mathbf{A}_1}$. To define an inverse of $\overline{\varphi}$ we multiply the corresponding terms in the tensor product. This completes the proof. □

Definition 4.4. *The $\mathbf{A}_1$-form of the highest weight module V^q with highest weight $\lambda \in P$ and highest weight vector v is the $\mathfrak{U}_{\mathbf{A}_1}$-module $V_{\mathbf{A}_1} = \mathfrak{U}_{\mathbf{A}_1} v$.*

For the rest of the section, by V^q we denote a highest weight module over $\mathfrak{U}_q \mathfrak{p}_n$ with highest weight $\lambda \in P$ and highest weight vector v. We can strengthen the above definition with the following proposition:

Proposition 4.5. *With the notation as above:*

$$V_{\mathbf{A}_1} = \mathfrak{U}^-_{\mathbf{A}_1} v.$$

Proof. In the light of Proposition 4.3, it suffices to show that $\mathfrak{U}^+_{\mathbf{A}_1} v = 0$ and $\mathfrak{U}^0_{\mathbf{A}_1} v = \mathbf{A}_1 v$. The former identity follows from the fact that v is a highest weight vector. For the latter identity we use following:

$$q^h v = q^{\lambda(h)} v, \qquad (q^h; 0)_q v = \frac{q^{\lambda(h)} - 1}{q - 1} v.$$

□

For each $\mu \in P$, we set $(V_{\mathbf{A}_1})_\mu = V_{\mathbf{A}_1} \cap V^q_\mu$. The following shows that the weight space decomposition of V^q carries over to $V_{\mathbf{A}_1}$.

Proposition 4.6. *$V_{\mathbf{A}_1}$ has the weight space decomposition $V_{\mathbf{A}_1} = \bigoplus_{\mu \leq \lambda} (V_{\mathbf{A}_1})_\mu$.*

Proof. The idea is standard but for reader's convenience we present the proof. Let $v = v_1 + \ldots + v_p \in V_{\mathbf{A}_1}$, where $v_j \in V^q_{\mu_j}$ and $\mu_j \in P$ for each $j \in \{1, 2, \ldots, p\}$. We can assume that μ_j are distinct due to the weight decomposition of V^q. Therefore, it is enough to show that $v_j \in V_{\mathbf{A}_1}$ for each j.

Fix an index j. For each $i \neq j$, choose $h_i \in P^\vee$ such that $\mu_j(h_i) \neq \mu_i(h_i)$. Let $u \in \mathfrak{U}_{\mathbf{A}_1}$ be defined by

$$u := \prod_{i \neq j} \frac{\left(q^{h_i}; -\mu_i(h_i)\right)_q}{\left(q^{\mu_j(h_i)}; -\mu_i(h_i)\right)_q}.$$

Then for each $i \neq j$:

$$\frac{\left(q^{h_i}; -\mu_i(h_i)\right)_q}{\left(q^{\mu_j(h_i)}; -\mu_i(h_i)\right)_q} v_k = \frac{q^{-\mu_i(h_i)} q^{h_i} - 1}{q^{-\mu_i(h_i)} q^{\mu_j(h_i)} - 1} v_k = \frac{q^{-\mu_i(h_i)} q^{\mu_k(h_i)} - 1}{q^{-\mu_i(h_i)} q^{\mu_j(h_i)} - 1} v_k,$$

where $k \in \{1, 2, \ldots, p\}$. Therefore $uv_j = v_j$ and $uv_i = 0$. Hence, $v_j = uv \in V_{\mathbf{A}_1}$. □

Proposition 4.7. *For each $\mu \in P$, the weight space $(V_{\mathbf{A}_1})_\mu$ is a free $\mathbf{A}_1$-module with* $\operatorname{rank}_{\mathbf{A}_1} (V_{\mathbf{A}_1})_\mu = \dim_{\mathbb{C}(q)} V^q_\mu$.

Proof. Since $\mathbf{A}_1$ is a principal ideal domain, every finitely generated torsion-free module over $\mathbf{A}_1$ is free. Notice that, for each $\mu \in P$, the weight space $(V_{\mathbf{A}_1})_\mu$ is finitely generated as an $\mathbf{A}_1$-module. The weight space is also torsion free, as otherwise it would contradict the fact that $\mathbf{A}_1$ is an integral domain. Thus, for each $\mu \in P$, the weight space $(V_{\mathbf{A}_1})_\mu$ is a free $\mathbf{A}_1$-module.

Since $\mathbb{C}(q)$ is the field of quotients of $\mathbf{A}_1$, a set of vectors of a $\mathbb{C}(q)$-vector space is linearly independent if and only if it is $\mathbf{A}_1$-linearly independent. Thus $\operatorname{rank}_{\mathbf{A}_1}(V_{\mathbf{A}_1})_\mu \leq \dim_{\mathbb{C}(q)} V^q_\mu$. Let f_ζ's be some monomials in f_i's, $f_{\overline{j}}$'s, and $F_{\overline{\ell}}$'s. Since $\{f_\zeta\}$ forms a linearly independent set over $\mathbb{C}(q)$, $\{f_\zeta v \mid \zeta \in Z\}$ is a $\mathbb{C}(q)$-basis of V^q_μ for an appropriate set Z. This basis is also contained in $(V_{\mathbf{A}_1})_\mu$ by definition. So we have that $\operatorname{rank}_{\mathbf{A}_1}(V_{\mathbf{A}_1})_\mu \geq \dim_{\mathbb{C}(q)} V^q_\mu$, which completes the proof. $\square$

Combining the previous two propositions gives us the following:

Corollary 4.8. *The map $\phi : \mathbb{C}(q) \otimes_{\mathbf{A}_1} V_{\mathbf{A}_1} \to V^q$, $f \otimes v \mapsto fv$, is an isomorphism of $\mathbb{C}(q)$-vector spaces.*

Let $\mathbf{J}_1$ be the maximal ideal of $\mathbf{A}_1$ generated by $q-1$. Then there is a canonical isomorphism of fields

$$\mathbf{A}_1/\mathbf{J}_1 \xrightarrow{\sim} \mathbb{C} \quad \text{given by } f(q) + \mathbf{J}_1 \longmapsto f(1).$$

Define the $\mathbb{C}$-vector spaces

$$\begin{aligned} U_1 &= (\mathbf{A}_1/\mathbf{J}_1) \otimes_{\mathbf{A}_1} \mathfrak{U}_{\mathbf{A}_1}, \\ V^1 &= (\mathbf{A}_1/\mathbf{J}_1) \otimes_{\mathbf{A}_1} V_{\mathbf{A}_1}. \end{aligned}$$

Note that since $V_{\mathbf{A}_1}$ is a $\mathfrak{U}_{\mathbf{A}_1}$-module, V^1 is naturally a U_1-module. Note that

$$U_1 \cong \mathfrak{U}_{\mathbf{A}_1}/\mathbf{J}_1\mathfrak{U}_{\mathbf{A}_1} \quad \text{and} \quad V^1 \cong V_{\mathbf{A}_1}/\mathbf{J}_1 V_{\mathbf{A}_1},$$

which gives rise to the following natural maps

$$\begin{aligned} \mathfrak{U}_{\mathbf{A}_1} &\longrightarrow \mathfrak{U}_{\mathbf{A}_1}/\mathbf{J}_1\mathfrak{U}_{\mathbf{A}_1} \cong U_1, \\ V_{\mathbf{A}_1} &\longrightarrow V_{\mathbf{A}_1}/\mathbf{J}_1 V_{\mathbf{A}_1} \cong V^1. \end{aligned}$$

Note that q is mapped to 1 under these maps, hence U_1 can be considered as the limit of $\mathfrak{U}_q\mathfrak{p}_n$ at $q=1$. The passage under these maps is referred to as taking the *classical limit*. We write $\overline{x}$ for the image of x under these maps.

Let $\overline{h} \in U_1$ denote the classical limit of the element $(q^h;0)_q \in \mathfrak{U}_{\mathbf{A}_1}$. The following is standard (see for example Lemma 3.4.3 in [HK]):

Lemma 4.9.

(a) For all $h \in P^\vee$, $\overline{q^h} = 1$

(b) For any $h, h' \in P^\vee$, $\overline{h+h'} = \overline{h} + \overline{h'}$.

This lemma shows that the image of $\mathfrak{U}^0_{\mathbf{A}_1}$ under the classical limit is quite close to $U^0 = \mathfrak{U}(\mathfrak{h})$.

For each $\mu \in P$, define $V^1_\mu = (\mathbf{A}_1/\mathbf{J}_1) \otimes_{\mathbf{A}_1} (V_{\mathbf{A}_1})_\mu \cong (V_{\mathbf{A}_1})_\mu/\mathbf{J}_1(V_{\mathbf{A}_1})_\mu$.

Lemma 4.10.

(a) For each $\mu \in P$, if $\{v_i \mid i = 1, ..., m\}$ is a basis of the free $\mathbf{A}_1$-module $(V_{\mathbf{A}_1})_\mu$, then $\{\overline{v}_i \mid i = 1, ..., m\}$ is a basis of the $\mathbb{C}$-vector space V^1_μ.

(b) For each $\mu \in P$, a subset $\{v_i \mid i = 1, ..., m\}$ of $(V_{\mathbf{A}_1})_\mu$ is $\mathbf{A}_1$-linearly independent if the $\{\overline{v}_i \mid i = 1, ..., m\} \subset V^1_\mu$ is $\mathbb{C}$-linearly independent.

Proof. We first prove that $\overline{v}_i$, $i = 1, ..., m$, span V^1_μ. For every v in $V^1_\mu = (\mathbf{A}_1/\mathbf{J}_1) \otimes_{\mathbf{A}_1} (V_{\mathbf{A}_1})_\mu$ we have

$$v = \sum_{i=1}^{m} a_i \otimes_{\mathbf{A}_1} v_i = \sum_{i=1}^{m} b_i (1 \otimes_{\mathbf{A}_1} v_i) = \sum_{i=1}^{m} b_i \overline{v}_i,$$

where $a_i = b_i + \mathbf{J}_1 \in \mathbf{A}_1/\mathbf{J}_1$, $\overline{v}_i = (1 \otimes_{\mathbf{A}_1} v_i)$. So the set $\{1 \otimes_{\mathbf{A}_1} v_i\}$ $\mathbb{C}$-spans V^1_μ. The linear independence follows from the fact that $\{v_i\}$, $i = 1, ..., m$, are $\mathbf{A}_1$-linearly independent.

We now prove part (b). Assume that $\sum_{i=1}^{m} c_i(q) v_i = 0$ for $c_i(q) \in \mathbf{A}_1$, with some $c_j(q) \neq 0$ for some j. Then, multiplying by an appropriate power of $q - 1$, we may assume that $c_j(1) \neq 0$. Applying the classical limit gives $\sum_{i=1}^{m} c_i(1) \overline{v}_i = 0$. This contradicts the linear independence of $\overline{v}_i$, $i = 1, ..., m$. □

The following proposition is the analogue of Propositions 4.6 and 4.7 for V^1.

Proposition 4.11.

(a) $V^1 = \bigoplus_{\mu \leq \lambda} V^1_\mu$

(b) For each $\mu \in P$, $\dim_{\mathbb{C}} V^1_\mu = \mathrm{rank}_{\mathbf{A}_1} (V_{\mathbf{A}_1})_\mu$

Proof. The first assertion follows from Proposition 4.6, while the second assertion follows from Lemma 4.10. □

The following theorem shows that the classical limit of $\mathfrak{U}_q\mathfrak{p}_n$ is isomorphic to $\mathfrak{U}\mathfrak{p}_n$.

Theorem 4.12.

(a) The elements $\overline{e_i}$, $\overline{e_{\bar{i}}}$, $\overline{f_i}$, $\overline{f_{\bar{i}}}$, $(i \in I)$ $\overline{F_{\bar{i}}}$, $(i \in I \cup \{n\})$ and $\overline{h}$ $(h \in P^\vee)$ satisfy the defining relations of $\mathfrak{U}\mathfrak{p}_n$. Moreover, there exists a $\mathbb{C}$-superalgebra isomorphism $\varphi : \mathfrak{U}\mathfrak{p}_n \longrightarrow U_1$ and the U_1-module V^1 has a $\mathfrak{U}\mathfrak{p}_n$-module structure.

(b) For each $\mu \in P$ and $h \in P^\vee$, the element $\overline{h}$ acts on V^1_μ as scalar multiplication by $\mu(h)$. So, V^1_μ is the μ-weight space of the $\mathfrak{U}\mathfrak{p}_n$-module V^1.

(c) As a $\mathfrak{U}\mathfrak{p}_n$-module, V^1 is a highest weight module with highest weight $\lambda \in P$ and highest weight vector $\overline{v}$.

Proof. To prove (a), we recall that by Theorem 4.1 in [AGG], there is a $\mathbb{C}$-superalgebra isomorphism

$$\psi : \mathfrak{U}\mathfrak{p}_n \to \mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n/(q-1)\mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n,$$

where $\mathcal{A}$ is the localization of $\mathbb{C}[q, q^{-1}]$ at the ideal generated by $q - 1$, and $\mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n$ is the $\mathcal{A}$-subalgebra of $\mathfrak{U}_q\mathfrak{p}_n$ generated by a set of elements τ_{ij} (for the precise definition of τ_{ij}, see §4 in [AGG]). The map $\theta : \mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n/(q-1)\mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n \to \mathfrak{U}_{\mathbf{A}_1}/\mathbf{A}_1\mathfrak{U}_{\mathbf{A}_1}$, defined by

$$\theta(\overline{\tau}_{-i,-i-1}) \mapsto \overline{e_i} \qquad \theta(\overline{\tau}_{-i,i+1}) \mapsto \overline{e_{\bar{i}}}$$

$$\theta(\overline{\tau}_{i,i+1}) \mapsto \overline{f_i} \qquad \theta(\overline{\tau}_{i,-i-1}) \mapsto \overline{f_{\bar{i}}}$$
$$\theta(\overline{\tau}_{i,-i}) \mapsto \overline{F_{\bar{i}}} \qquad \theta(\overline{\tau}_{ii}) \mapsto \overline{k}_i$$

is also a $\mathbb{C}$-superalgebra isomorphism. Indeed, this follows from the definition of the corresponding generators and the fact that the classical limit of t_{ij} and τ_{ij} coincide. Also, we have already established that $\sigma : \mathfrak{U}_{\mathbf{A}_1}/\mathbf{A}_1\mathfrak{U}_{\mathbf{A}_1} \to U_1$ is a $\mathbb{C}$-superalgebra isomorphism by the definition of U_1. Therefore, the map

$$\phi = \sigma \circ \theta \circ \psi : \mathfrak{U}\mathfrak{p}_n \longrightarrow U_1$$

is a $\mathbb{C}$-superalgebra isomorphism.

To prove (b), let $w \in (V_{\mathbf{A}_1})_\mu$ and $h \in P^\vee$. Then we have that

$$(q^h;0)_q w = \frac{q^h - 1}{q-1} w = \frac{q^{\mu(h)} - 1}{q-1} w.$$

Taking the classical limit of both sides gives $\overline{h}\overline{w} = \mu(h)\overline{w}$, as desired.

It remains to prove (c). From (b) we have that $\overline{h}\overline{v} = \lambda(h)\overline{v}$. Since v is highest vector of $V_{\mathbf{A}_1}$ with highest weight λ, it follows that $\overline{e_i v} = 0$ and $\overline{e_{\bar{i}} v} = 0$ for each $i \in I$. Therefore, (a) and Proposition 4.5 imply that $V^1 = U_1^- \overline{v} = \mathfrak{U}\mathfrak{p}_n^- \overline{v}$. Hence, by definition, V^1 is a highest weight $\mathfrak{U}\mathfrak{p}_n$-module with highest weight λ and highest weight vector $\overline{v}$. ☐

Proposition 4.13. $\operatorname{ch} V^1 = \operatorname{ch} V^q$

Proof. Propositions 4.7 and 4.11 imply that $\dim_{\mathbb{C}} V^1_\mu = \dim_{\mathbb{C}(q)} V^q_\mu$ for each $\mu \in P$. This, along with Theorem 4.12(b), gives us the desired result. ☐

Corollary 4.14. *$V^q(\lambda)$ is finite dimensional if and only if $\lambda \in \Lambda^+$.*

Proof. Let $V^q = V^q(\lambda)$ with highest weight vector v, and suppose that $\lambda \in \Lambda^+$. From Proposition 3.6, we have that $f_i^{\lambda(k_i)-\lambda(k_{i+1})+1} v = 0$ for all $i \in I$. Applying the classical limit leads to $\overline{f}_i^{\lambda(k_i)-\lambda(k_{i+1})+1}\overline{v} = 0$. Since V^1 is a highest weight module, Proposition 3.3 gives us that V^1 is finite dimensional. Thus, by Proposition 4.13, V^q is finite dimensional.

Conversely, suppose that $V^q(\lambda)$ is finite dimensional. By Proposition 4.13, we have that V^1 is also finite dimensional. By Proposition 3.4, we have that $\lambda \in \Lambda^+$. ☐

One peculiarity of the Lie algebra $\mathfrak{p}_n$ is that $\mathfrak{p}_n^+$ and $\mathfrak{p}_n^-$ do not have the same dimension: with our choice of roots, there are more negative ones than positive ones. This is also reflected in the asymmetry between $\mathfrak{U}_q\mathfrak{p}_n^+$ and $\mathfrak{U}_q\mathfrak{p}_n^-$. A natural question to ask is how making the reverse choice of positive and negative roots could have affected the results in the last two sections. In particular, a maximal vector v would be defined as a vector such that $\mathfrak{U}_q\mathfrak{p}_n^- v = 0$. The main results like Proposition 4.13 and Corollary 4.14 would still hold. This is not surprising considering, for instance, that Theorem 3.4 remains valid after reversing the choice of positive and negative roots. Other results like Proposition 3.3 would have to be modified accordingly, e.g. by replacing f_i by e_i.

Now that Verma modules and the notion of highest weight module have been defined for $\mathfrak{U}_q\mathfrak{p}_n$, it is natural to introduce the category $\mathcal{O}_q$ of representations of $\mathfrak{U}_q\mathfrak{p}_n$. For quantized enveloping algebras of semisimple Lie algebras, a definition of the category $\mathcal{O}_q$ is provided, for instance, in [AM], and is studied in *loc. cit.* mostly in the case when q is a root-of-unity. The definition of category $\mathcal{O}$ for Lie superalgebras with triangular decomposition is also given in *loc. cit.* We can combine both definitions into the following in the case of $\mathfrak{p}_n$.

Definition 4.15. *The category $\mathcal{O}_q$ for $\mathfrak{U}_q\mathfrak{p}_n$ consists of (left) modules M such that*

1. M *is finitely generated;*

2. M *is a weight module;*

3. M *is a locally finite module over* $\mathfrak{U}_q\mathfrak{p}_n^+$.

Verma modules $M^q(\lambda)$ belong to $\mathcal{O}_q$ and their irreducible quotients $V^q(\lambda)$ exhaust all the irreducible objects in $\mathcal{O}_q$ up to isomorphism.

For the Lie superalgebra $\mathfrak{p}_n$, one can consider both its category $\mathcal{O}$ and the category $\mathcal{O}_0$ of its even part $\mathfrak{gl}_n$, and use restriction and induction functors to relate both. Unfortunately, $\mathfrak{U}_q\mathfrak{gl}_n$ is not a subalgebra of $\mathfrak{U}_q\mathfrak{p}_n$ as the defining relations of $\mathfrak{U}_q\mathfrak{p}_n$ show. Moreover, the subalgebra of $\mathfrak{U}_q\mathfrak{p}_n$ generated by e_i, f_i, q^{k_i} and $q^{k_{i+1}}$ is not isomorphic to $\mathfrak{U}_q\mathfrak{gl}_2$, we don't know what is the center of $\mathfrak{U}_q\mathfrak{p}_n$ and what could replace the Weyl group that control in part the combinatorics of the category $\mathcal{O}$ for semisimple Lie algebras. We do not address here how to circumvent these difficulties.

5 Category of Tensor Representations of $\mathfrak{U}_q\mathfrak{p}_n$-modules

In this final section we discuss the category of tensor representations of $\mathfrak{U}_q\mathfrak{p}_n$. It is shown in [Mo] that the $\mathfrak{p}_n$-module $\mathbb{C}(n|n)^{\otimes k}$ is not completely reducible for any $k \geq 2$. We will prove a similar result for $\mathfrak{U}_q\mathfrak{p}_n$.

Let $V = \mathbb{C}_q(n|n)$. The action of the generators t_{ij} of $\mathfrak{U}_q\mathfrak{p}_n$ on V is given by the following formulas obtained in [AGG]:

$$\begin{aligned}
t_{ii}(u_a) &= \sum_{b=-n}^{n} q^{\delta_{bi}(1-2p(i))+\delta_{b,-i}(2p(i)-1)} E_{bb}(u_a);\\
t_{i,-i}(u_a) &= (q-q^{-1})\delta_{i>0} E_{-i,i}(u_a);\\
t_{ij}(u_a) &= (q-q^{-1})(-1)^{p(i)} \mathsf{E}_{ji}(u_a), \text{ if } |i| \neq |j|.
\end{aligned}$$

The action of the Drinfeld-Jimbo generators of $\mathfrak{U}_q\mathfrak{p}_n$ in (3) then follows. We can then extend this action to $V^{\otimes k}$ through comultiplication given in Lemma 2.3.

Recall also from [AGG] the $\mathfrak{U}_q\mathfrak{p}_n$-module homomorphisms $\mathfrak{c} : \mathbb{C}_q(n|n)^{\otimes 2} \longrightarrow \mathbb{C}_q(n|n)^{\otimes 2}$ and $\mathfrak{t} : \mathbb{C}_q(n|n)^{\otimes 2} \longrightarrow \mathbb{C}_q(n|n)^{\otimes 2}$, where

$$\begin{aligned}
\mathfrak{c} = &\sum_{a,b=-n}^{n} (-1)^{p(a)p(b)} E_{ab} \otimes E_{-a,-b},\\
\mathfrak{t} = &\sum_{i,j=-n}^{n} (-1)^{p(j)} E_{ij} \otimes E_{ji} + (q-1)\sum_{i=1}^{n} (E_{-i,i} \otimes E_{i,-i})\\
&+ (q-1)\sum_{i=1}^{n} (E_{ii} \otimes E_{ii}) - (q^{-1}-1)\sum_{i=1}^{n} (E_{i,-i} \otimes E_{-i,i})\\
&- (q^{-1}-1)\sum_{i=1}^{n} (E_{-i,-i} \otimes E_{-i,-i}) + (q-q^{-1})\sum_{i=1}^{n} (E_{ii} \otimes E_{-i,-i})\\
&+ (q-q^{-1})\sum_{|j|<|i|} (E_{jj} \otimes E_{ii}) + (q-q^{-1})\sum_{|j|<|i|} \Big((-1)^{p(i)p(j)} E_{ji} \otimes E_{-j,-i}\Big).
\end{aligned}$$

These maps are then extended to $\mathfrak{U}_q\mathfrak{p}_n$-module homomorphisms $\mathfrak{c}_i : \mathbb{C}_q(n|n)^{\otimes k} \longrightarrow \mathbb{C}_q(n|n)^{\otimes k}$ and $\mathfrak{t}_i : \mathbb{C}_q(n|n)^{\otimes k} \longrightarrow \mathbb{C}_q(n|n)^{\otimes k}$ by applying $\mathfrak{c}$ and $\mathfrak{t}$, respectively, to the i^{th} and $(i+1)^{th}$ tensors. We will refer to the map $\mathfrak{c}$ as the *contraction* map.

Remark 5.1. The contraction map in [Mo] requires a sign change. As a result some of the subsequent theorems have to be modified in [Mo]. Namely, the results in Sections 6.1 and 6.2 in [Mo] need to be corrected and the correct version can be obtained by taking $q = 1$ of our results in this section.

5.1 Maximal vectors in $V^{\otimes k}$

In this subsection we describe the complete set of linearly independent maximal vectors in the $\mathfrak{U}_q\mathfrak{p}_n$-module $V^{\otimes k}$. Following the ideas in Theorem 3.8 of [Mo], we will use a q-analogue of the Young symmetrizer defined in [Gy] to define these maximal vectors.

Let S_k be the symmetric group on the set $\{1, \dots, k\}$, and let $S := \{\mathsf{s}_1, \dots, \mathsf{s}_k\}$ where $\mathsf{s}_i = (i, i+1)$. Recall that the periplectic q-Brauer algebra $\mathfrak{B}_{q,k}$ is generated by t_i and c_i for $1 \le i \le k-1$ and satisfies a set of relations listed in Definition 5.1 of [AGG]. The action of $\mathfrak{B}_{q,k}$ on $\mathbb{C}_q(n|n)^{\otimes k}$ is given by t_i and c_i acting by $\mathfrak{t}_i$ and $\mathfrak{c}_i$, respectively.

We consider the Hecke algebra H_k as the subalgebra of $\mathfrak{B}_{q,k}$ generated by $\{h(\mathsf{s}_i) = \mathsf{t}_i \mid i = 1, 2, \dots, k-1\}$ subject to the following relations:

$$\begin{aligned}
&(h(\mathsf{s}_i) - q)(h(\mathsf{s}_i) + q^{-1}) = 0,\\
&h(\mathsf{s}_i)h(\mathsf{s}_{i+1})h(\mathsf{s}_i) = h(\mathsf{s}_{i+1})h(\mathsf{s}_i)h(\mathsf{s}_{i+1}).
\end{aligned}$$

If σ is a permutation having a reduced decomposition $\sigma = \mathsf{s}_{i_1} \cdots \mathsf{s}_{i_\ell}$ we set $h(\sigma) = h(\mathsf{s}_{i_1}) \cdots h(\mathsf{s}_{i_\ell})$. Then $h(\sigma)h(\sigma') = h(\sigma\sigma')$ if $\ell(\sigma\sigma') = \ell(\sigma) + \ell(\sigma')$, where $\sigma, \sigma' \in S_k$ and $\ell(\sigma)$ is the length of the permutation σ.

Define the following element of $\mathfrak{B}_{q,k}$:

$$\mathsf{c}_{r,s} := h(\sigma_{r,s})\mathsf{c}_1 h^{-1}(\sigma_{r,s}),$$

where $\sigma_{r,s} := (1, r)(2, s)$.

For $j \in \left\{1, \dots, \left\lfloor \frac{k}{2} \right\rfloor\right\}$ and two disjoint ordered subsets $\tilde{r} = \{r_1, \dots, r_j\}$ and $\tilde{s} = \{s_1, \dots, s_j\}$ of $\{1, \dots, k\}$ such that $r_i < s_i$ for all $i = 1, \dots, j$, set

$$\mathsf{c}_{\tilde{r},\tilde{s}} := \mathsf{c}_{r_1,s_1} \cdots \mathsf{c}_{r_j,s_j}, \qquad \mathsf{c}_{\emptyset,\emptyset} := \mathrm{id}.$$

We set $(\tilde{r}, \tilde{s}) = \{(r_1, s_1), \dots, (r_j, s_j)\}$ and denote by $\mathcal{P}(j)$ the set of all $(\tilde{r}, \tilde{s})$ such that the cardinality of both $\tilde{r}$ and $\tilde{s}$ equal j. Set $\mathcal{P} = \bigcup_{j=0}^{\lfloor \frac{k}{2} \rfloor} \mathcal{P}(j)$.

We follow the common definition of standard tableau as given for instance in [Gy]. If λ is a partition of N we write $\lambda \vdash N$. Let $\lambda \vdash N$ have length at most $2n$. Following [Gy], we define two standard tableaux $T_+ = T_+(\lambda)$ and $T_- = T_-(\lambda)$ depending on λ, where the entries of T_+ increase by one across the rows from left to right, and the entries of T_- increase by one down the columns. Let W_+ (respectively, W_-) be the group of all elements in S_N which permute the entries within each row of T_+ (respectively, each column of T_-).

Let

$$\begin{aligned}
\mathsf{e}_+ = \mathsf{e}_+(\lambda) &:= \sum_{\sigma \in W_+} q^{\ell(\sigma)} h(\sigma),\\
\mathsf{e}_- = \mathsf{e}_-(\lambda) &:= \sum_{\sigma \in W_-} (-q)^{-\ell(\sigma)} h(\sigma).
\end{aligned}$$

Note that for each $\mathsf{s} = \mathsf{s}_i$ in W_+,

$$\mathsf{e}_+ = \sum_{\substack{\sigma \in W_+ \\ \mathsf{s}\sigma > \sigma}} q^{\ell(\sigma)}(1 + qh(\mathsf{s}))h(\sigma).$$

Thus, we have that $(1 - q^{-1}h(\mathsf{s}))\mathsf{e}_+ = 0$, or in other words, $h(\mathsf{s})\mathsf{e}_+ = q\mathsf{e}_+$. Hence

$$h(\rho)\mathsf{e}_+ = q^{\ell(\rho)}\, \mathsf{e}_+$$

for $\rho \in W_+$. With the same reasoning we obtain analogous identities included in the following Lemma.

Lemma 5.2. *For $\rho \in W_+$ and $\rho' \in W_-$,*

$$\begin{aligned} h(\rho)\mathsf{e}_+ &= \mathsf{e}_+ h(\rho) = q^{\ell(\rho)}\mathsf{e}_+, \\ h(\rho')\mathsf{e}_- &= \mathsf{e}_- h(\rho') = (-q)^{-\ell(\rho')}\mathsf{e}_-. \end{aligned}$$

Let T be a standard tableaux of shape $\lambda \vdash k$. Denote by $\sigma_\pm^T$ the permutation that transforms $T_\pm$ to T. Note that the set $\sigma_+^T W_+ (\sigma_+^T)^{-1}$ (respectively, $\sigma_-^T W_- (\sigma_-^T)^{-1}$) consists of all permutations that permute the rows (respectively, columns) of T.

Define $x_T(q) \in H_k$ by

$$x_T(q) := h(\sigma_-^T)\, \mathsf{e}_- \left(h(\sigma_-^T)\right)^{-1} h(\sigma_+^T)\, \mathsf{e}_+ \left(h(\sigma_+^T)\right)^{-1}.$$

Note that there exists $\xi \in \mathbb{C}(q)$, depending on the shape λ of the tableau T, such that

$$x_T(q)^2 = \xi x_T(q).$$

Then the q-analogue of the Young symmetrizer as defined in [Gy] is

$$y_T(q) = \frac{1}{\xi} x_T(q).$$

In what follows for a subset A of $\{1, 2, ..., k\}$ by A^c we denote its complement. Denote by $\mathsf{ST}((\tilde{r} \cup \tilde{s})^c)$ the set of all standard tableaux of shape $\mu \vdash k - 2j$, for some $j \in \left\{0, 1, \dots, \left\lfloor \frac{k}{2} \right\rfloor\right\}$, with entries in $(\tilde{r} \cup \tilde{s})^c$, where $(\tilde{r}, \tilde{s}) \in \mathcal{P}(j)$.

For each $\tau \in \mathsf{ST}((\tilde{r} \cup \tilde{s})^c)$, define the associated simple tensor of τ by $w_{\tau,\tilde{r},\tilde{s}} := w_1 \otimes \ldots \otimes w_k$, where

$$w_i := \begin{cases} u_1 & \text{if } i \in \tilde{r} \\ u_{-1} & \text{if } i \in \tilde{s} \\ u_j & \text{if } j \in (\tilde{r} \cup \tilde{s})^c \text{ and } i \text{ is in } j\text{th row of } \tau. \end{cases}$$

We now prove a q-anologue of Theorem 3.8 in [Mo].

Theorem 5.3. *Let n and k be positive integers such that $n \geq k$. Then*

$$\{y_\tau \mathsf{c}_{\tilde{r},\tilde{s}} w_{\tau,\tilde{r},\tilde{s}} \mid (\tilde{r}, \tilde{s}) \in \mathcal{P},\, \tau \in \mathsf{ST}((\tilde{r} \cup \tilde{s})^c), \ell(\tau) \leq n\}$$

is a linearly independent set of maximal vectors in the $\mathfrak{U}_q\mathfrak{p}_n$-module $V^{\otimes k}$.

Proof. Let $w = w_{\tau,\tilde{r},\tilde{s}}$ and $\theta = y_\tau \mathsf{c}_{\tilde{r},\tilde{s}} w$. Note that the weight of θ is the same as the weight of w since $y_\tau c_{\tilde{r},\tilde{s}}$ commutes with the action of $\mathfrak{U}_q\mathfrak{p}_n$. Note that the fact that $\theta \neq 0$ and the linear independence property follow by applying the classical limit and using Theorem 3.8 in [Mo].

To show that θ is a maximal vector, it suffices to show that θ is annihilated by each root vector e_i and $e_{\bar{i}}$, $i \in I$. The action of e_i and $e_{\bar{i}}$ on w can be explicitly written as follows

$$
\begin{aligned}
e_{\bar{i}}(w_1 \otimes \ldots \otimes w_k) &= \sum_{a=1}^{k} (-1)^{p(w_1)+\ldots+p(w_{a-1})} q^{k_i} w_1 \otimes \ldots \otimes q^{k_i} w_{a-1} \otimes e_{\bar{i}} w_a \otimes q^{k_{i+1}} w_{a+1} \otimes \ldots \otimes q^{k_{i+1}} w_k, \\
e_i(w_1 \otimes \ldots \otimes w_k) &= \sum_{a=1}^{k} q^{k_i} w_1 \otimes \ldots \otimes q^{k_i} w_{a-1} \otimes e_i w_a \otimes q^{k_{i+1}} w_{a+1} \otimes \ldots \otimes q^{k_{i+1}} w_k \\
&\quad - \frac{q-q^{-1}}{2} \sum_{\substack{a,b=1 \\ a<b}}^{k} (-1)^{p(w_1)+\ldots+p(w_{a-1})} q^{k_i} w_1 \otimes \ldots \otimes q^{k_i} w_{a-1} \otimes e_{\bar{i}} w_a \otimes q^{k_{i+1}} w_{a+1} \otimes \ldots \\
&\quad \otimes q^{k_{i+1}} w_{b-1} \otimes F_{\overline{i+1}} w_b \otimes q^{k_{i+1}} w_{b+1} \otimes \ldots \otimes q^{k_{i+1}} w_k.
\end{aligned} \tag{9}
$$

Let $x = e_i$ or $x = e_{\bar{i}}$. We look at how x acts on $\mathsf{c}_{\tilde{r},\tilde{s}} w$. Note that $\mathsf{c}V^{\otimes 2}$ is the trivial $\mathfrak{U}_q\mathfrak{p}_n$-module $\mathbb{C}(q)$. Therefore the sum of the terms with x acting on a pair of contracted tensor factors is zero. Also, from (9), we see that the action of x on non-contracted tensor factors, or in other words where $w_i \in V_0$, those specific terms in the summation will be zero, except when $x = e_{j-1}$ for some $j \geq 2$. Thus x either annihilates $\mathsf{c}_{\tilde{r},\tilde{s}} w$ or produces a sum of tensors which are obtained by applying $x = e_{j-1}$, for some $j \geq 2$, to a factor unaffected by $\mathsf{c}_{\tilde{r},\tilde{s}}$. Each of those tensors has one factor whose subscript is in $(\tilde{r} \cup \tilde{s})^c$ and has been lowered by one, so u_j has been changed to u_{j-1}. Denote such a tensor by v. We want to show that $y_\tau \mathsf{v} = 0$.

Fix $\psi \in W_+$. There are two factors of the tensor $h(\sigma_+^\tau)h(\psi)h(\sigma_+^\tau)^{-1}\mathsf{v}$ that have the same u_j. Then there exists a transposition $(a,b) = (\sigma_-^\tau)\rho(\sigma_-^\tau)^{-1} \in (\sigma_-^\tau)W_-(\sigma_-^\tau)^{-1}$ which permutes these two factors. Using that $\ell(a,b) = \ell(\rho)$, we obtain

$$h(\sigma_-^\tau)\, h(\rho)\, h(\sigma_-^\tau)^{-1}\, h(\sigma_+^\tau)h(\psi)h(\sigma_+^\tau)^{-1}\mathsf{v} = h(a,b)\, h(\sigma_+^\tau)h(\psi)h(\sigma_+^\tau)^{-1}\mathsf{v} = q^{\ell(\rho)} h(\sigma_+^\tau)h(\psi)h(\sigma_+^\tau)^{-1}\mathsf{v}.$$

By Lemma 5.2, we have that for $\rho \in W_-$,

$$(-q)^{-\ell(\rho)}\mathsf{e}_- = \mathsf{e}_-\, h(\rho) = \sum_{\sigma \in W_-} (-q)^{-\ell(\sigma)} h(\sigma)\, h(\rho)$$

and hence

$$h(\sigma_-^\tau)\mathsf{e}_-\, h(\sigma_-^\tau)^{-1} = \sum_{\sigma \in W_-} (-q)^{-\ell(\sigma)+\ell(\rho)} h(\sigma_-^\tau)\, h(\sigma)\, h(\sigma_-^\tau)^{-1}\, h(\sigma_-^\tau)\, h(\rho)\, h(\sigma_-^\tau)^{-1}$$

Thus we have

$$
\begin{aligned}
h(\sigma_-^\tau)\mathsf{e}_-\, h(\sigma_-^\tau)^{-1} h(\sigma_+^\tau)h(\psi)h(\sigma_+^\tau)^{-1}\mathsf{v} &= \sum_{\sigma \in W_-} (-q)^{-\ell(\sigma)+\ell(\rho)} h(\sigma_-^\tau)\, h(\sigma)\, h(\sigma_-^\tau)^{-1}\, h(\sigma_-^\tau)\, h(\rho)\, h(\sigma_-^\tau)^{-1} h(\sigma_+^\tau)h(\psi)h(\sigma_+^\tau)^{-1}\mathsf{v} \\
&= \sum_{\sigma \in W_-} (-q)^{-\ell(\sigma)+\ell(\rho)} q^{\ell(\rho)} h(\sigma_-^\tau)\, h(\sigma)\, h(\sigma_-^\tau)^{-1}\, h(\sigma_+^\tau)h(\psi)h(\sigma_+^\tau)^{-1}\mathsf{v} \\
&= (-1)^{\ell(\rho)} q^{2\ell(\rho)} \sum_{\sigma \in W_-} (-q)^{-\ell(\sigma)} h(\sigma_-^\tau)\, h(\sigma)\, h(\sigma_-^\tau)^{-1}\, h(\sigma_+^\tau)h(\psi)h(\sigma_+^\tau)^{-1}\mathsf{v} \\
&= (-1)^{\ell(\rho)} q^{2\ell(\rho)} h(\sigma_-^\tau)\mathsf{e}_-\, h(\sigma_-^\tau)^{-1} h(\sigma_+^\tau)h(\psi)h(\sigma_+^\tau)^{-1}\mathsf{v}.
\end{aligned}
$$

Hence,

$$h(\sigma_-^\tau)\mathsf{e}_-\, h(\sigma_-^\tau)^{-1} h(\sigma_+^\tau)h(\psi)h(\sigma_+^\tau)^{-1}\mathsf{v} = 0$$

for each $\psi \in W_+$. Therefore, we have that

$$h(\sigma_-^\tau)\mathsf{e}_-\, h(\sigma_-^\tau)^{-1} h(\sigma_+^\tau)\mathsf{e}_+\, h(\sigma_+^\tau)^{-1}\mathsf{v} = \sum_{\psi \in W_+} q^{\ell(\psi)} h(\sigma_-^\tau)\mathsf{e}_-\, h(\sigma_-^\tau)^{-1} h(\sigma_+^\tau)h(\psi)h(\sigma_+^\tau)^{-1}\mathsf{v} = 0.$$

This concludes the proof of $y_\tau \mathsf{v} = 0$. $\square$

5.2 Decomposition of $V^{\otimes 2}$

By Theorem 5.3, the following vectors are linearly independent maximal vectors of the $\mathfrak{U}_q\mathfrak{p}_n$-module $V^{\otimes 2}$:

$$\begin{aligned}
\theta_1^q &= y_{\boxed{1}\boxed{2}}(u_1 \otimes u_1) = u_1 \otimes u_1,\\
\theta_2^q &= y_{\substack{\boxed{1}\\\boxed{2}}}(u_1 \otimes u_2) = \frac{1}{1+q^{-2}}(q^{-1}u_1 \otimes u_2 - u_2 \otimes u_1),\\
\theta_3^q &= \mathsf{c}_1(u_1 \otimes u_{-1}) = \sum_{i=-n}^{n} u_i \otimes u_{-i},
\end{aligned}$$

where

$$\begin{aligned}
y_{\boxed{1}\boxed{2}} &= \frac{1}{1+q^2}(1+q\mathsf{t}_1),\\
y_{\substack{\boxed{1}\\\boxed{2}}} &= \frac{1}{1+q^{-2}}(1-q^{-1}\mathsf{t}_1).
\end{aligned}$$

Proposition 5.4. *The vectors $\theta_1^q, \theta_2^q, \theta_3^q$ form a complete set of linearly independent maximal vectors of the $\mathfrak{U}_q\mathfrak{p}_n$-module $V^{\otimes 2}$. Moreover, $\theta_1^q, \theta_3^q \in y_{\boxed{1}\boxed{2}}V^{\otimes 2}$ and $\theta_2^q \in y_{\substack{\boxed{1}\\\boxed{2}}}V^{\otimes 2}$.*

Proof. Applying the classical limit to θ_i^q yields the linearly independent maximal vectors θ_i in [Mo] (after appropriate sign change for θ_3^q, see Remark 5.1). Since the complete set of linearly independent maximal vectors of the $\mathfrak{U}\mathfrak{p}_n$-module $V^{\otimes 2}$ has exactly three vectors, the result follows from Proposition 4.13. The second statement is subject to a direct verification. $\square$

In what follows we will show that $V^{\otimes 2}$ is isomorphic to the direct sum of the two indecomposable representations, $y_{\boxed{1}\boxed{2}}V^{\otimes 2}$ and $y_{\substack{\boxed{1}\\\boxed{2}}}V^{\otimes 2}$.

Proposition 5.5. *The module $y_{\substack{\boxed{1}\\\boxed{2}}}V^{\otimes 2}$ is reducible and indecomposable. More precisely, there is a non-split exact sequence of $\mathfrak{U}_q\mathfrak{p}_n$-modules*

$$0 \longrightarrow V^q(\epsilon_1+\epsilon_2) \longrightarrow y_{\substack{\boxed{1}\\\boxed{2}}}V^{\otimes 2} \longrightarrow V^q(0) \longrightarrow 0. \tag{10}$$

Proof. We have that

$$\mathsf{c}_1 y_{\substack{\boxed{1}\\\boxed{2}}}V^{\otimes 2} \subset \mathsf{c}_1 V^{\otimes 2} \cong \mathbb{C}(q).$$

However, since

$$\mathsf{c}_1 y_{\substack{\boxed{1}\\\boxed{2}}}(u_1 \otimes u_{-1}) = \frac{1}{1+q^{-2}}\mathsf{c}_1[q^{-2}u_1 \otimes u_{-1} - u_{-1} \otimes u_{-1}] = \theta_3^q \neq 0, \tag{11}$$

it follows that $\mathsf{c}_1 y_{\substack{\boxed{1}\\\boxed{2}}}V^{\otimes 2}$ cannot be zero. Therefore, $\mathsf{c}_1 y_{\substack{\boxed{1}\\\boxed{2}}}V^{\otimes 2} \cong \mathbb{C}(q)$. Consider the restriction of the contraction map c_1 to $y_{\substack{\boxed{1}\\\boxed{2}}}V^{\otimes 2}$, and let $\mathcal{N}$ denote the kernel of this restriction. Then

$$y_{\substack{\boxed{1}\\\boxed{2}}}V^{\otimes 2}/\mathcal{N} \cong \mathbb{C}(q) \cong V^q(0).$$

By Proposition 5.4, $\theta_2^q \in \mathcal{N}$ and $\theta_1^q, \theta_3^q \notin \mathcal{N}$. Hence, by the same proposition, θ_2^q is the only, up to a scalar multiple, maximal vector in $\mathcal{N}$, in particular, $\mathcal{N}$ is simple. Thus

$$V^q(\epsilon_1+\epsilon_2) \cong \mathfrak{U}_q\mathfrak{p}_n\theta_2^q = \mathcal{N} \subsetneq y_{\substack{\boxed{1}\\ \boxed{2}}}V^{\otimes 2}.$$

This implies the exact sequence (10). The sequence does not split because $y_{\substack{\boxed{1}\\ \boxed{2}}}V^{\otimes 2}$ has a unique up to a scalar multiple maximal vector. $\square$

Proposition 5.6. *The module $y_{\boxed{1}\boxed{2}}V^{\otimes 2}$ is reducible and indecomposable. More precisely, there is a non-split short exact sequences of $\mathfrak{U}_q\mathfrak{p}_n$-modules*

$$0 \longrightarrow V^q(0) \longrightarrow y_{\boxed{1}\boxed{2}}V^{\otimes 2} \longrightarrow V^q(2\epsilon_1) \longrightarrow 0. \tag{12}$$

Proof. We have that

$$y_{\boxed{1}\boxed{2}}\mathsf{c}_1 = \mathsf{c}_1.$$

This implies that

$$\mathsf{c}_1V^{\otimes 2} \subset y_{\boxed{1}\boxed{2}}V^{\otimes 2}.$$

Note that $\theta_1^q \notin \mathsf{c}_1V^{\otimes 2}$. However, (11) gives that $\theta_3^q \in \mathsf{c}_1(V^{\otimes 2})$. By Proposition 5.4, θ_3^q is the only, up to a scalar multiple, maximal vector in $\mathsf{c}_1V^{\otimes 2}$. Thus, we have that

$$V^q(0) \cong \mathfrak{U}_q\mathfrak{p}_n\theta_3^q \subset \mathsf{c}_1V^{\otimes 2} \subsetneq y_{\boxed{1}\boxed{2}}V^{\otimes 2}.$$

By Proposition 5.4, we have

$$y_{\boxed{1}\boxed{2}}V^{\otimes 2}/\mathfrak{U}_q\mathfrak{p}_n\theta_3^q \cong V^q(2\epsilon_1),$$

which implies the exact sequence (12). Since θ_1^q generates $y_{\boxed{1}\boxed{2}}V^{\otimes 2}$, the sequence does not split. $\square$

The following theorem is the main result of this subsection.

Theorem 5.7. *As a $\mathfrak{U}_q\mathfrak{p}_n$-module, we have the following decomposition*

$$V^{\otimes 2} = y_{\boxed{1}\boxed{2}}V^{\otimes 2} \oplus y_{\substack{\boxed{1}\\ \boxed{2}}}V^{\otimes 2},$$

where the submodules in the above decomposition are involved in the non-split short exact sequences (10), (12).

Proof. Let $T : V^{\otimes 2} \longrightarrow V^{\otimes 2}$ be the $\mathfrak{U}_q\mathfrak{p}_n$-module homomorphism defined by

$$T(v) = y_{\boxed{1}\boxed{2}}v.$$

Note that

$$y_{\boxed{1}\boxed{2}}y_{\substack{\boxed{1}\\ \boxed{2}}} = \frac{1}{(1+q^2)(1+q^{-2})}(1+qt_1)(1-q^{-1}t_1) = 0.$$

So, we have that $y_{\substack{\boxed{1}\\ \boxed{2}}}V^{\otimes 2} \subset \ker T$, which implies that $y_{\substack{\boxed{1}\\ \boxed{2}}}V^{\otimes 2} = \ker T$ since the only maximal vector of $V^{\otimes 2}$ in $\ker T$ is θ_2^q. Thus we have a short exact sequence

$$0 \longrightarrow y_{\substack{\boxed{1}\\ \boxed{2}}}V^{\otimes 2} \longrightarrow V^{\otimes 2} \longrightarrow y_{\boxed{1}\boxed{2}}V^{\otimes 2} \longrightarrow 0.$$

Using the embedding $\iota : y_{\boxed{1}\boxed{2}}V^{\otimes 2} \longrightarrow V^{\otimes 2}$ and that $T \circ \iota = \mathrm{id}$, we see that the sequence above splits. The remaining part of the theorem follows from Propositions 5.5 and 5.6. $\square$

5.3 Decomposition of $V^{\otimes 3}$

In this subsection we prove an analogous statement to Theorem 5.7 for $V^{\otimes 3}$. More precisely, we show that

$$V^{\otimes 3} = y_{\ytableaushort{123}} V^{\otimes 3} \oplus y_{\ytableaushort{12,3}} V^{\otimes 3} \oplus y_{\ytableaushort{13,2}} V^{\otimes 3} \oplus y_{\ytableaushort{1,2,3}} V^{\otimes 3} \tag{13}$$

is the decomposition of $V^{\otimes 3}$ into indecomposables, where

$$\begin{aligned}
y_{\ytableaushort{123}} &= \frac{1}{1+2q^2+2q^4+q^6}(1+q\mathsf{t}_1+q\mathsf{t}_2+q^2\mathsf{t}_1\mathsf{t}_2+q^2\mathsf{t}_2\mathsf{t}_1+q^3\mathsf{t}_1\mathsf{t}_2\mathsf{t}_1),\\
y_{\ytableaushort{12,3}} &= \frac{1}{q^{-2}+1+q^2}[1+q\mathsf{t}_1+(q-q^{-1})\mathsf{t}_2-\mathsf{t}_1\mathsf{t}_2+(q^2-1)\mathsf{t}_2\mathsf{t}_1-q\mathsf{t}_1\mathsf{t}_2\mathsf{t}_1],\\
y_{\ytableaushort{13,2}} &= \frac{1}{q^{-2}+1+q^2}(1-q^{-1}\mathsf{t}_1-q^2\mathsf{t}_2\mathsf{t}_1+q\mathsf{t}_1\mathsf{t}_2\mathsf{t}_1),\\
y_{\ytableaushort{1,2,3}} &= \frac{1}{1+2q^{-2}+2q^{-4}+q^{-6}}(1-q^{-1}\mathsf{t}_1-q^{-1}\mathsf{t}_2+q^{-2}\mathsf{t}_1\mathsf{t}_2+q^{-2}\mathsf{t}_2\mathsf{t}_1-q^{-3}\mathsf{t}_1\mathsf{t}_2\mathsf{t}_1).
\end{aligned}$$

With a slight of notation, Theorem 5.3 implies that the following are linearly independent maximal vectors of $V^{\otimes 3}$:

$$\begin{aligned}
\theta_1^q &= \mathsf{c}_1(u_1\otimes u_{-1}\otimes u_1),\\
\theta_2^q &= \mathsf{t}_2\mathsf{c}_1\mathsf{c}_2(u_1\otimes u_1\otimes u_{-1}),\\
\theta_3^q &= \mathsf{c}_2(u_1\otimes u_1\otimes u_{-1}),\\
\theta_4^q &= y_{\ytableaushort{123}}(u_1\otimes u_1\otimes u_1),\\
\theta_5^q &= y_{\ytableaushort{12,3}}(u_1\otimes u_1\otimes u_2),\\
\theta_6^q &= y_{\ytableaushort{13,2}}(u_1\otimes u_2\otimes u_1),\\
\theta_7^q &= y_{\ytableaushort{1,2,3}}(u_1\otimes u_2\otimes u_3).
\end{aligned}$$

The next proposition is proven in a similar way as Proposition 5.4.

Proposition 5.8. *The vectors θ_i^q, $i=1,...,7$, form a complete set of linearly independent maximal vectors of the $\mathfrak{U}_q\mathfrak{p}_n$-module $V^{\otimes 3}$. Furthermore,*

$$\begin{aligned}
\theta_4^q,\quad \theta_1^q+q\theta_2^q+q^2\theta_3^q &\in y_{\ytableaushort{123}}V^{\otimes 3},\\
\theta_5^q,\quad -\theta_1^q-(q-q^{-1})\theta_2^q+\theta_3^q &\in y_{\ytableaushort{12,3}}V^{\otimes 3},\\
\theta_6^q,\quad -q\theta_2^q+\theta_3^q &\in y_{\ytableaushort{13,2}}V^{\otimes 3},\\
\theta_7^q &\in y_{\ytableaushort{1,2,3}}V^{\otimes 3}.
\end{aligned}$$

We will now look into each of the submodules in the decomposition of (13).

Proposition 5.9. *The module* $y_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{3}\end{smallmatrix}}V^{\otimes 3}$ *is reducible and indecomposable. More precisely, we have the following non-split short exact sequence*

$$0 \longrightarrow V^q(\epsilon_1+\epsilon_2+\epsilon_3) \longrightarrow y_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{3}\end{smallmatrix}}V^{\otimes 3} \longrightarrow V^q(\epsilon_1) \longrightarrow 0. \tag{14}$$

Proof. Consider the contraction map c_1. Let $\mathcal{N}$ be the kernel of the restriction of c_1 to $y_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{3}\end{smallmatrix}}V^{\otimes 3}$. We have that

$$\mathsf{c}_1 y_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{3}\end{smallmatrix}}V^{\otimes 3} \subset \mathsf{c}_1 V^{\otimes 3} \cong V = \mathbb{C}_q(n|n)$$

and

$$\mathsf{c}_1 y_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{3}\end{smallmatrix}}(u_1\otimes u_{-1}\otimes u_1) = \frac{1}{1+2q^{-2}+2q^{-4}+q^{-6}}\mathsf{c}_1[(1+q^2-q^{-4})u_1\otimes u_{-1}\otimes u_1 - q^{-2}u_1\otimes u_1\otimes u_{-1}+u_{-1}\otimes u_1\otimes u_1] \neq 0.$$

By Proposition 5.8, θ_7^q is the only maximal vector in $\mathcal{N}$. Hence,

$$V^q(\epsilon_1+\epsilon_2+\epsilon_3) \cong \mathfrak{U}_q\mathfrak{p}_n\theta_7^q = \mathcal{N} \subsetneq y_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{3}\end{smallmatrix}}V^{\otimes 3}.$$

Moreover,

$$y_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{3}\end{smallmatrix}}V^{\otimes 3}/\mathfrak{U}_q\mathfrak{p}_n\theta_7^q \cong V^q(\epsilon_1) \cong \mathbb{C}_q(n|n),$$

which implies the exact sequence. The sequence does not split because $y_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{3}\end{smallmatrix}}V^{\otimes 3}$ has a unique up to a scalar multiple maximal vector. □

Proposition 5.10. *The module* $y_{\boxed{1}\boxed{2}\boxed{3}}V^{\otimes 3}$ *is reducible and indecomposable. More precisely, we have the following non-split short exact sequence*

$$0 \longrightarrow V^q(\epsilon_1) \longrightarrow y_{\boxed{1}\boxed{2}\boxed{3}}V^{\otimes 3} \longrightarrow V^q(3\epsilon_1) \longrightarrow 0. \tag{15}$$

Proof. Let $\mathsf{K} = \mathsf{c}_1\mathsf{c}_2 + q\mathsf{t}_2\mathsf{c}_1\mathsf{c}_2 + q^2\mathsf{c}_2$. Note that

$$y_{\boxed{1}\boxed{2}\boxed{3}}\mathsf{K} = \mathsf{K}.$$

It follows that

$$\mathsf{K}V^{\otimes 3} \subset y_{\boxed{1}\boxed{2}\boxed{3}}V^{\otimes 3}.$$

Note that $\theta_4^q \neq \mathsf{K}V^{\otimes 3}$. However, $\mathsf{K}(u_1\otimes u_1\otimes u_{-1}) = \theta_1^q + q\theta_2^q + q^2\theta_3^q$. By Proposition 5.8, $\theta_1^q + q\theta_2^q + q^2\theta_3^q$ is the only, up to a scalar multiple, maximal vector in $\mathsf{K}V^{\otimes 3}$. Thus, we have that

$$V^q(\epsilon_1) \cong \mathfrak{U}_q\mathfrak{p}_n(\theta_1^q + q\theta_2^q + q^2\theta_3^q) \subsetneq y_{\boxed{1}\boxed{2}\boxed{3}}V^{\otimes 3}.$$

By Proposition 5.8, we have

$$y_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{3}\end{smallmatrix}}V^{\otimes 3}/\mathfrak{U}_q\mathfrak{p}_n(\theta_1^q + q\theta_2^q + q^2\theta_3^q) \cong V^q(3\epsilon_1).$$

which implies the exact sequence (15). Since θ_4^q generates $y_{\boxed{1}\boxed{2}}V^{\otimes 2}$, the sequence does not split. □

Proposition 5.11. *The module* $y_{\ytableaushort{12,3}}V^{\otimes 3}$ *is completely reducible as a* $\mathfrak{U}_q\mathfrak{p}_n$*-module into a direct sum of irreducible* $\mathfrak{U}_q\mathfrak{p}_n$*-modules. More precisely, we have the following split short exact sequence*

$$0 \longrightarrow V^q(2\epsilon_1+\epsilon_2) \longrightarrow y_{\ytableaushort{12,3}}V^{\otimes 3} \longrightarrow V^q(\epsilon_1) \longrightarrow 0. \tag{16}$$

Proof. Consider the $\mathfrak{U}_q\mathfrak{p}_n$-module homomorphism $T : y_{\ytableaushort{12,3}}V^{\otimes 3} \to \mathsf{c}_2V^{\otimes 3}$ such that $T(y_{\ytableaushort{12,3}}v) = \mathsf{c}_2 y_{\ytableaushort{12,3}}v$. Since

$$\mathsf{c}_2 y_{\ytableaushort{12,3}}\mathsf{c}_2 = \frac{q^{-2}}{q^2+1+q^{-2}}\mathsf{c}_2,$$

we have that T is surjective and

$$\mathsf{c}_2 y_{\ytableaushort{12,3}}V^{\otimes 3} = \mathsf{c}_2V^{\otimes 3}.$$

Note that

$$y_{\ytableaushort{12,3}}\mathsf{c}_2(u_1\otimes u_1\otimes u_{-1}) = -\theta_1^q - (q-q^{-1})\theta_2^q + \theta_3^q.$$

Thus, $-\theta_1^q - (q-q^{-1})\theta_2^q + \theta_3^q \in y_{\ytableaushort{12,3}}\mathsf{c}_2V^{\otimes 3}$. However, $u_1\otimes u_1\otimes u_2 \notin \mathsf{c}_2V^{\otimes 3}$, which implies that $\theta_5^q \in y_{\ytableaushort{12,3}}\mathsf{c}_2V^{\otimes 3}$. So, by Proposition 5.8, we have that

$$V^q(\epsilon_1) \cong \mathfrak{U}_q\mathfrak{p}_n\big(-\theta_1^q - (q-q^{-1})\theta_2^q + \theta_3^q\big) = y_{\ytableaushort{12,3}}\mathsf{c}_2V^{\otimes 3}.$$

Since $\mathsf{c}_2V^{\otimes 3} \cong V^q(\epsilon_1)$, we have an inclusion map $i : \mathsf{c}_2V^{\otimes 3} \to y_{\ytableaushort{12,3}}V^{\otimes 3}$ such that $i \circ T = \mathrm{id}$. Therefore, the following short exact sequence splits:

$$0 \longrightarrow \ker T \longrightarrow y_{\ytableaushort{12,3}}V^{\otimes 3} \xrightarrow{T} \mathsf{c}_2V^{\otimes 3} \longrightarrow 0$$

By Proposition 5.8, since $\theta_5^q \in \ker T$, we have that $\ker T = \mathfrak{U}_q\mathfrak{p}_n\theta_5^q \cong V^q(2\epsilon_1+\epsilon_2)$. This implies the exact sequence (16). □

Proposition 5.12. *The module* $y_{\ytableaushort{13,2}}V^{\otimes 3}$ *is isomorphic to* $y_{\ytableaushort{12,3}}V^{\otimes 3}$ *as* $\mathfrak{U}_q\mathfrak{p}_n$*-modules. More precisely, we have the following split short exact sequence*

$$0 \longrightarrow V^q(2\epsilon_1+\epsilon_2) \longrightarrow y_{\ytableaushort{13,2}}V^{\otimes 3} \longrightarrow V^q(\epsilon_1) \longrightarrow 0. \tag{17}$$

Proof. Let $v_1 = \theta_6^q$, $w_1 = \theta_5^q$, $v_2 = -q\theta_2^q + \theta_3^q$, and $w_2 = -\theta_1^q - (q-q^{-1})\theta_2^q + \theta_3^q$. Consider the $\mathfrak{U}_q\mathfrak{p}_n$-module homomorphism

$$S : y_{\ytableaushort{12,3}}V^{\otimes 3} \longrightarrow y_{\ytableaushort{13,2}}V^{\otimes 3}$$

such that

$$S(w_1) = v_1 \qquad \text{and} \qquad S(w_2) = v_2.$$

By Propositions 5.8 and 5.11, S is an isomorphism. The short exact sequence (17) then follows from Proposition 5.11. □

The following theorem is analogous to Theorem 5.7, but in the case of $V^{\otimes 3}$.

Theorem 5.13. *As a $\mathfrak{U}_q\mathfrak{p}_n$-module, we have the following decomposition*

$$V^{\otimes 3} = y_{\ytableaushort{123}}V^{\otimes 3} \oplus y_{\ytableaushort{12,3}}V^{\otimes 3} \oplus y_{\ytableaushort{13,2}}V^{\otimes 3} \oplus y_{\ytableaushort{1,2,3}}V^{\otimes 3}, \tag{18}$$

where each submodule in the above decomposition are involved in either the non-split short exact sequences (14) *and* (15)*, or in the split short exact sequences* (16) *and* (17)*.*

Proof. Let $T : V^{\otimes 3} \longrightarrow V^{\otimes 3}$ be the $\mathfrak{U}_q\mathfrak{p}_n$-module homomorphism defined by

$$T(v) = y_{\ytableaushort{123}}v.$$

Note that

$$y_{\ytableaushort{123}}\mathsf{K} = 0,$$

where $\mathsf{K} = y_{\ytableaushort{12,3}}, y_{\ytableaushort{13,2}}, y_{\ytableaushort{1,2,3}}$. By Proposition 5.8, this implies that $-\theta_1^q - (q-q^{-1})\theta_2^q + \theta_3^q, -q\theta_2^q + \theta_3^q, \theta_q^5, \theta_q^6, \theta_q^7 \in \ker T$. We have the short exact sequence

$$0 \longrightarrow \ker T \longrightarrow V^{\otimes 3} \longrightarrow y_{\ytableaushort{123}}V^{\otimes 3} \longrightarrow 0.$$

Using the embedding $\iota : y_{\ytableaushort{123}}V^{\otimes 3} \longrightarrow V^{\otimes 3}$ and that $T \circ \iota = \mathrm{id}$, we see that the sequence above splits.

Let $T' : \ker T \longrightarrow \ker T$ be the $\mathfrak{U}_q\mathfrak{p}_n$-module homomorphism defined by

$$T(v) = y_{\ytableaushort{12,3}}v.$$

Note that

$$y_{\ytableaushort{12,3}}\mathsf{K} = 0,$$

where $\mathsf{K} = y_{\ytableaushort{13,2}}, y_{\ytableaushort{1,2,3}}$. By Proposition 5.8, this implies that $-q\theta_2^q + \theta_3^q, \theta_q^6, \theta_q^7 \in \ker T'$. Using the embedding $\iota' : y_{\ytableaushort{12,3}}V^{\otimes 3} \longrightarrow V^{\otimes 3}$ and that $T' \circ \iota' = \mathrm{id}$, we see that the short exact sequence

$$0 \longrightarrow \ker T' \longrightarrow \ker T \longrightarrow y_{\ytableaushort{12,3}}V^{\otimes 3} \longrightarrow 0.$$

splits. Using similar arguments, we have that $\ker T' = y_{\ytableaushort{13,2}}V^{\otimes 3} \oplus y_{\ytableaushort{1,2,3}}V^{\otimes 3}$, and thus the decomposition (18) follows.

The remaining part of the theorem follows from Propositions 5.9, 5.10, 5.11, and 5.12. $\square$

5.4 Reducibility of $V^{\otimes k}$

Theorem 5.14. *For every $k \geq 2$, the $\mathfrak{U}_q\mathfrak{p}_n$-module $V^{\otimes k}$ is not completely reducible.*

Proof. Fix $k \geq 2$, and assume for the sake of contradiction that $V^{\otimes k}$ is completely reducible. For any $r, s \in \{1, 2 \ldots, k-1\}$ such that $r \neq s$,

$$\mathsf{c}_{r,s} V^{\otimes k} \cong V^{\otimes k-2}.$$

Consecutive applications of $\mathsf{c}_{r,s}$ to $V^{\otimes k}$ for appropriate r and s will lead to a submodule M of $V^{\otimes k}$ that is isomorphic either to $V^{\otimes 2}$ or to $V^{\otimes 3}$. By Theorems 5.7 and 5.13, $V^{\otimes 2}$ and $V^{\otimes 3}$ are not completely reducible. This leads to a contradiction as submodules of completely reducible modules are also completely reducible. $\square$

References

[AGG] S. Ahmed, D. Grantcharov, and N. Guay, *Quantized enveloping superalgebra of type P*, Lett. Math. Phys. **111** (2021), no. 3, Paper No. 84, 17.

[AM] Andersen, Mazorchuck, *Category $\mathcal{O}$ for quantum groups*, Journal of the European Mathematical Society **17** (2015), no. 2, 405–431.

[BDEA$^+$1] M. Balagovic, Z. Daugherty, I. Entova-Aizenbud, I. Halacheva, J. Hennig, M. S. Im, G. Letzter, E. Norton, V. Serganova, C. Stroppel, *The affine VW supercategory*, Selecta Math. (N.S.) **26** (2020), no. 2, Paper No. 20, 42 pp.

[BDEA$^+$2] M. Balagovic, Z. Daugherty, I. Entova-Aizenbud, I. Halacheva, J. Hennig, M. S. Im, G. Letzter, E. Norton, V. Serganova, C. Stroppel, *Translation functors and decomposition numbers for the periplectic Lie superalgebra $\mathfrak{p}(n)$*, Math. Res. Lett. **26** (2019), no. 3, 643–710.

[CP] C. Chen, Y. Peng, *Affine periplectic Brauer algebras*, J. Algebra **501** (2018), 345–372.

[Co] K. Coulembier, *The periplectic Brauer algebra*, Proc. Lond. Math. Soc. (3) **117** (2018), no. 3, 441–482.

[CE1] K. Coulembier, M. Ehrig, *The periplectic Brauer algebra II: Decomposition multiplicities*, J. Comb. Algebra **2** (2018), no. 1, 19–46.

[CE2] K. Coulembier, M. Ehrig, *The periplectic Brauer algebra III: The Deligne category*, Algebr Represent Theory (2020). https://doi.org/10.1007/s10468-020-09976-8.

[DKM] N. Davidson, J. R. Kujawa, and R. Muth, *Howe Duality of Type P*, arXiv e-prints (2021), arXiv:2109.03984.

[Dr] V. Drinfel'd, *Quantum groups*, Proceedings of the International Congress of Mathematicians, Vol. 1 (Berkeley, California, 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.

[DHIN] Z. Daugherty, I. Halacheva, M.S. Im, and E. Norton, *On calibrated representations of the degenerate affine periplectic Brauer algebra*, arXiv:1905.05148.

[EAS1] Inna Entova-Aizenbud, Vera Serganova, *Deligne categories and the periplectic Lie superalgebra*, arXiv:1807.09478.

[EAS2] Inna Entova-Aizenbud, Vera Serganova, *Kac-Wakimoto conjecture for the periplectic Lie superalgebra*, arXiv:1905.04712.

[FRT] L. Faddeev, N. Reshetikhin, L. Takhtajan, *Quantization of Lie groups and Lie algebras* (Russian), Algebra i Analiz **1** (1989), no. 1, 178–206; translation in Leningrad Math. J. **1** (1990), no. 1, 193–225.

[GJKK] D. Grantcharov, J.H. Jung., S.-J. Kang, M. Kim, *Highest weight modules over quantum queer superalgebra $U_q(\mathfrak{q}(n))$*, Comm. Math. Phys. **296** (2010), no. 3, 827-860.

[GJKKK] D. Grantcharov, J.H. Jung., S.-J. Kang, M. Kashiwara, M. Kim, *Quantum Queer Superalgebra and Crystal Bases*, Proc. Japan Acad. Ser. A Math. Sci. **86** (2010), no. 10, 177–182.

[Gy] A. Gyoja, *A q-Analogue of Young Symmetrizer*, Osaka J. Math **23** (1986), 841-852.

[HIR] C. Hoyt, M.S. Im, S. Reif, *Denominator identities for the periplectic Lie superalgebra*, J. Algebra, **567** (2021), 459–474.

[HK] J. Hong, S.-J. Kang, *Introduction to Quantum Groups and Crystal Bases*, Graduate Studies in Mathematics **42**, American Mathematical Society, 2002.

[IN] M.S. Im, E. Norton, *Irreducible calibrated representations of periplectic Brauer algebras and hook representations of the symmetric group*, Journal of Algebra. (2020), 560. 442-485. 10.1016/j.jalgebra.2020.05.035.

[IRS] M. S. Im, S. Reif, V. Serganova, *Grothendieck rings of periplectic Lie superalgebras*, Mathematical Research Letters. **28** (2021). 1175-1195.

[K1] V. Kac, *Lie superalgebras*, Advances in Math. **26** (1977), 8–96.

[K2] V. Kac, *Characters of typical representations of classical Lie superalgebras* , Comm. Algebra **5** (1977), 889–897.

[KT] J. Kujawa, B. Tharp, *The marked Brauer category*, J. Lond. Math. Soc. **95** (2017), no. 2, 393–413.

[Mo] D. Moon, *Tensor product representations of the Lie superalgebra $\mathfrak{p}(n)$ and their centralizers*, Comm. Algebra **31** (2003), 2095–2140.

[M] I. M. Musson, *Lie Superalgebras and Enveloping Algebras*, Graduate Studies in Mathematics **131**, American Mathematical Society, 2012.

[Ol] G. Olshanski, *Quantized universal enveloping superalgebra of type Q and a super-extension of the Hecke algebra*, Lett. Math. Phys. **24** (1992), 93–102.

[Ser] V. Serganova, *On representations of the Lie superalgebra $\mathfrak{p}(n)$*, J. Algebra **258** (2002), 615–630.

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