TWISTED YANGIANS, TWISTED QUANTUM LOOP ALGEBRAS AND AFFINE HECKE ALGEBRAS OF TYPE BC

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Abstract. We study twisted Yangians of type AIII which have appeared in the literature under the name of reflection algebras. They admit q-versions which are new twisted quantum loop algebras. We explain how these can be defined equivalently either via the reflection equation or as coideal subalgebras of Yangians of $\mathfrak{gl}_n$ (resp. of quantum loop algebras of $\mathfrak{gl}_n$). The connection with affine Hecke algebras of type BC comes from a functor of Schur-Weyl type between their module categories.

1. Introduction

Yangians are quantum groups of affine type with a plethora of applications in theoretical physics. They are Hopf algebras which are quantizations of the enveloping algebra of $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$, where $\mathfrak{g}$ is a finite dimensional, complex, semisimple Lie algebra (or $\mathfrak{gl}_n$). Twisted Yangians appeared almost twenty years ago in the work of G. Olshanski [Ol] and have been quite studied since then for the classical symmetric pairs $(\mathfrak{gl}_n(\mathbb{C}), \mathfrak{o}_n(\mathbb{C}))$, $(\mathfrak{gl}_{2n}(\mathbb{C}), \mathfrak{sp}_n(\mathbb{C}))$, that is, $\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} = \mathfrak{o}_n(\mathbb{C})$, $\mathfrak{p} = \mathrm{sym}_n(\mathbb{C})$ in the first case (where $\mathrm{sym}_n(\mathbb{C})$ is the space of $n \times n$ symmetric matrices), and $\mathfrak{k} = \mathfrak{sp}_n(\mathbb{C})$, $\mathfrak{p} = \mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{so}_n(\mathbb{C}) \oplus \mathfrak{so}_n(\mathbb{C})$ (as vector spaces only) in the second case. These are two of the three families of classical symmetric pairs of type A. In this paper, we focus on type AIII, which is the symmetric pair $(\mathfrak{gl}_n, \mathfrak{gl}_p \oplus \mathfrak{gl}_{n-p})$ with $0 \leq p \leq n-1$. (The numbering of these types originates from the classification of Riemannian symmetric spaces due to E. Cartan.) More precisely, $\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} = \mathfrak{gl}_p \oplus \mathfrak{gl}_{n-p}$ and $\mathfrak{p} = M_{p,n-p}(\mathbb{C}) \oplus M_{n-p,p}(\mathbb{C})$.

The twisted Yangians of type AIII were studied in [MoRa] under the name of reflection algebras (where they were denoted $B(n,l)$, l playing the role of p here), following the work of E. Sklyanin [Sk], and even more general twisted Yangians are the subject of the articles [Ma1, Ma2] of N. MacKay. However, it was not yet known if these two kinds of twisted Yangians for the symmetric pair $(\mathfrak{gl}_n, \mathfrak{gl}_p \oplus \mathfrak{gl}_{n-p})$ were (almost) isomorphic. This is one of the results of this paper (see Theorem 3.2 for the precise statement). In Section 3.5, we give a presentation in terms of generators and relations of MacKay's twisted Yangians when $n = 2p$. It should be noted that we consider twisted Yangians which depend on two deformation parameters: they appear, a priori, to be new algebras, but this is not really the case since one is a rescaling parameter and the dependency on the other parameter can be eliminated via a simple isomorphism (see Corollary 3.1). However, the two parameters in question are important for the construction of the Drinfeld functor in Section 4. The twisted Yangians are also coideal subalgebras inside the Yangian of $\mathfrak{gl}_n$, but are not Hopf algebras: see Proposition 3.2. An interesting recent paper about coideal subalgebras from the point of view of Manin triples is [BeCr].

The impetus for this paper came from a desire to generalize to the twisted Yangians of type AIII the work of S. Khoroshkin and M. Nazarov [KhNa1, KhNa2, KhNa3, KhNa4]. A starting point is the joint paper [EFM] of the third author in which a functor is constructed from a category of Harish-Chandra modules for the symmetric pair $(\mathfrak{gl}_n, \mathfrak{gl}_p \oplus \mathfrak{gl}_{n-p})$ to the category of modules over a degenerate affine Hecke algebra of type BC, extending the construction in [ArSu], which is originally due to I. Cherednik [Ch2] and was used by S. Khoroshkin and M. Nazarov in their aforementioned work. A second ingredient used by these authors (in the case of $\mathfrak{gl}_n$) is a functor, due originally to V. Drinfeld [Dr], which generalizes the classical Schur-Weyl functor to Yangians and degenerate affine Hecke algebras of type A. Another of our results is the construction of an analog of the Drinfeld functor from

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modules over the degenerate affine Hecke algebra of type BC to the category of left modules over a twisted Yangian of type AIII - see Theorem 4.1. (It should be noted, as pointed out in [KhNa3, KhNa4], that no such functor exists for the other two classical symmetric pairs of type A above.) It is simpler to obtain this functor using MacKay's presentation of twisted Yangians, but we are also able to define it in terms of the generators used by A. Molev and E. Ragoucy in [MoRa]: this is done in Theorem 4.2.

In trying to extend the work of S. Khoroshkin and M. Nazarov, one of the first obstacles is that the composite of the two functors discussed in the previous paragraph does not seem to correspond to a homomorphism from the twisted Yangian of type AIII to an algebra of the form $\mathfrak{U}\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{PD}(\mathbb{C}^l \otimes \mathbb{C}^k)$, where $\mathfrak{g}$ should be a Lie algebra part of a certain Howe dual pair and $\mathcal{PD}(\mathbb{C}^l \otimes \mathbb{C}^k)$ is the algebra of polynomial differential operators on $\mathbb{C}^l \otimes \mathbb{C}^k$. It is not clear what is the proper substitute for $\mathfrak{U}\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{PD}(\mathbb{C}^l \otimes \mathbb{C}^k)$, so we do not have an analogue of proposition 1.3 in [KhNa1] in type AIII. This appears to be essentially due to the fact that no simple formula like (1.14) in [KhNa1] is known for the Drinfeld functor studied below. The extension of the work of S. Khoroshkin and M. Nazarov to the symmetric pair $(\mathfrak{gl}_n, \mathfrak{gl}_p \oplus \mathfrak{gl}_{n-p})$ will hopefully be the subject of future work. At least, we can provide one application of the Drinfeld functor, namely, the construction in Section 5 of a Fock space representation of the twisted Yangian of type AIII, thus extending work of D. Uglov for the Yangian of $\mathfrak{gl}_n$ [Ug].

In the second part of the current article, we introduce new twisted quantum loop algebras of type AIII which can be viewed as q-versions of the twisted Yangians of type AIII. This answers at least partially a question raised in [MRS]. Recently, a general framework for understanding twisted quantum loop algebras has been developed by S. Kolb via quantum symmetric pairs for Kac-Moody algebras [Ko]. We prove that our new twisted quantum loop algebras can be defined equivalently as either coideal subalgebras of the quantum loop algebra of $\mathfrak{gl}_n$ or using a reflection equation with parameters (Theorem 6.3). We also show that their quasi-classical limit is the enveloping algebra of a certain twisted loop algebra (Corollary 6.1). The second main result related to these new twisted quantum loop algebras $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$ is the construction of a Drinfeld functor between categories of modules over affine Hecke algebras of type BC and over $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$ (Theorem 7.2). The two parameters on which this affine Hecke algebra depends match exactly the two parameters q and ξ which enter the formula for the embedding of $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$ in $\mathfrak{U}\mathfrak{gl}_n$. In the last section, we determine a family of central elements in $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$ via an approach similar to the one used in [MRS] for twisted q-Yangians.

It is suggested in [NSS] to call generalized Onsager algebra any twisted loop algebra obtained as the fixed-point set of an automorphism ρ of $\mathfrak{sl}_n(\mathbb{C}[t,t^{-1}])$ of the form $\rho(X \otimes p(t)) = \rho_0(X) \otimes p(t^{-1})$, where $p(t) \in \mathbb{C}[t,t^{-1}]$ and ρ_0 is an automorphism of $\mathfrak{sl}_n$. ($\mathfrak{sl}_n$ could be replaced by $\mathfrak{gl}_n$.) Our twisted quantum loop algebras can thus be viewed as generalized q-Onsager algebras. Other such algebras are studied in [BaBe, BeFo], where a very broad class of reflection algebras associated to quantum affine algebras is considered. The paper [NSS] provides general results about irreducible finite dimensional representations of equivariant map algebras and the classification of such representations for generalized Onsager algebras is contained there: we include this result in Theorem 6.1.

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2. Affine Hecke algebras of type BC

We need to recall a couple of definitions and results about the affine Hecke algebras of type BC and their degenerate version. The symmetric group on l elements will be denoted $\mathfrak{S}_l$ and we set $\Gamma = \mathbb{Z}/2\mathbb{Z}$, so that the wreath product $W_l = \Gamma \wr \mathfrak{S}_l$ is the Weyl group of type BC_l. The non-reduced root system of type BC_l consists of the following set of vectors:

$$\{\pm e_i + e_j, \pm e_i - e_j | 1 \le i \ne j \le l\} \cup \{\pm e_i, \pm 2e_i | 1 \le i \le l\} \subset \mathbb{R}^l = \mathrm{Span}_{\mathbb{R}}\{e_1, \dots, e_l\},$$

where $\{e_i\}_{i=1}^l$ is the standard basis of $\mathbb{R}^l$. Let σ_{ij} be the reflection corresponding to the root $e_i - e_j$, set $\sigma_i = \sigma_{i,i+1}$ and let γ_i be the reflection corresponding to e_i.

Definition 2.1. *Let $\kappa_1, \kappa_2 \in \mathbb{C}$. The degenerate affine Hecke algebra $\mathtt{H}^l_{\kappa_1,\kappa_2}$ of type BC_l is the algebra generated by the group algebra $\mathbb{C}[W_l]$ and a set of pairwise commuting elements $y_1, \ldots, y_l$ such that*

$$\sigma_i y_i - y_{i+1}\sigma_i = \kappa_1 \text{ for } 1 \le i \le l-1, \;\; \sigma_i y_j = y_j \sigma_i \text{ if } j \neq i, i+1,$$
$$\gamma_l y_l + y_l \gamma_l = \kappa_2, \;\; \gamma_l y_j = y_j \gamma_l \text{ if } j \neq l.$$

Lemma 2.1. *The subalgebra of $\mathtt{H}^l_{\kappa_1,\kappa_2}$ generated by $y_i, 1 \le i \le l$, and $\mathfrak{S}_l$ is isomorphic to the degenerate affine Hecke algebra $\mathtt{H}^l_{\kappa_1}$ of type GL_l.*

Note that, for any $1 \le l_2 \le l$, we have an embedding $\iota_2 : \mathtt{H}^{l_2}_{\kappa_1,\kappa_2} \hookrightarrow \mathtt{H}^l_{\kappa_1,\kappa_2}$ by considering the generators $y_{l-l_2+1}, \ldots, y_l, \gamma_{l-l_2+1}, \ldots, \gamma_l$ and $\sigma_{l-l_2+1}, \ldots, \sigma_{l-1}$ of $\mathtt{H}^l_{\kappa_1,\kappa_2}$. For any $1 \le l_1 \le l$, we also have an embedding $\iota_1 : \mathtt{H}^{l_1}_{\kappa_1} \hookrightarrow \mathtt{H}^l_{\kappa_1,\kappa_2}$ by considering the generators $y_1, \ldots, y_{l_1}, \sigma_1, \ldots, \sigma_{l_1-1}$ of $\mathtt{H}^l_{\kappa_1,\kappa_2}$. Moreover, if $l_1 + l_2 \le l$, we can combine ι_1 and ι_2 to obtain an embedding $\iota_1 \otimes \iota_2 : \mathtt{H}^{l_1}_{\kappa_1} \otimes_{\mathbb{C}} \mathtt{H}^{l_2}_{\kappa_1,\kappa_2} \hookrightarrow \mathtt{H}^l_{\kappa_1,\kappa_2}$. However, $\iota_1 \otimes \iota_2$ does not extend to an embedding $\mathtt{H}^{l_1}_{\kappa_1,\kappa_2} \otimes_{\mathbb{C}} \mathtt{H}^{l_2}_{\kappa_1,\kappa_2} \hookrightarrow \mathtt{H}^l_{\kappa_1,\kappa_2}$ because, if $i < j$, $[\gamma_i, y_j] = \kappa_1 \sigma_{ij}(\gamma_i - \gamma_j)$.

We will need an equivalent definition of the degenerate affine Hecke algebra $\mathtt{H}^l_{\kappa_1,\kappa_2}$.

Lemma 2.2 ([EFM], lemma 3.1)**.** *$\mathtt{H}^l_{\kappa_1,\kappa_2}$ is isomorphic to the algebra generated by elements $\widetilde{y}_i, 1 \le i \le l$, and by $\mathbb{C}[W_l]$ with the following relations:*

$$\sigma_i \widetilde{y}_i = \widetilde{y}_{i+1}\sigma_i, \;\; \sigma_i \widetilde{y}_j = \widetilde{y}_j \sigma_i \text{ if } j \neq i, i+1, \;\; \widetilde{y}_l \gamma_l = -\gamma_l \widetilde{y}_l, \;\; \widetilde{y}_i \gamma_l = \gamma_l \widetilde{y}_i \text{ if } i \neq l,$$

$$\begin{aligned}[\widetilde{y}_i, \widetilde{y}_j] \;&=\; \frac{\kappa_1\kappa_2}{2}\sigma_{ij}(\gamma_j - \gamma_i) + \frac{\kappa_1^2}{4}\sum_{\substack{k=1\\k\neq i,j}}^{l} \big((\sigma_{jk}\sigma_{ik} - \sigma_{ik}\sigma_{jk}) \\ &\quad + \sigma_{ik}\sigma_{jk}(-\gamma_i\gamma_k + \gamma_i\gamma_j + \gamma_j\gamma_k) - \sigma_{jk}\sigma_{ik}(\gamma_i\gamma_j - \gamma_j\gamma_k + \gamma_i\gamma_k)\big). \qquad (1)\end{aligned}$$

Proof. The connection between the two presentations is given by

$$\widetilde{y}_i = y_i - \frac{\kappa_2}{2}\gamma_i - \frac{\kappa_1}{2}\sum_{k=i+1}^{l}\sigma_{ik} + \frac{\kappa_1}{2}\sum_{k=1}^{i-1}\sigma_{ik} - \frac{\kappa_1}{2}\sum_{\substack{k=1\\k\neq i}}^{l}\sigma_{ik}\gamma_i\gamma_k.$$

□

Lemma 2.3 ([Lu], 3.12)**.** *The center of the degenerate affine Hecke algebra $\mathtt{H}^l_{\kappa_1}$ (resp. of $\mathtt{H}^l_{\kappa_1,\kappa_2}$) is generated by the $\mathfrak{S}_l$-symmetric polynomials in the variables $y_1, \ldots, y_l$ (resp. in the variables $y_1^2, \ldots, y_l^2$).*

Let us now move on to the non-degenerate case. Strictly speaking, the next definition is the one for the affine Hecke algebra of type $\mathfrak{gl}_l$, which is also called the extended affine Hecke algebra of type A_{l-1}. To simplify the terminology and the notation, we will say that it is the one of type A_{l-1}.

Definition 2.2. *Let $\kappa \in \mathbb{C}^\times$. The affine Hecke algebra of type A_{l-1}, denoted $\mathbf{H}^l_\kappa$, is the unital associative algebra with generators $\sigma_1^{\pm 1}, \ldots, \sigma_{l-1}^{\pm 1}$, $Y_1^{\pm 1}, \ldots, Y_l^{\pm 1}$ satisfying the relations:*

(a) $\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1$ *for* $1 \le i \le l-1$, $\quad Y_jY_j^{-1} = Y_j^{-1}Y_j = 1$, $\quad Y_iY_j = Y_jY_i$ *for all* $1 \le i, j \le l$;
(b) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ *if* $1 \le i \le l-2$, $\quad \sigma_i\sigma_j = \sigma_j\sigma_i$ *if* $|i-j| > 1$ (*braid relations of type A*);
(c) $(\sigma_i + 1)(\sigma_i - \kappa^2) = 0$ *for* $1 \le i \le l-1$ (*Hecke relations*);
(d) $Y_j\sigma_i = \sigma_iY_j$ *if* $j \neq i, i+1$, $\quad \sigma_iY_i\sigma_i = \kappa Y_{i+1}$ $1 \le i \le l-1$.

If we delete the generators $Y_i^{\pm 1}$ and the corresponding relations, we obtain a subalgebra $\mathcal{H}^l_\kappa$ which is the finite Hecke algebra of type A_{l-1}. If we also delete the Hecke relations, we get the group ring of the braid group $\mathtt{B}^A_l$.

Definition 2.3. *Let $\kappa_1, \kappa_2 \in \mathbb{C}^\times$. The affine Hecke algebra $\mathbf{H}^l_{\kappa_1,\kappa_2}$ of type B_l is the associative algebra with generators $\sigma_1^{\pm 1}, \ldots, \sigma_l^{\pm 1}$, $Y_1^{\pm 1}, \ldots, Y_l^{\pm 1}$ such that*

(a) the generators $\sigma_1^{\pm 1}, \ldots, \sigma_{l-1}^{\pm 1}$, $Y_1^{\pm 1}, \ldots, Y_l^{\pm 1}$ *satisfy the same relations as those in the definition of* $\mathbf{H}_{\kappa_1}^l$;
(b) $\sigma_l \sigma_l^{-1} = \sigma_l^{-1}\sigma_l = 1$;
(c) $\sigma_l\sigma_{l-1}\sigma_l\sigma_{l-1} = \sigma_{l-1}\sigma_l\sigma_{l-1}\sigma_l$, $\sigma_l\sigma_i = \sigma_i\sigma_l$ *if* $i \neq l-1$ *(braid relations of type B);*
(d) $(\sigma_l + 1)(\sigma_l - \kappa_2^2) = 0$ *(Hecke relation);*
(e) $\sigma_l Y_l \sigma_l = \kappa_1^{2l}\kappa_2^2 Y_l^{-1}$, $\sigma_l Y_i = Y_i \sigma_l$ *if* $i \neq l$.

If we delete the generators $Y_i^{\pm 1}$ *and the corresponding relations, we obtain a subalgebra* $\mathcal{H}_{\kappa_1,\kappa_2}^l$ *which is the finite Hecke algebra of type* B_l. *If we do not impose the Hecke relations on* σ_i *for* $i = 1, \ldots, l$, *we get the group ring of the braid group* B_l^B *of type* B.

We will need the following lemma later.

Lemma 2.4 (See [KaLu], 4.4). *(1) Let* $\mathbb{C}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}]^{\mathfrak{S}_l}$ *be the ring of Laurent polynomials invariant under the permutation action of* $\mathfrak{S}_l$. *Then* $f(\kappa^{\mp 1}Y_1^{\pm 1}, \ldots, \kappa^{\mp l}Y_l^{\pm 1})$ *lies in the center of the affine Hecke algebra* $\mathbf{H}_\kappa^l$ *for any* $f(x_1, \ldots, x_l) \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}]^{\mathfrak{S}_l}$.

(2) Let $\mathbb{C}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}]^{W_l}$ *be the polynomials invariant under the action of* W_l. *Then* $f(\kappa_1^{\mp 1}Y_1^{\pm 1}, \ldots, \kappa_1^{\mp l}Y_l^{\pm 1})$ *lies in the center of the affine Hecke algebra* $\mathbf{H}_{\kappa_1,\kappa_2}^l$ *for any* $f(x_1^{\pm 1}, \ldots, x_l^{\pm 1}) \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_l^{\pm}]^{W_l}$. *Here* σ_i *acts by permutation of the indices and* $\sigma_l(x_l^{\pm 1}) = x_l^{\mp 1}, \sigma_l(x_i^{\pm 1}) = x_i^{\pm 1}$ *if* $i = 1, \ldots, l-1$.

3. Twisted Yangians of type AIII and the reflection equation with parameters

3.1. Yangians for $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{sl}_n(\mathbb{C})$.

Definition 3.1. *Suppose that* $n \geq 3$, $\lambda \in \mathbb{C}$. *Let* $\{z_\alpha\}_{\alpha \in I}$ *be an orthonormal basis of* $\mathfrak{sl}_n(\mathbb{C})$ *with respect to the Killing form and indexed by some set* I. *The Yangian* $Y_\lambda(\mathfrak{sl}_n)$ *is the complex, unital, associative algebra generated by elements* $z, J(z)$ *for* $z \in \mathfrak{sl}_n(\mathbb{C})$ *satisfying the relations*

$$J(az_1 + bz_2) = aJ(z_1) + bJ(z_2), \quad [J(z_1), z_2] = J([z_1, z_2]),$$

$$[J(z_1), J([z_2, z_3])] + [J(z_2), J([z_3, z_1])] + [J(z_3), J([z_1, z_2])] = \lambda^2 \sum_{\alpha,\beta,\gamma \in I} \Big([z_1, z_\alpha], \big[[z_2, z_\beta], [z_3, z_\gamma]\big]\Big)\{z_\alpha, z_\beta, z_\gamma\}$$

where $\{z_\alpha, z_\beta, z_\gamma\} = \dfrac{1}{24}\sum_{\sigma \in \mathfrak{S}_3} z_{\sigma(\alpha)} z_{\sigma(\beta)} z_{\sigma(\gamma)}$ *and* $\mathfrak{S}_3$ *is the permutation group of* $\{\alpha, \beta, \gamma\}$.

Note that $Y_{\lambda_1}(\mathfrak{sl}_n) \cong Y_{\lambda_2}(\mathfrak{sl}_n)$ if $\lambda_1\lambda_2 \neq 0$. It will be more convenient to work with the following slightly bigger algebra.

Definition 3.2. *We let* $\widetilde{Y}_\lambda(\mathfrak{gl}_n)$ *be the algebra which is defined exactly as* $Y_\lambda(\mathfrak{sl}_n)$, *except that the elements* z *can be taken in all of* $\mathfrak{gl}_n(\mathbb{C})$ *and the set* $\{z_\alpha\}_{\alpha \in I}$ *should be an orthonormal basis of* $\mathfrak{gl}_n(\mathbb{C})$ *with respect to the Killing form.*

Definition 3.3. *The Yangian* $Y(\mathfrak{gl}_n)$ *is the complex, unital, associative algebra generated by elements* $T_{ij}^{(r)}$ *for* $1 \leq i, j \leq n, r \in \mathbb{Z}_{\geq 0}$ *with* $T_{ij}^{(0)} = \delta_{ij}$ *and satisfying the following relation:*

$$[T_{ij}(u), T_{st}(v)] = \frac{1}{u - v}\big(T_{sj}(u)T_{it}(v) - T_{sj}(v)T_{it}(u)\big), \tag{2}$$

where $T_{ij}(u) = \sum_{r=0}^\infty T_{ij}^{(r)} u^{-r} \in Y(\mathfrak{gl}_n)[[u^{-1}]]$.

The defining relations of $Y(\mathfrak{gl}_n)$ can be rewritten in terms of the R-matrix

$$R(u) = 1 - \frac{1}{u}\sum_{i,j=1}^n E_{ij} \otimes E_{ji}, \text{ where } E_{ij} \text{ is the usual elementary matrix}$$

and $T(u) = \sum_{i,j=1}^n T_{ij}(u) \otimes E_{ij} \in Y(\mathfrak{gl}_n)[[u^{-1}]] \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ as

$$R(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u - v).$$

Proposition 3.1. [MNO] *If $\lambda \neq 0$, $Y_\lambda(\mathfrak{sl}_n)$ is isomorphic to the subalgebra of $Y(\mathfrak{gl}_n)$ (denoted $Y(\mathfrak{sl}_n)$) which consists of the elements fixed under all automorphisms of the type $T(u) \mapsto f(u)T(u)$ where $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. This isomorphism can be extended to an embedding $\widetilde{Y}_\lambda(\mathfrak{gl}_n) \hookrightarrow Y(\mathfrak{gl}_n)$. Moreover, if Z denotes the center of $Y(\mathfrak{gl}_n)$, then $Y(\mathfrak{gl}_n) \cong Z \otimes_{\mathbb{C}} Y(\mathfrak{sl}_n)$.*

3.2. **Symmetric pair of type AIII in classical Lie theory.** Let $n \geq 2$ and let $1 \leq p \leq n/2$ with p an integer. Denote by Θ_p the $n \times n$ diagonal matrix $\Theta_p = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$, where $\epsilon_i = 1$ for $i = 1, \ldots, p$ and $\epsilon_i = -1$ for $i = p+1, \ldots, n$. Θ_p can be used to construct a Lie algebra involution θ of $\mathfrak{gl}_n(\mathbb{C})$: we set $\theta(X) = \Theta_p X \Theta_p$ for $X \in \mathfrak{gl}_n(\mathbb{C})$. Let $\mathfrak{k} = \{X \in \mathfrak{gl}_n(\mathbb{C}) | \theta(X) = X\} \cong \mathfrak{gl}_p \oplus \mathfrak{gl}_{n-p}$ and $\mathfrak{p} = \{X \in \mathfrak{gl}_n(\mathbb{C}) | \theta(X) = -X\}$, so that $\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{k} \oplus \mathfrak{p}$. The involution θ restricts to $\mathfrak{sl}_n(\mathbb{C})$ and we set $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{sl}_n(\mathbb{C})$.

Definition 3.4. *The pair $(\mathfrak{gl}_n(\mathbb{C}), \mathfrak{k})$ (or $(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{k}_0)$) is called the symmetric pair of type AIII. (This terminology comes from the classification of Riemannian symmetric spaces by E. Cartan.)*

Definition 3.5. *The twisted current Lie algebra $\mathfrak{sl}_n^\theta(\mathbb{C}[t])$ is equal to $\{X \otimes p(t) \in \mathfrak{sl}_n(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t] | \theta(X) \otimes p(t) = X \otimes p(-t)\}$. $\mathfrak{gl}_n^\theta(\mathbb{C}[t])$ is defined similarly.*

3.3. **Twisted Yangians of type AIII.** We will denote by $\widetilde{T}_{ij}(u)$ the matrix entries of $T^{-1}(u)$.

Definition 3.6. *Assume that $\tau_1 \neq 0$. The twisted Yangian of type AIII, $B_{\tau_1,\tau_2}(n,p)$, is the subalgebra of $Y(\mathfrak{gl}_n)$ generated by $b_{ij}^{(r)}, 1 \leq i,j \leq n$, $r \in \mathbb{Z}_{\geq 0}$ with $b_{ij}^{(0)} = \epsilon_i \delta_{ij}$ and, if $r \geq 1$,*

$$b_{ij}^{(r)} = \tau_1^{r-1} \sum_{s=0}^{r} \sum_{k=1}^{n} (-1)^{r-s} \epsilon_k T_{ik}^{(s)} \widetilde{T}_{kj}^{(r-s)} - \tau_1^{r-1}\tau_2 \sum_{s=0}^{r-1} \sum_{k=1}^{n} (-1)^{r-s} T_{ik}^{(s)} \widetilde{T}_{kj}^{(r-s-1)}.$$

Definition 3.7. *$SB_{\tau_1,\tau_2}(n,p)$ is defined as the intersection of $B_{\tau_1,\tau_2}(n,p)$ with $Y(\mathfrak{sl}_n)$.*

Set $b_{ij}(u) = \delta_{ij}\epsilon_i + \sum_{r=1}^{\infty} \tau_1^{1-r} b_{ij}^{(r)} u^{-r}$ and $B(u) = \sum_{i,j=1}^{n} b_{ij}(u) \otimes E_{ij} \in B_{\tau_1,\tau_2}(n,p)[[u^{-1}]] \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$. Then we can express the embedding $B_{\tau_1,\tau_2}(n,p) \hookrightarrow Y(\mathfrak{gl}_n)$ via

$$B(u) = T(u)\Theta_{p,\tau_2}(u)T^{-1}(-u),$$

where $\Theta_{p,\tau_2}(u) = 1 \otimes \left(\Theta_p + \tau_2 u^{-1}\right) \in Y(\mathfrak{gl}_n)[[u^{-1}]] \otimes_{\mathbb{C}} \mathrm{End}\,(\mathbb{C}^n)$.

It follows immediately that the equation

$$B(u)B(-u) = (1 - \tau_2^2 u^{-2}) \tag{3}$$

is satisfied.

Proposition 3.2 (Proposition 3.3 in [MoRa]). *$B_{\tau_1,\tau_2}(n,p)$ is a left coideal subalgebra in $Y(\mathfrak{gl}_n)$ with coproduct given by*

$$\Delta(b_{ij}(u)) = \sum_{s,t=1}^{n} T_{is}(u)\widetilde{T}_{tj}(-u) \otimes b_{st}(u).$$

Furthermore, we have the following result.

Proposition 3.3. *The twisted Yangian $B_{\tau_1,\tau_2}(n,p)$ satisfies the reflection equation*

$$R(u-v)B_1(u)R(u+v)B_2(v) = B_2(v)R(u+v)B_1(u)R(u-v), \tag{4}$$

where $B_1(u) = \sum_{ij} b_{ij}(u) \otimes E_{ij} \otimes 1$, $B_2(u) = \sum_{ij} b_{ij}(u) \otimes 1 \otimes E_{ij} \in B_{\tau_1,\tau_2}(n,p)[[u^{-1}]] \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)^{\otimes 2}$.

Proof. Following the same steps as in the proof of theorem 3.1 in [MoRa], we are reduced to proving the following equality:

$$R(u-v)\Theta_{p,\tau_2}^1(u)R(u+v)\Theta_{p,\tau_2}^2(v) = \Theta_{p,\tau_2}^2(v)R(u+v)\Theta_{p,\tau_2}^1(u)R(u-v).$$

This can be proved by direct calculation as follows:

$$
\begin{aligned}
&(u^2-v^2)R(u-v)\Theta^1_{p,\tau_2}(u)R(u+v)\Theta^2_{p,\tau_2}(v)\\
&\quad=\Big((u-v)-\sum_{i,j=1}^n E_{ij}\otimes E_{ji}\Big)\Big(\sum_{i=1}^n(\epsilon_i+\tau_2u^{-1})E_{ii}\otimes \mathrm{Id}\Big)\Big((u+v)-\sum_{i,j=1}^n E_{ij}\otimes E_{ji}\Big)\Big(\sum_{i=1}^n(\epsilon_i+\tau_2v^{-1})\mathrm{Id}\otimes E_{ii}\Big)\\
&\quad=\sum_{i,j=1}^n(u^2-v^2)(\epsilon_i+\tau_2u^{-1})(\epsilon_j+\tau_2v^{-1})E_{ii}\otimes E_{jj}-\sum_{i,j=1}^n(u+v)(\epsilon_i+\tau_2u^{-1})(\epsilon_j+\tau_2v^{-1})E_{ji}\otimes E_{ij}\\
&\qquad-\sum_{i,j=1}^n(u-v)(\epsilon_i+\tau_2u^{-1})(\epsilon_i+\tau_2v^{-1})E_{ij}\otimes E_{ji}+\sum_{i,j=1}^n(\epsilon_i+\tau_2u^{-1})(\epsilon_i+\tau_2v^{-1})E_{jj}\otimes E_{ii}\\
&\quad=\sum_{i,j=1}^n(\epsilon_j+\tau_2v^{-1})(\tau_2u^{-1}(1+u^2-v^2)+\epsilon_j+\epsilon_i(u^2-v^2))E_{ii}\otimes E_{jj}\\
&\qquad-\sum_{i,j=1}^n(\epsilon_i+\tau_2v^{-1})(2\tau_2+\epsilon_i(u-v)+\epsilon_j(u+v))E_{ij}\otimes E_{ji},
\end{aligned}
$$

$$
\begin{aligned}
&(u^2-v^2)\Theta^2_{p,\tau_2}(v)R(u+v)\Theta^1_{p,\tau_2}(u)R(u-v)\\
&\quad=\Big(\sum_{i=1}^n(\epsilon_i+\tau_2v^{-1})\mathrm{Id}\otimes E_{ii}\Big)\Big((u+v)-\sum_{i,j=1}^n E_{ij}\otimes E_{ji}\Big)\Big(\sum_{i=1}^n(\epsilon_i+\tau_2u^{-1})E_{ii}\otimes \mathrm{Id}\Big)\Big((u-v)-\sum_{i,j=1}^n E_{ij}\otimes E_{ji}\Big)\\
&\quad=\sum_{i,j=1}^n(u^2-v^2)(\epsilon_j+\tau_2v^{-1})(\epsilon_i+\tau_2u^{-1})E_{ii}\otimes E_{jj}-\sum_{i,j=1}^n(u-v)(\epsilon_i+\tau_2v^{-1})(\epsilon_i+\tau_2u^{-1})E_{ji}\otimes E_{ij}\\
&\qquad-\sum_{i,j=1}^n(u+v)(\epsilon_j+\tau_2v^{-1})(\epsilon_i+\tau_2u^{-1})E_{ij}\otimes E_{ji}+\sum_{i,j=1}^n(\epsilon_i+\tau_2v^{-1})(\epsilon_i+\tau_2u^{-1})E_{jj}\otimes E_{ii}\\
&\quad=\sum_{i,j=1}^n(\epsilon_j+\tau_2v^{-1})(\tau_2u^{-1}(1+u^2-v^2)+\epsilon_j+\epsilon_i(u^2-v^2))E_{ii}\otimes E_{jj}\\
&\qquad-\sum_{i,j=1}^n(\epsilon_j+\tau_2v^{-1})(2\tau_2+\epsilon_j(u-v)+\epsilon_i(u+v))E_{ij}\otimes E_{ji}.
\end{aligned}
$$

Notice that

$$(\epsilon_i+\tau_2v^{-1})(2\tau_2+\epsilon_i(u-v)+\epsilon_j(u+v))-(\epsilon_j+\tau_2v^{-1})(2\tau_2+\epsilon_j(u-v)+\epsilon_i(u+v))=0,$$

which implies the conclusion. ☐

Theorem 3.1. *Assume $\tau_1\neq 0$. The reflection equation (4), the unitary relation (3), and $b^{(0)}_{ij}=\delta_{ij}\epsilon_i$ $(1\le i,j\le n)$ are the defining relations for the twisted Yangian $B_{\tau_1,\tau_2}(n,p)$.*

Proof. The argument is the same as in the proof of theorem 3.1 in [MoRa]. ☐

In [MoRa] (see also [Sk]), the authors also consider the algebra $\widetilde{\mathcal{B}}(n,p)$ which is generated by $b^{(r)}_{ij}$ with defining relation given only by the reflection equation (4). They prove that, in $\widetilde{\mathcal{B}}(n,p)$, $\widetilde{\mathcal{B}}(u)\widetilde{\mathcal{B}}(-u)=f(u)\mathrm{Id}$ where $f(u)$ is an even series in u^{-1} with coefficients in the center of $\widetilde{\mathcal{B}}(n,p)$, $\widetilde{\mathcal{B}}(u)$ is the matrix $\sum_{i,j=1}^n\sum_{r=0}^\infty \widetilde{\mathsf{b}}^{(r)}_{ij}u^{-r}\otimes E_{ij}\in \widetilde{\mathcal{B}}(n,p)[[u^{-1}]]\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ and $\{\widetilde{\mathsf{b}}^{(r)}_{ij}\}$ is the set of generators of $\widetilde{\mathcal{B}}(n,p)$. The quotient of $\widetilde{\mathcal{B}}(n,p)$ by the ideal $(f(u)-1)$ is the reflection algebra denoted $\mathcal{B}(n,p)$ in [MoRa]. Our twisted Yangian $B_{\tau_1,\tau_2}(n,p)$ is exactly the quotient of $\widetilde{\mathcal{B}}(n,p)$ by the relation $B(u)B(-u)=(1-\tau_2^2u^{-2})\mathrm{Id}$. In particular, when $\tau_2=0$, we get the algebra $\mathcal{B}(n,p)$ studied in [MoRa]. More generally, we have the following corollary.

Corollary 3.1. *For $\tau_1 \neq 0$ and any $\tau_2 \in \mathbb{C}$, we have an isomorphism $B_{\tau_1,\tau_2}(n,p) \cong \mathcal{B}(n,p)$.*

Proof. Set $g(u) = 1 - \tau_2 u^{-1}$. An isomorphism $\psi : B_{\tau_1,\tau_2}(n,p) \xrightarrow{\sim} \mathcal{B}(n,p)$ is given by $\psi(B(u)) = g(u)\mathcal{B}(u)$, where $\mathcal{B}(u)$ is defined similarly to $\widetilde{\mathcal{B}}(u)$. □

Corollary 3.2 (PBW basis for twisted reflection algebra)**.** *The set of ordered monomials (under arbitrary total ordering) in the generators*

$$b_{ij}^{(2k-1)}, \quad 0 \leq i,j \leq p \ \text{or} \quad p+1 \leq i,j \leq n,$$

$$b_{ij}^{(2k)}, \quad 1 \leq i \leq p < j \leq n \ \text{or} \quad 1 \leq j \leq p < i \leq n,$$

for $k \geq 1$ form a PBW-type basis of the twisted reflection algebra $B_{\tau_1,\tau_2}(n,p)$.

Proof. Let $\{\mathsf{b}_{ij}^{(r)}\}$ be the set of generators of $\mathcal{B}(n,p)$ as in [MoRa]. Using the isomorphism in Corollary 3.1, we have $\psi(b_{ij}^{(0)}) = \mathsf{b}_{ij}^{(0)}$, $\psi(b_{ij}^{(r)}) = \tau_1^{r-1}(\mathsf{b}_{ij}^{(r)} - \tau_2 \mathsf{b}_{ij}^{(r-1)})$ for $r \geq 1$. The corollary follows from corollary 3.2 in [MoRa]. □

Now define a filtration on $B_{\tau_1,\tau_2}(n,p)$ by setting $\deg(b_{ij}^{(r)}) = r-1$ for $r \geq 1$ and $\deg(b_{ij}^{(0)}) = 0$. Then we can define the associated graded algebra $\mathrm{gr}B_{\tau_1,\tau_2}(n,p)$. Using the isomorphism ψ in Corollary 3.1 and the results from [MoRa] Section 3.2, we have the following consequence.

Corollary 3.3. *We have an isomorphism of algebras*

$$\mathfrak{U}\mathfrak{gl}_n^\theta(\mathbb{C}[t]) \xrightarrow{\sim} \mathrm{gr}B_{\tau_1,\tau_2}(n,p), \qquad \tau_1^{r-1}(\epsilon_j + (-1)^{r-1}\epsilon_i)E_{ij}t^{r-1} \mapsto \bar{b}_{ij}^{(r)}.$$

3.4. **MacKay's twisted Yangians of type AIII.** The twisted Yangians that we study in this subsection were introduced by N. MacKay in [Ma1, Ma2]. One novelty here is that the algebras that we define are a bit more general because they depend on two deformation parameters.

Definition 3.8. [Ma1, Ma2] *The MacKay twisted Yangian $\widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})$ of type AIII is the subalgebra of the Yangian $\widetilde{Y}_{\lambda_1}(\mathfrak{gl}_n)$ generated by the elements $E_{ab} \in \mathfrak{k}$ and by $\widetilde{J}(E_{ij})$ with $E_{ij} \in \mathfrak{p}$, where*

$$\widetilde{J}(E_{ij}) = J(E_{ij}) - \epsilon_i \left(\frac{\lambda_2}{2} + \frac{\lambda_1(n-2p)}{4} \right) E_{ij} - \frac{\lambda_1}{4}[C, E_{ij}],$$

where $C = \sum_{i,j=1}^p E_{ij}E_{ji} + \sum_{i,j=p+1}^n E_{ij}E_{ji}$ is the quadratic Casimir operator of $\mathfrak{k}$. We will denote by $Y_{\lambda_1,\lambda_2}(\mathfrak{sl}_n,\mathfrak{k}_0)$ the algebra obtained by allowing only the matrices $E_{ij}, E_{ii} - E_{jj} \in \mathfrak{k}_0$ for $i \neq j$.

Note that, if $\lambda_1 \neq 0$, then $\widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k}) \cong \widetilde{Y}_{1,\frac{\lambda_2}{\lambda_1}}(\mathfrak{gl}_n,\mathfrak{k})$ by rescaling the generators.

Lemma 3.1. *The MacKay twisted Yangian $\widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})$ is a left coideal subalgebra of the Yangian $\widetilde{Y}_\lambda(\mathfrak{gl}_n)$; that is, $\Delta(\widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})) \subset \widetilde{Y}_{\lambda_1}(\mathfrak{gl}_n) \otimes_{\mathbb{C}} \widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})$ where Δ is the coproduct on $Y(\mathfrak{gl}_n)$.*

Proof. The proof is essentially contained in subsection 2.3 in [DMS]. □

Lemma 3.2. *The following relation holds in $\mathfrak{U}\mathfrak{gl}_n$: $\sum_{k=1}^n \epsilon_k E_{ik}E_{kj} = \frac{2p-n}{2}E_{ij} + \frac{\epsilon_i}{2}[C, E_{ij}]$, where $1 \leq i \leq p < j \leq n$ or $1 \leq j \leq p < i \leq n$.*

Proof. Assume that i, j satisfy one of the two conditions in the statement of this lemma:

$$\begin{aligned}
[C, E_{ij}] &= \left(\sum_{h,k=1}^{p} + \sum_{h,k=p+1}^{n}\right)[E_{hk}E_{kh}, E_{ij}] \\
&= \left(\sum_{h,k=1}^{p} + \sum_{h,k=p+1}^{n}\right)([E_{hk}, E_{ij}]E_{kh} + E_{hk}[E_{kh}, E_{ij}]) \\
&= \epsilon_i \sum_{k=1}^{p}(E_{ik}E_{kj} + E_{kj}E_{ik}) - \epsilon_i \sum_{k=p+1}^{n}(E_{ik}E_{kj} + E_{kj}E_{ik}).
\end{aligned}$$

So

$$\begin{aligned}
&\sum_{k=1}^{p} E_{ik}E_{kj} - \sum_{k=p+1}^{n} E_{ik}E_{kj} \\
&\quad = \frac{1}{2}\sum_{k=1}^{p}(E_{ik}E_{kj} + [E_{ik}, E_{kj}] + E_{kj}E_{ik}) - \frac{1}{2}\sum_{k=p+1}^{n}(E_{ik}E_{kj} + [E_{ik}, E_{kj}] + E_{kj}E_{ik}) \\
&\quad = \frac{2p-n}{2}E_{ij} + \frac{\epsilon_i}{2}[C, E_{ij}].
\end{aligned}$$

□

Proposition 3.4. *Let $\lambda_1 = 1$, $\tau_2 = (n-2p)/2 + \lambda_2/2$. The algebra $\widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n, \mathfrak{k})$ is isomorphic to the subalgebra of $B_{\tau_1,\tau_2}(n,p)$ generated by $b_{ij}^{(1)}$ with $1 \le i, j \le n$ and by $b_{ij}^{(2)}$ for all $1 \le i \le p < j \le n$ and all $1 \le j \le p < i \le n$.*

Proof. The two twisted Yangians $\widetilde{Y}_{\lambda_1=1,\lambda_2}(\mathfrak{gl}_n, \mathfrak{k})$ and $B_{\tau_1,\tau_2}(n,p)$ are subalgebras of $Y(\mathfrak{gl}_n)$, and to understand how they are connected, we need to use the following relation in $Y(\mathfrak{gl}_n)$ [ChPr1] when $i \neq j$:

$$J(E_{ij}) = T_{ij}^{(2)} - \frac{1}{2}\sum_{k=1}^{n} E_{ik}E_{kj}, \tag{5}$$

where we have identified E_{ik}, E_{kj} with $T_{ik}^{(1)}, T_{kj}^{(1)}$.

We need to know the coefficients of $\widetilde{T}_{kj}(u)$. A general expression for the coefficients of $\widetilde{T}_{kj}(u)$ can be found in [MoRa], and the first few terms are the following (see also [MNO]):

$$\widetilde{T}_{ij}^{(1)} = -T_{ij}^{(1)}, \quad \widetilde{T}_{ij}^{(2)} = -T_{ij}^{(2)} + \sum_{k=1}^{n} T_{ik}^{(1)}T_{kj}^{(1)}. \tag{6}$$

We can now compute that

$$b_{ij}^{(1)} = \epsilon_j T_{ij}^{(1)} + \epsilon_i T_{ij}^{(1)} + \tau_2\delta_{ij} = (\epsilon_i + \epsilon_j)E_{ij} + \tau_2\delta_{ij}. \tag{7}$$

When $1 \le i \le p < j \le n$ or $1 \le j \le p < i \le n$, we have

$$\begin{aligned}
b_{ij}^{(2)} &= (\epsilon_j - \epsilon_i)\tau_1 T_{ij}^{(2)} + 2\tau_1\tau_2 T_{ij}^{(1)} + \epsilon_i\tau_1 \sum_{k=1}^{n} T_{ik}^{(1)}T_{kj}^{(1)} + \tau_1 \sum_{k=1}^{n} \epsilon_k T_{ik}^{(1)}T_{kj}^{(1)} \\
&= -2\tau_1\epsilon_i J(E_{ij}) + 2\tau_1\tau_2 E_{ij} + \tau_1 \sum_{k=1}^{n} \epsilon_k E_{ik}E_{kj} \\
&\quad \text{(using Lemma 3.2)} \\
&= -2\epsilon_i\tau_1 \left(J(E_{ij}) + \epsilon_i\left(\frac{(n-2p)}{4} - \tau_2\right)E_{ij} - \frac{1}{4}[C, E_{ij}]\right) = -2\epsilon_i\tau_1\widetilde{J}(E_{ij}).
\end{aligned} \tag{8}$$

We have embeddings $\widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k}) \hookrightarrow Y(\mathfrak{gl}_n)$ and $B_{\tau_1,\tau_2}(n,p) \hookrightarrow Y(\mathfrak{gl}_n)$ and the preceding computations show that the image of the former lands in the image of the latter if $\tau_2 = \dfrac{(n-2p)}{2} + \dfrac{\lambda_2}{2}$ (and $\tau_1 \neq 0$ as usual) and that it can be identified with the subalgebra of $B_{\tau_1,\tau_2}(n,p)$ generated by all $b_{ij}^{(1)}$ with $1 \leq i,j \leq n$, and by $b_{ij}^{(2)}$ with $1 \leq i \leq p < j \leq n, 1 \leq j \leq p < i \leq n$. $\square$

Corollary 3.4. *Suppose that $n \geq 3$ and λ_2, τ_2 are related as in Proposition 3.4. Then $Y_{1,\lambda_2}(\mathfrak{sl}_n, \mathfrak{k}_0)$ is equal to $SB_{\tau_1,\tau_2}(n,p)$.*

Proof. By definition, the algebra $Y_{1,\lambda_2}(\mathfrak{sl}_n,\mathfrak{k}_0)$ is generated by elements in $\mathfrak{k}_0$ and by $\widetilde{J}(E_{ij})$ with $E_{ij} \in \mathfrak{p}$.

Then from (7), (8), it can be identified with the subalgebra of $SB_{\tau_1,\tau_2}(n,p)$ generated by

$$b_{ij}^{(1)}\ (1 \leq i \neq j \leq p, p+1 \leq i \neq j \leq n),\ b_{ii}^{(1)} - b_{jj}^{(1)}\ (1 \leq i \neq j \leq n),\ b_{ij}^{(2)}\ (1 \leq i \leq p < j \leq n,\ 1 \leq j \leq p < i \leq n).$$

Thus $Y_{1,\lambda_2}(\mathfrak{sl}_n,\mathfrak{k}_0)$ is contained in $SB_{\tau_1,\tau_2}(n,p)$. Both have filtrations inherited from the one on $Y(\mathfrak{gl}_n)$ obtained by giving $t_{ij}^{(r)}$ degree r. So we have $\mathrm{gr}(Y_{1,\lambda_2}(\mathfrak{sl}_n,\mathfrak{k}_0)) \subset \mathrm{gr}(SB_{\tau_1,\tau_2}(n,p)) \cong \mathfrak{U}\mathfrak{sl}_n^{\theta}(\mathbb{C}[t])$, where the last isomorphism can be deduced from Section 3 in [MoRa] and from Corollary 3.1.

$\mathfrak{U}\mathfrak{sl}_n^{\theta}(\mathbb{C}[t])$ is generated as an algebra by its subspaces spanned by $\mathfrak{k}_0$ and $\mathfrak{p} \otimes_{\mathbb{C}} \mathbb{C}t$, so $\mathrm{gr}(Y_{1,\lambda_2}(\mathfrak{sl}_n,\mathfrak{k}_0))$ contains generators of $\mathfrak{U}\mathfrak{sl}_n^{\theta}(\mathbb{C}[t])$ and hence both are equal. It follows that $\mathrm{gr}(Y_{1,\lambda_2}(\mathfrak{sl}_n,\mathfrak{k}_0)) = \mathrm{gr}(SB_{\tau_1,\tau_2}(n,p))$ and therefore $Y_{1,\lambda_2}(\mathfrak{sl}_n,\mathfrak{k}_0) = SB_{\tau_1,\tau_2}(n,p)$. $\square$

3.5. **Presentation of the twisted Yangian of type AIII by generators and relations.** In this section, we give a presentation in terms of generators and relations of the Mackay twisted Yangian $\widetilde{Y}_{\lambda_1=1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})$ with $n=2p$. The initial idea which led to this section is the observation that an isomorphism given explicitly in [GHL] allows one to view $\mathfrak{sl}_{2p}^{\theta}(\mathbb{C}[t])$ as being isomorphic to $\mathfrak{sl}_p(\mathbb{C}[t] \rtimes \Gamma)$. We can then apply ideas from [Gu2, Gu3] to obtain a presentation for a deformation of the enveloping algebra of $\mathfrak{sl}_p(\mathbb{C}[t]\rtimes\Gamma) \oplus \mathbb{C}I_p$ (I_p being the identity matrix) which turns out to be isomorphic to $\widetilde{Y}_{\lambda_1=1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})$ for $\mathfrak{k} = \mathfrak{gl}_p(\mathbb{C}) \oplus \mathfrak{gl}_p(\mathbb{C})$.

We introduce an action of Γ on the polynomial ring $\mathbb{C}[t]$ (where $\Gamma = \mathbb{Z}/2\mathbb{Z}$): if $\gamma \in \Gamma$, $\gamma \neq 1_\Gamma$ and $p(t) \in \mathbb{C}[t]$, then $\gamma(p(t)) = p(-t)$. We can form the semi-direct product (also called smash product) $\mathbb{C}[t]\rtimes\Gamma$. Moreover, the kernel of the universal central extension of $\mathfrak{sl}_p(\mathbb{C}[t]\rtimes\Gamma)$ is isomorphic to $HC_1(\mathbb{C}[t]\rtimes\Gamma)$ and it is known that $HC_1(\mathbb{C}[t]\rtimes\Gamma)=0$. Since $\mathbb{C}[t]\rtimes\Gamma$ is not a commutative ring, it may be a good idea to recall the following definition.

Definition 3.9. *Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$, not necessarily commutative. The Lie algebra $\mathfrak{sl}_p(\mathcal{A})$ is defined as $\mathfrak{sl}_p(\mathcal{A}) = [\mathfrak{gl}_p(\mathcal{A}), \mathfrak{gl}_p(\mathcal{A})]$. $\mathfrak{sl}_p(\mathcal{A})$ is also the space of matrices in $\mathfrak{gl}_p(\mathcal{A})$ with trace in $[\mathcal{A},\mathcal{A}]$.*

Let us assume that $p \geq 4$.

Proposition 3.5. [Gu2] *The Lie algebra $\mathfrak{sl}_p(\mathbb{C}[t]\rtimes\Gamma)$ is isomorphic to the Lie algebra generated by the elements $F_{ab}(t), F_{ab}(\gamma)$ for $\gamma\in\Gamma$, $1 \leq a \neq b \leq p$ and satisfying the following relations: If $1 \leq a,b,c \leq p$ are all distinct and $1 \leq a,c,d \leq p$ are also all distinct, and if $\gamma,\gamma_1,\gamma_2 \in \Gamma$, then*

$$[F_{ab}(t),F_{bc}(t)] = [F_{ad}(t),F_{dc}(t)], \qquad [F_{ab}(1),F_{bc}(t)] = F_{ac}(t);$$

$$[F_{ab}(\gamma^i),F_{bc}(t)] = [F_{ad}((-1)^i t),F_{dc}(\gamma)]\ \ \gamma\neq 1_\Gamma, i=0,1, \quad [F_{ab}(\gamma_1),F_{bc}(\gamma_2)] = F_{ac}(\gamma_1\gamma_2).$$

If $1 \leq a,b,c,d \leq p$ and $a\neq b\neq c\neq d\neq a$, then

$$[F_{ab}(t),F_{cd}(t)]=0,\ \ [F_{ab}(\gamma),F_{cd}(t)]=0,\ \ [F_{ab}(\gamma_1),F_{cd}(\gamma_2)]=0.$$

In [GHL], it was explained that $\mathfrak{sl}_p(\mathbb{C}[t,t^{-1}]\rtimes\Gamma)$ is isomorphic to $\mathfrak{sl}_n(\mathbb{C}[w,w^{-1}])$ (w should be thought of as u^2), so that $\mathfrak{sl}_p(\mathbb{C}[t]\rtimes\Gamma)$ gets identified with the Lie subalgebra $\mathfrak{g}$ of $\mathfrak{sl}_n(\mathbb{C}[w,w^{-1}])$ spanned by all matrices of the form $E_{ij}w^r, (E_{ii}-E_{jj})w^r$ for any $1 \leq i \neq j \leq n$ if $r \geq 1$ and all the matrices $E_{ij}, E_{ii}-E_{jj}$ with $1 \leq i \neq j \leq n$ except those with $p+1 \leq i \leq n, 1 \leq j \leq p$. The next proposition was missed in [GHL].

Proposition 3.6. *$\mathfrak{sl}_n(\mathbb{C}[w^{\pm1}])$ is isomorphic to $\mathfrak{sl}_n^{\theta}(\mathbb{C}[t^{\pm1}])$ and $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_n^{\theta}(\mathbb{C}[t])$.*

Proof. An isomorphism $\rho : \mathfrak{sl}_n(\mathbb{C}[w,w^{-1}]) \xrightarrow{\sim} \mathfrak{sl}_n^{\theta}(\mathbb{C}[t,t^{-1}])$ can be described explicitly in the following way. If $\epsilon_i = \epsilon_j$, then $\rho(E_{ij}w^r) = E_{ij}t^{2r}$; if $\epsilon_i = -1$ and $\epsilon_j = 1$, then $\rho(E_{ij}w^r) = E_{ij}t^{2r+1}$; if $\epsilon_i = 1$ and $\epsilon_j = -1$, then $\rho(E_{ij}w^r) = E_{ij}t^{2r-1}$. □

Let $\mathbf{e}_0$ and $\mathbf{e}_1$ be the two primitive idempotents of Γ. Composing ρ with the isomorphism $\mathfrak{sl}_p(\mathbb{C}[t,t^{-1}] \rtimes \Gamma) \xrightarrow{\sim} \mathfrak{sl}_n(\mathbb{C}[w,w^{-1}])$ given in [GHL] yields the isomorphism $\tau : \mathfrak{sl}_p(\mathbb{C}[t,t^{-1}] \rtimes \Gamma) \xrightarrow{\sim} \mathfrak{sl}_n^{\theta}(\mathbb{C}[t,t^{-1}])$ given by $\tau(E_{ij}t^{2r}\mathbf{e}_0) = E_{ij}t^{2r}$, $\tau(E_{ij}t^{2r}\mathbf{e}_1) = E_{i+p,j+p}t^{2r}$, $\tau(E_{ij}t^{2r+1}\mathbf{e}_0) = E_{i+p,j}t^{2r+1}$, $\tau(E_{ij}t^{2r+1}\mathbf{e}_1) = E_{i,j+p}t^{2r+1}$, which restricts to an isomorphism $\mathfrak{sl}_p(\mathbb{C}[t] \rtimes \Gamma) \xrightarrow{\sim} \mathfrak{sl}_n^{\theta}(\mathbb{C}[t])$.

The previous proposition implies that $Y_{\lambda_1,\lambda_2}(\mathfrak{sl}_n,\mathfrak{k}_0)$ can be viewed as a deformation of $\mathfrak{U}\mathfrak{sl}_p(\mathbb{C}[t] \rtimes \Gamma)$; this is part of the content of Theorem 3.2 below. We set $H_{ab}(\gamma^i) = F_{aa}(\gamma^i) - F_{bb}(\gamma^i), i = 0,1$.

Definition 3.10. *We denote by $Y^p_{\lambda_2}$ the algebra generated by elements $F_{ab}(t)$ with $1 \le a \ne b \le p$ and by $F_{ab}(\gamma)$ for $\gamma \in \Gamma$, $1 \le a,b \le p$ and satisfying $[F_{ab}(1),F_{bc}(t)] = F_{ac}(t) = [F_{ab}(t),F_{bc}(1)]$ if $c \ne a,b$ and the following relations:*

If $1 \le a,b,c \le p$ are all distinct and $1 \le a,c,d \le p$ are also all distinct, then

$$\begin{aligned}[F_{ab}(t),F_{bc}(t)] - [F_{ad}(t),F_{dc}(t)] &= \sum_{j=1}^{p}\Big(H_{bd}(\mathbf{e}_1)F_{aj}(\mathbf{e}_0)F_{jc}(\mathbf{e}_0) - \big(F_{bj}(\mathbf{e}_0)F_{jb}(\mathbf{e}_0) - F_{dj}(\mathbf{e}_0)F_{jd}(\mathbf{e}_0)\big)F_{ac}(\mathbf{e}_1)\Big)\\ &\quad+\sum_{j=1}^{p}\Big(H_{bd}(\mathbf{e}_0)F_{aj}(\mathbf{e}_1)F_{jc}(\mathbf{e}_1) - \big(F_{bj}(\mathbf{e}_1)F_{jb}(\mathbf{e}_1) - F_{dj}(\mathbf{e}_1)F_{jd}(\mathbf{e}_1)\big)F_{ac}(\mathbf{e}_0)\Big)\\ &\quad+2\lambda_2(H_{bd}(\mathbf{e}_1)F_{ac}(\mathbf{e}_0) - H_{bd}(\mathbf{e}_0)F_{ac}(\mathbf{e}_1)).\end{aligned}$$

For $a \ne b \ne c, a \ne d \ne c$ and for $\gamma,\gamma_1,\gamma_2 \in \Gamma$,

$$[F_{ab}(\gamma^i),F_{bc}(t)] = [F_{ad}((-1)^i t),F_{dc}(\gamma)] \;\; \gamma \ne 1_\Gamma, i = 0,1, \quad [F_{ab}(\gamma_1),F_{bc}(\gamma_2)] = F_{ac}(\gamma_1\gamma_2).$$

If $1 \le a,b,c,d \le p$ and $a \ne b \ne c \ne d \ne a$, then

$$\begin{aligned}[F_{ab}(t),F_{cd}(t)] &= \sum_{j=1}^{p}\big(F_{cj}(\mathbf{e}_1)F_{jb}(\mathbf{e}_1)F_{ad}(\mathbf{e}_0) - F_{cb}(\mathbf{e}_1)F_{aj}(\mathbf{e}_0)F_{jd}(\mathbf{e}_0)\big) - \lambda_2 F_{ad}(\mathbf{e}_0)F_{cb}(\mathbf{e}_1)\\ &\quad+\sum_{j=1}^{p}\big(F_{cj}(\mathbf{e}_0)F_{jb}(\mathbf{e}_0)F_{ad}(\mathbf{e}_1) - F_{cb}(\mathbf{e}_0)F_{aj}(\mathbf{e}_1)F_{jd}(\mathbf{e}_1)\big) + \lambda_2 F_{ad}(\mathbf{e}_1)F_{cb}(\mathbf{e}_0).\end{aligned}$$

If $a \ne d, b \ne c$, then

$$[F_{ab}(\gamma),F_{cd}(t)] = 0, \;\; [F_{ab}(\gamma_1),F_{cd}(\gamma_2)] = 0.$$

Finally, $I_p = \sum_{i=1}^{p} F_{ii}(1)$ is central.

Theorem 3.2. *The algebras $Y^p_{\lambda_2}$ and $\widetilde{Y}_{\lambda_1=1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})$ are isomorphic.*

Proof. We can define an algebra epimorphism $\psi : Y^p_{\lambda_2} \twoheadrightarrow \widetilde{Y}_{\lambda_1=1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})$ by setting $\psi(F_{ab}(1_\Gamma)) = E_{ab} + E_{a+p,b+p}$, $\psi(F_{ab}(\gamma)) = E_{ab} - E_{a+p,b+p}$ for $1 \le a,b \le p$ and $\psi(F_{ab}(t)) = \widetilde{J}(E_{a,b+p}) + \widetilde{J}(E_{a+p,b})$ for $1 \le a \ne b \le p$. To check that this does indeed define an algebra homomorphism, one should use equation (8) along with the reflection equation and the unitary condition. We will not include the relevant computations. Passing to the associated graded algebras, we obtain a homomorphism $\mathrm{gr}(\psi) : \mathrm{gr}(Y^p_{\lambda_2}) \twoheadrightarrow \mathrm{gr}(\widetilde{Y}_{\lambda_1=1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k}))$. Here, the filtration on $\widetilde{Y}_{\lambda_1=1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})$ is the one induced from the filtration on $\widetilde{Y}_{\lambda_1=1}(\mathfrak{gl}_n)$ obtained by giving $J(E_{ab})$ degree one. We thus have an embedding $\mathrm{gr}(\widetilde{Y}_{\lambda_1=1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})) \hookrightarrow \mathrm{gr}(\widetilde{Y}_{\lambda_1=1}(\mathfrak{gl}_n))$ and a quotient map $\mathfrak{U}(\mathfrak{sl}_p(\mathbb{C}[t] \rtimes \Gamma) \oplus \mathbb{C}I_p) \twoheadrightarrow \mathrm{gr}(Y^p_{\lambda_2})$ (see Proposition 3.5). The composite of all these maps is the monomorphism $\mathfrak{U}(\mathfrak{sl}_p(\mathbb{C}[t] \rtimes \Gamma) \oplus \mathbb{C}I_p) \hookrightarrow \mathfrak{U}(\mathfrak{sl}_n(\mathbb{C}[t]) \oplus \mathbb{C}I_n)$ which identifies $\mathfrak{U}(\mathfrak{sl}_p(\mathbb{C}[t] \rtimes \Gamma) \oplus \mathbb{C}I_p)$ with $\mathfrak{U}(\mathfrak{sl}_n^{\theta}(\mathbb{C}[t]) \oplus \mathbb{C}I_n)$. (See Proposition 3.6 and the paragraph just below it.) Therefore, $\mathrm{gr}(\psi)$ is an isomorphism, hence so is ψ. □

4. Drinfeld functors

4.1. **Preliminaries.** Let the symmetric group $\mathfrak{S}_l$ act on the space $(\mathbb{C}^n)^{\otimes l}$ by permutation of the l tensor components. If $E_{ij} \in \mathfrak{gl}_n(\mathbb{C})$ and $\mathbf{v} \in (\mathbb{C}^n)^{\otimes l}$, we denote by $E_{ij}^{(k)}(\mathbf{v})$ the element obtained by applying E_{ij} to the k-th factor in the tensor product. Then, as an element in $\mathrm{End}_{\mathbb{C}}((\mathbb{C}^n)^{\otimes l})$, the permutation $\sigma_{ks} \in \mathfrak{S}_l$ equals $\sum_{i,j=1}^n E_{ij}^{(k)} E_{ji}^{(s)}$. Set $\mathtt{P}_k = \sum_{i,j=1}^n E_{ij}^{(k)} E_{ji}^{(k+1)} \in \mathrm{End}_{\mathbb{C}}((\mathbb{C}^n)^{\otimes l})$ and $\mathtt{I}_k = \sum_{i,j=1}^n E_{ij}^{(k)} \otimes E_{ij} \in \mathrm{End}_{\mathbb{C}}((\mathbb{C}^n)^{\otimes l}) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$.

Lemma 4.1. *View $\mathtt{P}_k$ as the linear operator $\mathtt{P}_k \otimes 1 \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)^{\otimes l} \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$. We have*

$$[\mathtt{I}_{k+1}, \mathtt{P}_k] = -[\mathtt{I}_k, \mathtt{I}_{k+1}], \quad [\mathtt{I}_k, \mathtt{P}_k] = [\mathtt{I}_k, \mathtt{I}_{k+1}].$$

Proof. The proof is based on direct calculations:

$$[\mathtt{I}_{k+1}, \mathtt{P}_k] = \sum_{s,t,r=1}^{n} E_{rt}^{(k)} \otimes E_{sr}^{(k+1)} \otimes E_{st} - \sum_{s,t,m=1}^{n} E_{sm}^{(k)} \otimes E_{mt}^{(k+1)} \otimes E_{st},$$

$$[\mathtt{I}_k, \mathtt{I}_{k+1}] = \sum_{s,t,m=1}^{n} E_{st}^{(k)} \otimes E_{tm}^{(k+1)} \otimes E_{sm} - \sum_{s,t,r=1}^{n} E_{st}^{(k)} \otimes E_{rs}^{(k+1)} \otimes E_{rt},$$

and $[\mathtt{I}_k, \mathtt{P}_k] = \mathtt{I}_k \mathtt{P}_k - \mathtt{P}_k \mathtt{I}_k = \mathtt{P}_k \mathtt{I}_{k+1} - \mathtt{I}_{k+1} \mathtt{P}_k = -[\mathtt{I}_{k+1}, \mathtt{P}_k]$. □

Let $\gamma_k \in W_l$ act on $(\mathbb{C}^n)^{\otimes l}$ by multiplication on the k-th component by the matrix Θ_p (this operator will be denoted by $\Theta_p^{(k)}$). This defines a W_l-module structure on $(\mathbb{C}^n)^{\otimes l}$. Thus for any $\mathtt{H}_{\kappa_1}^l$-module (resp. $\mathtt{H}_{\kappa_1,\kappa_2}^l$-module) M, the space $M \otimes_{\mathbb{C}} (\mathbb{C}^n)^{\otimes l}$ has an $\mathfrak{S}_l$ (resp. W_l) module structure obtained from the diagonal action. From now on, let $\varepsilon = 1$ or -1.

4.2. **Drinfeld functor for $Y(\mathfrak{gl}_n)$.** In this section, we recall the construction of the Drinfeld functor in type A.

Let M be any $\mathtt{H}_{\kappa_1}^l$-module and set $\widetilde{\mathsf{D}}^A(M) = M \otimes_{\mathbb{C}} (\mathbb{C}^n)^{\otimes l}$. For constants λ and c_k $(k = 1, \ldots, l)$, define

$$\mathtt{T}^\lambda(u) = \mathtt{T}_1^\lambda(u) \cdots \mathtt{T}_l^\lambda(u) \in \mathtt{H}_{\kappa_1}^l[[u^{-1}]] \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}((\mathbb{C}^n)^{\otimes l}) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n),$$

where $\mathtt{T}_k^\lambda(u) = 1 + \dfrac{1}{u - \lambda y_k + c_k} \otimes \mathtt{I}_k, k = 1, \ldots, l.$

Then the map $T(u) \mapsto \mathtt{T}^\lambda(u)$ defines an action of $Y(\mathfrak{gl}_n)$ on $\widetilde{\mathsf{D}}^A(M)$. As was mentioned in Section 4.1, $\widetilde{\mathsf{D}}^A(M)$ has a $\mathfrak{S}_l$-module structure. Define the space $\mathsf{D}^{A,\varepsilon}(M)$ as

$$\mathsf{D}^{A,\varepsilon}(M) = \widetilde{\mathsf{D}}^A(M) / \sum_{i=1}^{l-1} \mathrm{Im}(\sigma_i - \varepsilon), \text{ where Im means image.}$$

Proposition 4.1 ([Ar] proposition 2, [Dr] theorem 1)**.** *Assume $c_k = c$ for $(k = 1, \ldots, l)$ and $\kappa_1 \neq 0$. Let M be any left $\mathtt{H}_{\kappa_1}^l$-module. When $\lambda = \varepsilon/\kappa_1$, the map $T(u) \to \mathtt{T}^\lambda(u)$ defines an action of $Y(\mathfrak{gl}_n)$ on $\mathsf{D}^{A,\varepsilon}(M)$. Thus we have a functor*

$$\mathsf{D}^{A,\varepsilon} : \mathtt{H}_{\kappa_1}^l - \mathrm{mod}_L \longrightarrow Y(\mathfrak{gl}_n) - \mathrm{mod}_L, \quad M \mapsto \mathsf{D}^{A,\varepsilon}(M).$$

When $\varepsilon = -1$, a condensed version of the proof is contained in [Ar]; we give a few more details for completeness.

Proof of Proposition 4.1. We need the following relations:

$$\mathtt{I}_i \sigma_j = \sigma_j \mathtt{I}_i, \text{ if } j \neq i-1, i, \quad \mathtt{I}_i \sigma_i = \sigma_i \mathtt{I}_{i+1}, \; \mathtt{I}_i \sigma_{i-1} = \sigma_{i-1} \mathtt{I}_{i-1}.$$

We can write $\prod_{k=1}^l (u + c - \lambda y_k) \mathtt{T}_1^\lambda(u) \mathtt{T}_2^\lambda(u) \cdots \mathtt{T}_l^\lambda(u) = \widetilde{\mathtt{T}}_1^\lambda(u) \cdots \widetilde{\mathtt{T}}_l^\lambda(u)$, where $\widetilde{\mathtt{T}}_k^\lambda(u) = u + c - \lambda y_k + 1 \otimes \mathtt{I}_k$. Since $\prod_{k=1}^l (u + c - \lambda y_k)$ is in the center of $\mathtt{H}_{\kappa_1}^l$ (Lemma 2.3), it is enough to show that, for $\lambda = \varepsilon/\kappa_1$, the image of

$\widetilde{\mathtt{T}}_1^\lambda(u)\cdots\widetilde{\mathtt{T}}_l^\lambda(u)\sigma_k-\sigma_k\widetilde{\mathtt{T}}_1^\lambda(u)\cdots\widetilde{\mathtt{T}}_l^\lambda(u)$ is contained in the image of $\sigma_k-\varepsilon$. First notice that σ_k commutes with $\widetilde{\mathtt{T}}_s^\lambda(u)$ for $s\neq k,k+1$. We also have

$$\begin{aligned}
&\widetilde{\mathtt{T}}_k^\lambda(u)\widetilde{\mathtt{T}}_{k+1}^\lambda(u)\sigma_k\\
&\quad=\widetilde{\mathtt{T}}_k^\lambda(u)\left(\sigma_k(u+c)-\sigma_k\lambda y_k+\lambda\kappa_1\otimes\mathtt{P}_k+\sigma_k 1\otimes\mathtt{I}_k\right)\\
&\quad=\widetilde{\mathtt{T}}_k^\lambda(u)\sigma_k(u+c-\lambda y_k+1\otimes\mathtt{I}_k)+\widetilde{\mathtt{T}}_k^\lambda(u)\lambda\kappa_1\otimes\mathtt{P}_k\\
&\quad=(\sigma_k(u+c)-\lambda\sigma_k y_{k+1}-\lambda\kappa_1\otimes\mathtt{P}_k+\sigma_k 1\otimes\mathtt{I}_{k+1})(u+c-\lambda y_k+1\otimes\mathtt{I}_k)+\widetilde{\mathtt{T}}_k^\lambda(u)\lambda\kappa_1\otimes\mathtt{P}_k\\
&\quad=\sigma_k(u+c-\lambda y_{k+1}+1\otimes\mathtt{I}_{k+1})(u+c-\lambda y_k+1\otimes\mathtt{I}_k)-\lambda\kappa_1\otimes\mathtt{P}_k(u+c-\lambda y_k+1\otimes\mathtt{I}_k)+\widetilde{\mathtt{T}}_k^\lambda(u)\lambda\kappa_1\otimes\mathtt{P}_k\\
&\quad=\sigma_k\widetilde{\mathtt{T}}_k^\lambda(u)\widetilde{\mathtt{T}}_{k+1}^\lambda(u)+\sigma_k\otimes[\mathtt{I}_{k+1},\mathtt{I}_k]+\lambda\kappa_1\otimes[\mathtt{I}_k,\mathtt{P}_k]\\
&\quad=\sigma_k\widetilde{\mathtt{T}}_k^\lambda(u)\widetilde{\mathtt{T}}_{k+1}^\lambda(u)+(\sigma_k-\lambda\kappa_1)1\otimes[\mathtt{I}_{k+1},\mathtt{I}_k]\quad\text{(by Lemma 4.1).}
\end{aligned}$$

Thus we have

$$\widetilde{\mathtt{T}}_1^\lambda(u)\cdots\widetilde{\mathtt{T}}_l^\lambda(u)\sigma_k-\sigma_k\widetilde{\mathtt{T}}_1^\lambda(u)\cdots\widetilde{\mathtt{T}}_l^\lambda(u)=(\sigma_k-\lambda\kappa_1)\widetilde{\mathtt{T}}_1^\lambda(u)\cdots\widetilde{\mathtt{T}}_{k-1}^\lambda(u)[\mathtt{I}_{k+1},\mathtt{I}_k]\widetilde{\mathtt{T}}_{k+2}^\lambda(u)\cdots\widetilde{\mathtt{T}}_l^\lambda(u).$$

We get the desired conclusion when $\lambda\kappa_1=\epsilon$.

As for homomorphisms, suppose that $f\in\mathrm{Hom}_{\mathtt{H}_{\kappa_1}^l}(M_1,M_2)$. Then f extends to a homomorphism $f\otimes 1:\widetilde{\mathsf{D}}^A(M_1)\longrightarrow\widetilde{\mathsf{D}}^A(M_2)$; since f is a homomorphism of modules over $\mathtt{H}_{\kappa_1}^l$, $(f\otimes 1)(\sum_{i=1}^{l-1}\mathrm{Im}(\sigma_i-\epsilon))\subset\sum_{i=1}^{l-1}\mathrm{Im}(\sigma_i-\epsilon)$. □

4.3. **Drinfeld functor for MacKay's twisted Yangians.** In this section, we explain how to construct a functor from the category of modules over the degenerate affine Hecke algebra of type BC_l to the category of modules for the MacKay twisted Yangian $\widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})$.

Consider a left module M over $\mathtt{H}_{\kappa_1,\kappa_2}^l$. From Section 4.2, since $\mathtt{H}_{\kappa_1,\kappa_2}^l$ contains $\mathtt{H}_{\kappa_1}^l$, we know that $\mathsf{D}^{A,\varepsilon}(M)$ is a left module over $Y(\mathfrak{gl}_n)$. So, by restriction, it is also a left module over $\widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})$.

Now consider the following space:

$$\mathsf{D}^{BC,\varepsilon}(M)=\mathsf{D}^{A,\varepsilon}(M)/\mathrm{Im}(\gamma_l-\varepsilon)=M\otimes_{\mathbb{C}}(\mathbb{C}^n)^{\otimes l}/\left(\sum_{k=1}^{l-1}\mathrm{Im}(\sigma_k-\varepsilon)+\mathrm{Im}(\gamma_l-\varepsilon)\right).$$

$\mathsf{D}^{BC,\varepsilon}(M)$ is not a left module over $Y(\mathfrak{gl}_n)$, but we have the following result which is an analog of theorem 1 in [Dr] and of Proposition 4.1.

Theorem 4.1. *$\mathsf{D}^{BC,\varepsilon}(M)$ is a left module over $\widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})$ if $\lambda_1=\kappa_1$ and $\lambda_2=\kappa_2$. Therefore, we have a functor*

$$\mathsf{D}^{BC,\varepsilon}:\mathtt{H}_{\kappa_1,\kappa_2}^l-\mathrm{mod}_L\longrightarrow\widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})-\mathrm{mod}_L.$$

If $f\in\mathrm{Hom}_{\mathtt{H}_{\kappa_1,\kappa_2}^l}(M_1,M_2)$, then $\mathsf{D}^{BC,\varepsilon}(f)\in\mathrm{Hom}_{\widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})}(\mathsf{D}^{BC,\varepsilon}(M_1),\mathsf{D}^{BC,\varepsilon}(M_2))$ is defined by $\mathsf{D}^{BC,\varepsilon}(f)(m\otimes\mathbf{v})=f(m)\otimes\mathbf{v}$. Moreover, if $p,n-p\geq l+1$, $\mathsf{D}^{BC,\varepsilon}$ provides an equivalence between the category of finite dimensional modules over $\mathtt{H}_{\kappa_1,\kappa_2}^l$ and the category of finite dimensional modules over $\widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})$ which are of level l in the sense that they decompose over $\mathfrak{gl}_p\oplus\mathfrak{gl}_{n-p}$ as direct sums of submodules of $(\mathbb{C}^n)^{\otimes l}$ (so in particular the identity matrix I_n acts by the scalar l).

Proof. It is enough to show that the generators $\widetilde{J}(E_{ij})\in\widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})$, which act on $\mathsf{D}^{A,\varepsilon}(M)$ by Proposition 4.1, descend to operators on $\mathsf{D}^{BC,\varepsilon}(M)$. Set

$$R_k=-\frac{\kappa_2}{2}\gamma_k-\frac{\kappa_1}{2}\sum_{\substack{j=1\\ j\neq k}}^{l}\sigma_{jk}\gamma_k\gamma_j.$$

It can be deduced from [Dr] (or from formula (5) and the proof of Proposition 4.1) that $J(E_{ij})$ with $1 \le i,j \le n$ acts on $\mathsf{D}^{A,\varepsilon}(M)$ in the following way:

$$J(E_{ij})(m\otimes \mathbf{v}) = \varepsilon\lambda_1 \sum_{k=1}^{l}\left(\frac{1}{\kappa_1}y_k + \frac{1}{2}\sum_{s=1}^{k-1}\sigma_{sk} - \frac{1}{2}\sum_{s=k+1}^{l}\sigma_{sk}\right) m\otimes E_{ij}^{(k)}(\mathbf{v}).$$

For $E_{ij}\in\mathfrak{p}$, we can make the following computations:

$$\begin{aligned}
\sum_{k=1}^{l} m\otimes R_k E_{ij}^{(k)}(\mathbf{v}) &= -\frac{1}{2}\sum_{k=1}^{l} m\otimes\left(\kappa_2\Theta_p^{(k)} + \kappa_1\sum_{\substack{h=1\\h\neq k}}^{l}\sum_{s,t=1}^{n} E_{st}^{(h)}E_{ts}^{(k)}\Theta_p^{(h)}\Theta_p^{(k)}\right)E_{ij}^{(k)}(\mathbf{v})\\
&= -\frac{\kappa_2}{2}\sum_{k=1}^{l} m\otimes\epsilon_i E_{ij}^{(k)}(\mathbf{v}) - \frac{\kappa_1}{2}\sum_{k=1}^{l}\sum_{\substack{h=1\\h\neq k}}^{l}\sum_{s,t=1}^{n} m\otimes\big((E_{st})^{(h)}\epsilon_t(E_{ts}E_{ij})^{(k)}\epsilon_s\big)(\mathbf{v})\\
&= -\frac{\kappa_2\epsilon_i}{2}E_{ij}(m\otimes\mathbf{v}) - \frac{\kappa_1}{2}\sum_{k=1}^{l}\sum_{\substack{h=1\\h\neq k}}^{l}\sum_{t=1}^{n}\epsilon_t\epsilon_i m\otimes(E_{it})^{(h)}E_{tj}^{(k)}(\mathbf{v})\\
&= -\frac{\kappa_2\epsilon_i}{2}E_{ij}(m\otimes\mathbf{v}) - \frac{\kappa_1}{2}\sum_{k,h=1}^{l}\sum_{t=1}^{n}\epsilon_t\epsilon_i m\otimes\big(E_{it}^{(h)}E_{tj}^{(k)}\big)(\mathbf{v}) + \frac{\kappa_1}{2}\sum_{k=1}^{l}\sum_{t=1}^{n}\epsilon_t\epsilon_i m\otimes E_{it}^{(k)}E_{tj}^{(k)}(\mathbf{v})\\
&= -\frac{\epsilon_i}{2}(\kappa_2+\kappa_1(n-2p))E_{ij}(m\otimes\mathbf{v}) - \frac{\kappa_1\epsilon_i}{2}\left(\sum_{t=1}^{p}E_{it}E_{tj} - \sum_{t=p+1}^{n}E_{it}E_{tj}\right)(m\otimes\mathbf{v})\\
&\quad\text{(using Lemma 3.2)}\\
&= -\frac{\epsilon_i}{2}(\kappa_2+\kappa_1(n-2p))E_{ij}(m\otimes\mathbf{v}) + \frac{\kappa_1\epsilon_i}{2}\left(\frac{n-2p}{2}E_{ij} - \frac{\epsilon_i}{2}[C,E_{ij}]\right)(m\otimes\mathbf{v})\\
&= -\left(\epsilon_i\left(\frac{\kappa_2}{2} + \frac{\kappa_1(n-2p)}{4}\right)E_{ij} + \frac{\kappa_1}{4}[C,E_{ij}]\right)(m\otimes\mathbf{v}).
\end{aligned}$$

Since

$$\widetilde{J}(E_{ij}) = J(E_{ij}) - \epsilon_i\left(\frac{\lambda_2}{2} + \frac{\lambda_1(n-2p)}{4}\right)E_{ij} - \frac{\lambda_1}{4}[C,E_{ij}], \tag{9}$$

we see that if $\lambda_1=\kappa_1, \lambda_2=\kappa_2$, we can define the action of $\widetilde{J}(E_{ij})$ on $\mathsf{D}^{A,\varepsilon}(M)$ to be

$$\widetilde{J}(E_{ij})(m\otimes\mathbf{v}) = \varepsilon\sum_{k=1}^{l}\left(y_k + \frac{\lambda_1}{2}\sum_{s=1}^{k-1}\sigma_{sk} - \frac{\lambda_1}{2}\sum_{s=k+1}^{l}\sigma_{sk} + R_k\right)m\otimes E_{ij}^{(k)}\mathbf{v} = \varepsilon\sum_{k=1}^{l}\widetilde{y}_k m\otimes E_{ij}^{(k)}\mathbf{v}. \tag{10}$$

We now want to see that it descends to an operator on $\mathsf{D}^{BC,\varepsilon}(M)$, so we have to show that it stabilizes the subspace V_l spanned by $\gamma_l m\otimes\mathbf{v} - \varepsilon m\otimes\Theta_p^{(l)}\mathbf{v}$ for any $m\in M, \mathbf{v}\in(\mathbb{C}^n)^{\otimes l}$:

$$\begin{aligned}
\widetilde{J}(E_{ij})(\gamma_l m\otimes\mathbf{v} - \varepsilon m\otimes\Theta_p^{(l)}\mathbf{v}) &= \varepsilon\sum_{k=1}^{l}(\widetilde{y}_k\gamma_l m\otimes E_{ij}^{(k)}\mathbf{v} - \varepsilon\widetilde{y}_k m\otimes E_{ij}^{(k)}\Theta_p^{(l)}(\mathbf{v}))\\
&= \varepsilon\sum_{k=1}^{l}(-1)^{\delta_{kl}}\big(\gamma_l\widetilde{y}_k m\otimes E_{ij}^{(k)}\mathbf{v} - \varepsilon\widetilde{y}_k m\otimes\Theta_p^{(l)}E_{ij}^{(k)}(\mathbf{v})\big)\in V_l.
\end{aligned}$$

We have thus shown that $\widetilde{J}(E_{ij})$ descends to an operator on $\mathsf{D}^{BC,\varepsilon}(M)$. The space $\mathsf{D}^{BC,\varepsilon}(M)$ is thus a left module over the MacKay twisted Yangian $\widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n,\mathfrak{k})$.

The proof that $\mathsf{D}^{BC,\varepsilon}$ provides an equivalence of categories follows from similar arguments as those used in [ChPr2, Gu1, VaVa], so we just outline the main ideas. The tensor space $(\mathbb{C}^n)^{\otimes l}$ decomposes as a direct sum of irreducible modules over $GL_p(\mathbb{C}) \times GL_{n-p}(\mathbb{C}) \times W_l$ (see [ATY] for a precise statement), hence any left module N over $\widetilde{Y}_{\lambda_1,\lambda_2}(\mathfrak{gl}_n, \mathfrak{k})$ of level l is of the form $N = M \otimes_{\mathbb{C}} (\mathbb{C}^n)^{\otimes l} / \sum_{i=1}^{l-1} \mathrm{Im}(\sigma_i - \varepsilon) + \mathrm{Im}(\gamma_l - \varepsilon)$ for some W_l-module M. To transform M into a module over $\mathtt{H}^l_{\kappa_1,\kappa_2}$, we need to find commuting operators $\mathcal{Y}_k \in \mathrm{End}_{\mathbb{C}}(M)$ for $k = 1, \dots, l$ such that $\mathcal{Y}_1, \dots, \mathcal{Y}_l, \sigma_1, \dots, \sigma_{l-1}, \gamma_l$ satisfy the defining relations of $\mathtt{H}^l_{\kappa_1,\kappa_2}$ given in Lemma 2.2. To determine how $\mathcal{Y}_j$ should act on $m \in M$, pick a primitive tensor $\mathbf{v} = v_{k_1} \otimes \cdots \otimes v_{k_l}$ with $k_1, \dots, k_l$ all distinct and $\{v_1, \dots, v_n\}$ the standard basis of $\mathbb{C}^n$ and consider how $\widetilde{J}(E_{i,k_j})$ acts on it, where $i \neq k_1, \dots, k_l$. (Here, the assumption that $n \geq \max\{p, n-p\} \geq l+1$ is needed.) To deduce that the commutator $[\mathcal{Y}_i, \mathcal{Y}_j]$ (for, say, $i < j$) is given by formula (1), one should apply $[\widetilde{J}(E_{a_i,b_i}), \widetilde{J}(E_{a_j,b_j})]$ to $m \otimes \mathbf{v}$ for an appropriate choice of $a_i, b_i, a_j, b_j, \mathbf{v}$ (for instance, take $b_i = k_i, b_j = k_j$ and $k_1, \dots, k_l, a_i, a_j$ all distinct). The assumption $p, n-p \geq l+1$ ensures that $\mathsf{D}^{BC,\varepsilon}(M)$ is non-zero if M is non-zero: see [ATY]. □

4.4. **Drinfeld functor for twisted Yangians of type AIII.** In this section, we will construct a functor from the category of left modules over the degenerate affine Hecke algebra of type BC to the category of left modules over the twisted Yangian of type AIII which was introduced in Section 3.3. We will use the same notation for this functor as in the previous section.

For any $\beta \in \mathbb{C}$, denote by $B_\beta(n,p)$ the twisted Yangian $B_{\tau_1,\tau_2}(n,p)$ with parameters $\tau_1 = 1, \tau_2 = \beta$. For any left $\mathtt{H}^l_{\kappa_1,\kappa_2}$-module M, view it as an $\mathtt{H}^l_{\kappa_1}$-module and set $\mathsf{D}^{BC,\varepsilon}(M) = \mathsf{D}^{A,\varepsilon}(M)/\mathrm{Im}(\gamma_l - \varepsilon)$.

For $k = 1, \dots, l$, define the following elements in $\mathtt{H}^l_{\kappa_1,\kappa_2}[[u^{-1}]] \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}((\mathbb{C}^n)^{\otimes l}) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$:

$$\mathtt{T}^\lambda_k(u) = 1 + \frac{1}{u - n/2 - \lambda y_k} \otimes \mathtt{I}_k, \quad \mathtt{S}^\lambda_k(u) = 1 - \frac{1}{u + n/2 - \lambda y_k} \otimes \mathtt{I}_k.$$

Lemma 4.2. *We have* $\mathtt{T}^\lambda_k(u)\mathtt{S}^\lambda_k(u) = 1$.

Proof. Using $\mathtt{I}_k \cdot \mathtt{I}_k = n\mathtt{I}_k$, we get

$$\begin{aligned} \mathtt{T}^\lambda_k(u)\mathtt{S}^\lambda_k(u) &= \left(1 + \frac{1}{u - n/2 - \lambda y_k} \otimes \mathtt{I}_k\right)\left(1 - \frac{1}{u + n/2 - \lambda y_k} \otimes \mathtt{I}_k\right) \\ &= 1 + \frac{u + n/2 - \lambda y_k - u + n/2 + \lambda y_k - n}{(u - n/2 - \lambda y_k)(u + n/2 - \lambda y_k)} \otimes \mathtt{I}_k = 1. \end{aligned}$$

□

Set

$$\mathtt{B}^\lambda(u) = \mathtt{T}^\lambda_1(u) \cdots \mathtt{T}^\lambda_l(u)\Theta_{p,\beta}(u)\mathtt{S}^\lambda_l(-u) \cdots \mathtt{S}^\lambda_1(-u). \tag{11}$$

Here we regard $\mathtt{B}^\lambda(u)$ as an $n \times n$-matrix whose entries, denoted $\mathtt{b}^\lambda_{ij}(u)$, are power series in u^{-1}. $\Theta_{p,\beta}(u)$ is shorter notation for $1 \otimes 1 \otimes \Theta_{p,\beta}(u) \in \mathtt{H}^l_{\kappa_1,\kappa_2}[[u^{-1}]] \otimes_{\mathbb{C}} \mathrm{End}\,((\mathbb{C}^n)^{\otimes l}) \otimes_{\mathbb{C}} \mathrm{End}\,(\mathbb{C}^n)$.

From Proposition 4.1, when $\lambda = \frac{\varepsilon}{\kappa_1}$, the map $T(u) \to \mathtt{T}^\lambda(u)$ defines an action of $Y(\mathfrak{gl}_n)$ on $\mathsf{D}^{A,\varepsilon}(M)$, so from the definition of $B_\beta(n,p)$ and Lemma 4.2, the map $B(u) \to \mathtt{B}^\lambda(u)$ defines a representation of $B_\beta(n,p)$ on the same space.

Theorem 4.2. *Let M be any $\mathtt{H}^l_{\kappa_1,\kappa_2}$-module and $\beta = \dfrac{\kappa_2}{2\kappa_1} + \dfrac{n-2p}{2}$. If $\lambda = \varepsilon/\kappa_1$, the map $B(u) \to \mathtt{B}^\lambda(u)$ defines a representation of the twisted Yangian $B_\beta(n,p)$ on the space $\mathsf{D}^{BC,\varepsilon}(M)$.*

Proof. Assume $\varepsilon = 1$. (The proof is similar when $\varepsilon = -1$.) From Lemma 2.3, we know that the element $\prod_{k=1}^l \big((u - n/2)^2 - \lambda^2 y_k^2\big)$ lies in the center of $\mathtt{H}^l_{\kappa_1,\kappa_2}$. Thus we can multiply both sides of (11) by this central element and we get

$$\widetilde{\mathtt{B}}^\lambda(u) = \prod_{k=1}^l ((u - n/2)^2 - \lambda^2 y_k^2)\mathtt{B}^\lambda(u) = \widetilde{\mathtt{T}}^\lambda_1(u) \cdots \widetilde{\mathtt{T}}^\lambda_l(u)\Theta_{p,\beta}(u)\widetilde{\mathtt{S}}^\lambda_l(-u) \cdots \widetilde{\mathtt{S}}^\lambda_1(-u),$$

where $\widetilde{\mathtt{T}}_k^\lambda(u) = u - n/2 - \lambda y_k + 1 \otimes \mathtt{I}_k, \widetilde{\mathtt{S}}_k^\lambda(-u) = u - n/2 + \lambda y_k + 1 \otimes \mathtt{I}_k$. It is enough to show that the commutator of $\widetilde{\mathtt{B}}^\lambda(u)$ and γ_l on $\mathsf{D}^{A,\varepsilon}(M)$ has its image contained in the image of $\gamma_l - 1$.

For each $\mathtt{I}_k$, let $\mathtt{I}_k^a = \sum_{1\le i,j\le p} E_{ij}^{(k)} \otimes E_{ij} + \sum_{p<i,j\le n} E_{ij}^{(k)} \otimes E_{ij}$ and $\mathtt{I}_k^b = \mathtt{I}_k - \mathtt{I}_k^a$. Notice that we have the following commutation relations:

$$\gamma_l \mathtt{I}_j = \mathtt{I}_j\gamma_l, \text{ if } j \neq l, \quad \mathtt{I}_l^a\gamma_l = \gamma_l\mathtt{I}_l^a, \mathtt{I}_l^b\gamma_l = -\gamma_l\mathtt{I}_l^b, \quad [\gamma_l, \Theta_{p,\beta}(u)] = 0.$$

From these, we can see that γ_l commutes with $\widetilde{\mathtt{T}}_k^\lambda$ and $\widetilde{\mathtt{S}}_k^\lambda$ if $k \neq l$.

Thus we only need to check the commutator of γ_l and $\widetilde{\mathtt{T}}_l^\lambda(u)\Theta_{p,\beta}(u)\widetilde{\mathtt{S}}_l^\lambda(-u)$ on $\mathsf{D}^{A,\varepsilon}(M)$. Set $\mathtt{R} = u - n/2 + 1 \otimes \mathtt{I}_l^a$. Then $[\mathtt{R}, \gamma_l] = 0$, $[\mathtt{R}, \Theta_{p,\beta}(u)] = 0$.

$$\begin{aligned}
&\widetilde{\mathtt{T}}_l^\lambda(u)\Theta_{p,\beta}(u)\widetilde{\mathtt{S}}_l^\lambda(-u)\gamma_l \\
&\qquad = (\mathtt{R} - \lambda y_l + 1 \otimes \mathtt{I}_l^b)\Theta_{p,\beta}(u)(\mathtt{R} + \lambda y_l + 1 \otimes \mathtt{I}_l^b)\gamma_l \\
&\qquad = (\mathtt{R} - \lambda y_l + 1 \otimes \mathtt{I}_l^b)\Theta_{p,\beta}(u)(\gamma_l\mathtt{R} + \lambda(-\gamma_l y_l + \kappa_2) \otimes \Theta_p^{(l)} - \gamma_l(1 \otimes \mathtt{I}_l^b)) \\
&\qquad = (\mathtt{R} - \lambda y_l + 1 \otimes \mathtt{I}_l^b)\Theta_{p,\beta}(u)\gamma_l(\mathtt{R} - \lambda y_l - 1 \otimes \mathtt{I}_l^b) + (\mathtt{R} - \lambda y_l + 1 \otimes \mathtt{I}_l^b)\Theta_{p,\beta}(u)(\lambda\kappa_2 \otimes \Theta_p^{(l)}) \\
&\qquad = (\gamma_l\mathtt{R} - \lambda(-\gamma_l y_l + \kappa_2) \otimes \Theta_p^{(l)} - \gamma_l(1 \otimes \mathtt{I}_l^b))\Theta_{p,\beta}(u)(\mathtt{R} - \lambda y_l - 1 \otimes \mathtt{I}_l^b) \\
&\qquad\quad + (\mathtt{R} - \lambda y_l + 1 \otimes \mathtt{I}_l^b)\Theta_{p,\beta}(u)(\lambda\kappa_2 \otimes \Theta_p^{(l)}) \\
&\qquad = \gamma_l(\mathtt{R} + \lambda y_l - 1 \otimes \mathtt{I}_l^b)\Theta_{p,\beta}(u)(\mathtt{R} - \lambda y_l - 1 \otimes \mathtt{I}_l^b) - (\lambda\kappa_2 \otimes \Theta_p^{(l)})\Theta_{p,\beta}(u)(\mathtt{R} - \lambda y_l - 1 \otimes \mathtt{I}_l^b) \\
&\qquad\quad + (\mathtt{R} - \lambda y_l + 1 \otimes \mathtt{I}_l^b)\Theta_{p,\beta}(u)(\lambda\kappa_2 \otimes \Theta_p^{(l)}).
\end{aligned}$$

Since

$$\begin{aligned}
&(\mathtt{R} + \lambda y_l - 1 \otimes \mathtt{I}_l^b)\Theta_{p,\beta}(u)(\mathtt{R} - \lambda y_l - 1 \otimes \mathtt{I}_l^b) - \widetilde{\mathtt{T}}_l^\lambda(u)\Theta_{p,\beta}(u)\widetilde{\mathtt{S}}_l^\lambda(u) \\
&\qquad = (\mathtt{R} + \lambda y_l - 1 \otimes \mathtt{I}_l^b)\Theta_{p,\beta}(u)(\mathtt{R} - \lambda y_l - 1 \otimes \mathtt{I}_l^b) - (\mathtt{R} - \lambda y_l + 1 \otimes \mathtt{I}_l^b)\Theta_{p,\beta}(u)(\mathtt{R} + \lambda y_l + 1 \otimes \mathtt{I}_l^b) \\
&\qquad = -2\left(\mathtt{R}\Theta_{p,\beta}(u)(1 \otimes \mathtt{I}_l^b) + (1 \otimes \mathtt{I}_l^b)\Theta_{p,\beta}(u)\mathtt{R}\right) \\
&\qquad = -2\left((\Theta_p + \beta u^{-1})(u - n/2 + 1 \otimes \mathtt{I}_l^a)(1 \otimes \mathtt{I}_l^b) + (-\Theta_p + \beta u^{-1})(1 \otimes \mathtt{I}_l^b)(u - n/2 + 1 \otimes \mathtt{I}_l^a)\right) \\
&\qquad = -2\left(\beta(2 - nu^{-1})\mathtt{I}_l^b + \Theta_p(\mathtt{I}_l^a\mathtt{I}_l^b - \mathtt{I}_l^b\mathtt{I}_l^a) + \beta u^{-1}(\mathtt{I}_l^a\mathtt{I}_l^b + \mathtt{I}_l^b\mathtt{I}_l^a)\right) \\
&\qquad = -2\beta(2 - nu^{-1})(1 \otimes \mathtt{I}_l^b) - 2(\Theta_p + \beta u^{-1})\left(p\sum_{i\le p, j>p} E_{ij}^{(l)} \otimes E_{ij} + (n-p)\sum_{i>p, j\le p} E_{ij}^{(l)} \otimes E_{ij}\right) \\
&\qquad\quad + 2(\Theta_p - \beta u^{-1})\left((n-p)\sum_{i\le p, j>p} E_{ij}^{(l)} \otimes E_{ij} + p\sum_{i>p, j\le p} E_{ij}^{(l)} \otimes E_{ij}\right) \\
&\qquad = -2(2\beta + 2p - n)(1 \otimes \mathtt{I}_l^b)
\end{aligned}$$

and

$$\begin{aligned}
&-(\lambda\kappa_2 \otimes \Theta_p^{(l)})\Theta_{p,\beta}(u)(\mathtt{R} - \lambda y_l - 1 \otimes \mathtt{I}_l^b) + (\mathtt{R} - \lambda y_l + 1 \otimes \mathtt{I}_l^b)\Theta_{p,\beta}(u)(\lambda\kappa_2 \otimes \Theta_p^{(l)}) \\
&\qquad = (\lambda\kappa_2 \otimes \Theta_p^{(l)})\Theta_{p,\beta}(u)(1 \otimes \mathtt{I}_l^b) + (1 \otimes \mathtt{I}_l^b)\Theta_{p,\beta}(u)(\lambda\kappa_2 \otimes \Theta_p^{(l)}) \\
&\qquad = 2\lambda\kappa_2(1 \otimes \mathtt{I}_l^b),
\end{aligned}$$

we have

$$\widetilde{\mathtt{T}}_l^\lambda(u)\Theta_{p,\beta}(u)\widetilde{\mathtt{S}}_l^\lambda(-u)\gamma_l - \gamma_l\widetilde{\mathtt{T}}_l^\lambda(u)\Theta_{p,\beta}(u)\widetilde{\mathtt{S}}_l^\lambda(-u) = -2\gamma_l(2\beta + 2p - n)(1 \otimes \mathtt{I}_l^b) + 2\lambda\kappa_2(1 \otimes \mathtt{I}_l^b) = 2\lambda\kappa_2(1 - \gamma_l)(1 \otimes \mathtt{I}_l^b),$$

because $\beta = \dfrac{\kappa_2}{2\kappa_1} + \dfrac{n-2p}{2}$ and $\lambda = \frac{\varepsilon}{\kappa_1}$. This proves that the entries of the coefficients of $\mathtt{T}_l^\lambda(u)\Theta_{p,\beta}(u)\mathtt{S}_l^\lambda(-u)$ descend to endomorphisms of $\mathsf{D}^{BC,\varepsilon}(M)$. □

The Drinfeld functor is compatible with the coproduct in the following sense. Recall Lemma 3.1 and the observation after Lemma 2.1.

Proposition 4.2. *Choose $l_1, l_2 \in \mathbb{Z}_{\geq 1}$ such that $l_1 + l_2 = l$. Let M_1 be an $\mathtt{H}^{l_1}_{\kappa_1}$-module and let M_2 be an $\mathtt{H}^{l_2}_{\kappa_1,\kappa_2}$-module. Set $M_1 \odot M_2 = \mathtt{H}^{l}_{\kappa_1,\kappa_2} \otimes_{\mathtt{H}^{l_1}_{\kappa_1} \otimes \mathtt{H}^{l_2}_{\kappa_1,\kappa_2}} (M_1 \otimes_{\mathbb{C}} M_2)$, which is an $\mathtt{H}^{l}_{\kappa_1,\kappa_2}$-module. Then $\mathsf{D}^{BC,\varepsilon}(M_1 \odot M_2)$ is isomorphic to $\mathsf{D}^{A,\varepsilon}(M_1) \otimes_{\mathbb{C}} \mathsf{D}^{BC,\varepsilon}(M_2)$ as a module over $B_\beta(n,p)$, where $\mathsf{D}^{A,\varepsilon}(M_1) \otimes_{\mathbb{C}} \mathsf{D}^{BC,\varepsilon}(M_2)$ becomes a left module over $B_\beta(n,p)$ via the coideal structure given in Proposition 3.2.*

Proof. As W_l-modules, $\mathtt{H}^{l}_{\kappa_1,\kappa_2} \cong \mathbb{C}[W_l] \otimes_{\mathbb{C}[\mathfrak{S}_{l_1}] \otimes_{\mathbb{C}} \mathbb{C}[W_{l_2}]} \mathtt{H}^{l_1}_{\kappa_1} \otimes_{\mathbb{C}} \mathtt{H}^{l_2}_{\kappa_1,\kappa_2}$, so $\mathsf{D}^{BC,\varepsilon}(M_1 \odot M_2) \cong \mathsf{D}^{A,\varepsilon}(M_1) \otimes_{\mathbb{C}} \mathsf{D}^{BC,\varepsilon}(M_2)$ as modules over $\mathfrak{gl}_p \oplus \mathfrak{gl}_{n-p}$. To complete the proof, it is enough to check that the action of $\mathsf{B}^{\lambda}(u)$ on $\mathsf{D}^{BC,\varepsilon}(M_1 \odot M_2)$ comes from the coproduct Δ: this follows from Proposition 3.2 and formula (11). ☐

The following theorem was proved in [Na1] by M. Nazarov. An analogous result also holds for Yangians in type A [Ar] and for super Yangians of type Q_n [Na2]. Actually, the proof presented below is similar to the proof of theorem 5.5 in [Na2].

Theorem 4.3. [Na1] *Let $\kappa_1, \kappa_2, \lambda, \beta$ be as in Theorem 4.2. Let M be an irreducible module over $\mathtt{H}^{l}_{\kappa_1,\kappa_2}$. Then $\mathsf{D}^{BC,\varepsilon}(M)$ is either 0 or an irreducible module over $B_\beta(n,p)$.*

Proof. One of the ideas is to reduce to the case of the twisted current algebra. Suppose that $\mathsf{D}^{BC,\varepsilon}(M) \neq \{0\}$ for some irreducible module M over $\mathtt{H}^{l}_{\kappa_1,\kappa_2}$. We want to show that $\mathsf{D}^{BC,\varepsilon}(M)$ is irreducible. Let $N_0 \subset \mathsf{D}^{BC,\varepsilon}(M)$ be a subspace preserved by the action of $B_\beta(n,p)$. Since we have $\mathfrak{gl}_p(\mathbb{C}) \oplus \mathfrak{gl}_{n-p}(\mathbb{C}) \subset B_\beta(n,p)$, N_0 is also preserved by $\mathfrak{gl}_p(\mathbb{C}) \oplus \mathfrak{gl}_{n-p}(\mathbb{C})$. From the classical Schur-Weyl duality between $\mathfrak{gl}_p(\mathbb{C}) \oplus \mathfrak{gl}_{n-p}(\mathbb{C})$ and W_l (see [ATY]), there exists a W_l-submodule M_0 of M such that $N_0 = \mathsf{D}^{BC,\varepsilon}(M_0)$. Assume that for any non-zero vector $b \in M_0$ the image of the subspace $\mathbb{C}b \otimes_{\mathbb{C}} (\mathbb{C}^n)^{\otimes l}$ is not zero in N_0. Notice that since M is irreducible, we have $\mathtt{H}^{l}_{\kappa_1,\kappa_2} \cdot M_0 = M$. We only need to show that N_0 generates $\mathsf{D}^{BC,\varepsilon}(M)$ under the action of $B_\beta(n,p)$. Let $M' = \mathtt{H}^{l}_{\kappa_1,\kappa_2} \otimes_{\mathbb{C}[W_l]} M_0 \cong \mathbb{C}[y_1, \ldots, y_l] \otimes_{\mathbb{C}} M_0$ be the left module over $\mathtt{H}^{l}_{\kappa_1,\kappa_2}$ induced from M_0. Then M is a quotient of M'. After identifying M_0 with the submodule $1 \otimes M_0 \subset M'$, we see that it is enough to show that $\mathsf{D}^{BC,\varepsilon}(1 \otimes M_0)$ generates $\mathsf{D}^{BC,\varepsilon}(M')$ under the action of $B_\beta(n,p)$.

Define a grading on $\mathtt{H}^{l}_{\kappa_1,\kappa_2}$ by letting $\deg(y_i) = 1$ for $i = 1, \ldots, l$, and $\deg(\sigma) = 0$ for $\sigma \in W_l$. Then M' becomes a filtered module. This induces a filtration on $M' \otimes_{\mathbb{C}} (\mathbb{C}^n)^{\otimes l}$ and so on the quotient $\mathsf{D}^{BC,\varepsilon}(M')$ which is compatible with the one on $B_\beta(n,p)$ defined in Section 3.3. After passing to the associated graded spaces and using Corollary 3.3, we are reduced to proving the theorem with $\mathtt{H}^{l}_{\kappa_1,\kappa_2}$ replaced by $\operatorname{gr} \mathtt{H}^{l}_{\kappa_1,\kappa_2} \cong \mathbb{C}[\bar{y}_1, \ldots, \bar{y}_l] \rtimes W_l$, $B_\beta(n,p)$ replaced by $\operatorname{gr} B_\beta(n,p) \cong \mathfrak{U}\mathfrak{gl}_n^{\theta}(\mathbb{C}[t])$ and M' replaced by $\mathbb{C}[\bar{y}_1, \ldots, \bar{y}_l] \otimes_{\mathbb{C}} M_0$. Here we use $\bar{y}_i$ to denote the image of the element y_i in $\operatorname{gr} \mathtt{H}^{l}_{\kappa_1,\kappa_2}$. Set $W = \mathsf{D}^{BC,\varepsilon}(\mathbb{C}[\bar{y}_1, \ldots, \bar{y}_l] \otimes_{\mathbb{C}} M_0)$. We only need to show that $\mathsf{D}^{BC,\varepsilon}(1 \otimes M_0)$ generates W under the action of $\mathfrak{U}\mathfrak{gl}_n^{\theta}(\mathbb{C}[t])$. The rest of the proof follows the argument in [Na2]. ☐

In light of the previous theorem, it may be useful to have a criterion which gives a sufficient condition for certain modules over the degenerate affine Hecke algebra to be irreducible. Such a criterion for principal series modules is proved in [KrRa]; see also [Ka] for the analogous result for affine Hecke algebras.

Definition 4.1. *Let $\mathbf{a} = (a_1, \ldots, a_l) \in \mathbb{C}^l$ and let $\mathbb{C}_{\mathbf{a}} = \mathbb{C}[y_1, \ldots, y_l]/(y_i - a_i)_{i=1}^{l}$. The module $\mathtt{H}^{l}_{\kappa_1,\kappa_2} \otimes_{\mathbb{C}[y_1,\ldots,y_l]} \mathbb{C}_{\mathbf{a}}$ is called a principal series module and is denoted $M_{\mathbf{a}}$.*

Theorem 4.4. [KrRa] *$M_{\mathbf{a}}$ is irreducible if and only if $a_i \neq \kappa_2 \ \forall\, i = 1, \ldots, l$ and $a_i \pm a_j \neq \kappa_1, -\kappa_1 \ \forall\, 1 \leq i < j \leq l$.*

Combining Theorems 4.3 and 4.4, we obtain a family of irreducible finite dimensional representations of $B_\beta(n,p)$.

We thank M. Nazarov for bringing Kato's theorem to our attention.

4.5. **Compatibility of the two Drinfeld functors.** In this section, we will show that the construction in Section 4.4 recovers the one in Section 4.3. This is to be expected in light of Proposition 3.4, but doesn't follow immediately from this isomorphism since it is necessary to find an explicit formula for the operator through which $\mathtt{b}_{ij}^{(2)}$ acts (if possible in terms of $\widetilde{y}_k$ instead of y_k) and then compare it with the formula we already have for $\widetilde{J}(E_{ij})$, namely (10), which involves $\widetilde{y}_k$.

Let us denote by "≡" the equivalence modulo u^{-3}. We have

$$\begin{aligned}\mathtt{B}^{\lambda}(u) \equiv\;& \left(1+u^{-1}\otimes\mathtt{I}_1+u^{-2}\left(\frac{n}{2}+\lambda y_1\right)\otimes\mathtt{I}_1\right)\cdots\left(1+u^{-1}\otimes\mathtt{I}_l+u^{-2}\left(\frac{n}{2}+\lambda y_l\right)\otimes\mathtt{I}_l\right)\\ &\cdot(\Theta_p+\beta u^{-1})\cdot\left(1+u^{-1}\otimes\mathtt{I}_l+u^{-2}\left(\frac{n}{2}-\lambda y_l\right)\otimes\mathtt{I}_l\right)\cdots\left(1+u^{-1}\otimes\mathtt{I}_1+u^{-2}\left(\frac{n}{2}-\lambda y_1\right)\otimes\mathtt{I}_1\right)\\ \equiv\;& \Theta_p+u^{-1}\left(\beta+1\otimes\sum_{k=1}^{l}(\mathtt{I}_k\Theta_p+\Theta_p\mathtt{I}_k)\right)+u^{-2}\left(2\beta\left(1\otimes\sum_{k=1}^{l}\mathtt{I}_k\right)+\sum_{1\le k<s\le l}(1\otimes\mathtt{I}_k\mathtt{I}_s)\Theta_p\right.\\ &+\sum_{l\ge k>s\ge 1}\Theta_p(1\otimes\mathtt{I}_k\mathtt{I}_s)+\left(1\otimes\sum_{k=1}^{l}\mathtt{I}_k\right)\Theta_p\left(1\otimes\sum_{s=1}^{l}\mathtt{I}_s\right)\\ &\left.+\sum_{k=1}^{l}\left(\Theta_p\left(\left(\frac{n}{2}-\lambda y_k\right)\otimes\mathtt{I}_k\right)+\left(\left(\frac{n}{2}+\lambda y_k\right)\otimes\mathtt{I}_k\right)\Theta_p\right)\right).\end{aligned} \tag{12}$$

So if we set $\mathtt{B}^{\lambda}(u)=\sum_{i,j=1}^{n}\mathtt{b}_{ij}^{\lambda}(u)\otimes E_{ij}$ and $\mathtt{b}_{ij}^{\lambda}(u)=\sum_{r=0}^{\infty}\mathtt{b}_{ij}^{(r)}u^{-r}$, we have

$$\mathtt{b}_{ij}^{(0)}=\epsilon_i\delta_{ij},\quad \mathtt{b}_{ij}^{(1)}=\beta\delta_{ij}+(\epsilon_i+\epsilon_j)\sum_{k=1}^{l}1\otimes E_{ij}^{(k)}. \tag{13}$$

In order to calculate $\mathtt{b}_{ij}^{(2)}$, we need the following lemma.

Lemma 4.3. *Assume $1\le i\le p<j\le n$ or $1\le j\le p<i\le n$. As operators on $(\mathbb{C}^n)^{\otimes l}$, we have*

$$\sum_{k=1}^{l}\left(-\sum_{t=k+1}^{l}\sigma_{tk}+\sum_{t=1}^{k-1}\sigma_{tk}\right)E_{ij}^{(k)}=-\epsilon_i\left(\sum_{1\le k<s\le l}(\mathtt{I}_k\mathtt{I}_s)\Theta_p+\sum_{l\ge k>s\ge1}\Theta_p(\mathtt{I}_k\mathtt{I}_s)\right)_{ij} \tag{14}$$

$$\sum_{k=1}^{l}\left(\sum_{\substack{t=1\\t\neq k}}^{l}\sigma_{kt}\gamma_t\gamma_k+(2p-n)\gamma_k\right)E_{ij}^{(k)}=\epsilon_i\left(\left(\sum_{k=1}^{l}\mathtt{I}_k\right)\Theta_p\left(\sum_{s=1}^{l}\mathtt{I}_s\right)\right)_{ij}. \tag{15}$$

Here by $(\cdot)_{ij}$, we mean the (i,j)-th entry, i.e., for an element $\mathtt{G}\in\mathrm{End}_{\mathbb{C}}((\mathbb{C}^n)^{\otimes l})\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$, $\mathtt{G}=\sum_{i,j=1}^{n}(\mathtt{G})_{ij}\otimes E_{ij}$.

Proof. The proof is based on direct calculations. Since $\sigma_{sk}=\sum_{m_1,m_2=1}^{n}E_{m_1,m_2}^{(s)}\otimes E_{m_2,m_1}^{(k)}$, we can rewrite the left hand side of (14) as

$$-\sum_{s>k}\sum_{m=1}^{n}E_{im}^{(s)}E_{mj}^{(k)}+\sum_{s<k}\sum_{m=1}^{n}E_{im}^{(s)}E_{mj}^{(k)}.$$

On the other hand, we have

$$\sum_{k>s}\Theta_p(\mathtt{I}_k\mathtt{I}_s)+\sum_{k<s}(\mathtt{I}_k\mathtt{I}_s)\Theta_p=\sum_{k>s}\sum_{i,j,m=1}^{n}\epsilon_iE_{im}^{(k)}E_{mj}^{(s)}\otimes E_{ij}+\sum_{k<s}\sum_{i,j,m=1}^{n}\epsilon_jE_{im}^{(k)}E_{mj}^{(s)}\otimes E_{ij}$$

which implies the equality (14) (after switching k and s) since $\epsilon_i=-\epsilon_j$.

Similarly, the left hand side of (15) can be written as

$$\sum_{\substack{k,s=1\\ s\neq k}}^{l}\sum_{m=1}^{n}\epsilon_m\epsilon_i E_{mj}^{(k)}E_{im}^{(s)}+(2p-n)\sum_{k=1}^{l}\epsilon_i E_{ij}^{(k)}.$$

On the other hand, we have

$$\left(\sum_{k=1}^{l}\mathtt{I}_k\right)\Theta_p\left(\sum_{s=1}^{l}\mathtt{I}_s\right)=\sum_{s\neq k}\sum_{i,j,m=1}^{n}\epsilon_m E_{im}^{(k)}E_{mj}^{(s)}\otimes E_{ij}+\sum_{k=1}^{l}\sum_{i,j,m=1}^{n}\epsilon_m E_{ij}^{(k)}\otimes E_{ij}$$

which implies the equality (15). □

Now from (12) and the previous lemma, we have, for $1\leq i\leq p<j\leq n$ or $1\leq j\leq p<i\leq n$,

$$\begin{aligned}\mathtt{b}_{ij}^{(2)} &= 2\beta\sum_{k=1}^{l}E_{ij}^{(k)}-\epsilon_i\varepsilon\sum_{k=1}^{l}\left(-\sum_{t=k+1}^{l}\sigma_{tk}+\sum_{t=1}^{k-1}\sigma_{tk}\right)\otimes E_{ij}^{(k)}\\ &\quad+\epsilon_i\varepsilon\sum_{k=1}^{l}\left(\sum_{\substack{t=1\\ t\neq k}}^{l}\sigma_{kt}\gamma_t\gamma_k+(2p-n)\gamma_k\right)\otimes E_{ij}^{(k)}-2\lambda\epsilon_i\sum_{k=1}^{l}y_k\otimes E_{ij}^{(k)}\\ &= -2\epsilon_i\sum_{k=1}^{l}\left(\lambda y_k+\frac{\varepsilon}{2}\sum_{t=1}^{k-1}\sigma_{tk}-\frac{\varepsilon}{2}\sum_{t=k+1}^{l}\sigma_{tk}-\frac{\varepsilon}{2}\sum_{\substack{t=1\\ t\neq k}}^{l}\sigma_{kt}\gamma_t\gamma_k+\varepsilon\left(-\frac{2p-n}{2}-\beta\right)\gamma_k\right)\otimes E_{ij}^{(k)}.\end{aligned}$$

If we take $\lambda=\varepsilon/\kappa_1$, $\beta=\kappa_2/2\kappa_1-(2p-n)/2$, we have

$$\begin{aligned}\varepsilon\kappa_1\mathtt{b}_{ij}^{(2)} &= -2\epsilon_i\sum_{k=1}^{l}\left(y_k+\frac{\kappa_1}{2}\sum_{t=1}^{k-1}\sigma_{tk}-\frac{\kappa_1}{2}\sum_{t=k+1}^{l}\sigma_{tk}-\frac{\kappa_1}{2}\sum_{\substack{t=1\\ t\neq k}}^{l}\sigma_{kt}\gamma_t\gamma_k-\frac{\kappa_2}{2}\gamma_k\right)\otimes E_{ij}^{(k)}\\ &= -2\epsilon_i\sum_{k=1}^{l}\widetilde{y}_k\otimes E_{ij}^{(k)}.\end{aligned}\tag{16}$$

Comparing (7), (8), (10) and (16), we have the compatibility of the Drinfeld functor for the twisted Yangian of type AIII and of the Drinfeld functor for MacKay's twisted Yangian.

5. Fock space representation for the twisted Yangian $B_{\tau_1,\tau_2}(n,p)$

5.1. **Preliminaries.** We will need to work with a different presentation of the degenerate affine Hecke algebra of type BC_l.

Definition 5.1. *Let $\check{\mathtt{H}}^l_{\kappa_1,\kappa_2}$ be the algebra generated by the group algebra $\mathbb{C}[W_l]$ and a set of pairwise commuting elements $\check{y}_1,\ldots,\check{y}_l$ such that*

$$\sigma_i\check{y}_i-\check{y}_{i+1}\sigma_i=\kappa_1,\ \ \sigma_i\check{y}_j=\check{y}_j\sigma_i\ \text{ if } j\neq i,i+1,\tag{17}$$

$$\gamma_1\check{y}_1+\check{y}_1\gamma_1=\kappa_2,\ \ \gamma_1\check{y}_j=\check{y}_j\gamma_1\ \text{ if } j\neq 1.\tag{18}$$

Lemma 5.1. *$\check{\mathtt{H}}^l_{\kappa_1,\kappa_2}$ is isomorphic to $\mathtt{H}^l_{-\kappa_1,\kappa_2}$.*

Proof. An isomorphism $\mathtt{H}^l_{-\kappa_1,\kappa_2}\longrightarrow\check{\mathtt{H}}^l_{\kappa_1,\kappa_2}$ is given by $y_i\mapsto\check{y}_{l-i+1},\gamma_i\mapsto\gamma_{l-i+1},1\leq i\leq l$ and by $\sigma_i\mapsto\sigma_{l-i}$ for $1\leq i\leq l-1$. □

$\check{\mathtt{H}}^l_{\kappa_1,\kappa_2}$ acts on $P_l = \mathbb{C}[z_1^{\pm 1},\ldots,z_l^{\pm 1}]$ in the following way: the action of $\sigma \in \mathfrak{S}_l$ is by permuting the variables $z_i^{\pm 1}$; the action of γ_i is by sending $z_i^{\pm 1}$ to $z_i^{\mp 1}$; the commuting elements $\check{y}_1,\ldots,\check{y}_l$ act via trigonometric Cherednik-Dunkl operators $D_{i,l}$ of type C_l:

$$\begin{aligned} D_{i,l} &= z_i\frac{\partial}{\partial z_i} + \sum_{1\le k<i\le l}\frac{\kappa_1}{1-z_iz_k^{-1}}(1-\sigma_{ik}) - \sum_{1\le i<k\le l}\frac{\kappa_1}{1-z_i^{-1}z_k}(1-\sigma_{ik}) \\ &\quad + \sum_{1\le k\ne i\le l}\frac{\kappa_1}{1-z_kz_i}(1-\sigma_{ik}\gamma_i\gamma_k) - \frac{\kappa_2}{1-z_i^2}(1-\gamma_i) - (i-1)\kappa_1 + \frac{\kappa_2}{2}. \end{aligned}$$

Remark 5.1. *Here we make a rather unusual choice of simple roots for a root system of type C_l: $\epsilon_i - \epsilon_{i+1}$ for $i = 1,\ldots,l-1$ and $-2\epsilon_1$.*

Because of Lemma 5.1, any module over $\check{\mathtt{H}}^l_{\kappa_1,\kappa_2}$ can be viewed as a module over $\mathtt{H}^l_{-\kappa_1,\kappa_2}$. We thus have a version of the Drinfeld functor with $\check{\mathtt{H}}^l_{\kappa_1,\kappa_2}$ instead of $\mathtt{H}^l_{\kappa_1,\kappa_2}$. However, it will be necessary for us not only to change the degerate affine Hecke algebra, but also to consider a slightly different twisted Yangian.

Definition 5.2. *Let $\check{B}_\beta(n,p)$ be the coideal subalgebra of $Y(\mathfrak{gl}_n)$ generated by the coefficients $\check{b}_{ij}^{(r)}$ of the entries of $\check{B}(u)$ which is given by $\check{B}(u) = T^{-1}(-u)\Theta_{p,\beta}T(u)$.*

$\check{B}_\beta(n,p)$ is isomorphic as an algebra (but not as a coideal subalgebra) to $B_\beta(n,p)$ via the automorphism of the Yangian given by $T(u)\mapsto T(-u)^{-1}$.

Set $\varepsilon = -1$, $\kappa_1 = 1$, $\lambda = 1$ and $\beta = \frac{\kappa_2}{2} + \frac{n-2p}{2}$. (Here we view κ_1 and κ_2 as the parameters of $\check{\mathtt{H}}^l_{\kappa_1,\kappa_2}$.) If we set

$$D^{\mathrm{BC}_l,-1}(P_l) = P_l \otimes_{\mathbb{C}} (\mathbb{C}^n)^{\otimes l} \Big/ \left(\sum_{k=1}^{l-1}\mathrm{Im}(\sigma_k+1) + \mathrm{Im}(\gamma_1+1)\right),$$

an analog of Theorem 4.2 holds and we have an algebra homomorphism

$$\rho_{l,-1} : \check{B}_\beta(n,p) \to \mathrm{End}_{\mathbb{C}}\big(D^{\mathrm{BC}_l,-1}(P_l)\big)$$

which is given by $\rho_{l,-1}(\check{B}(u)) = \check{\mathtt{S}}_l(-u)\check{\mathtt{S}}_{l-1}(-u)\cdots\check{\mathtt{S}}_1(-u)\Theta_{p,\beta}(u)\check{\mathtt{T}}_1(u)\check{\mathtt{T}}_2(u)\cdots\check{\mathtt{T}}_l(u)$ as in equation (11) with $\lambda = 1$. $\check{\mathtt{T}}_i(u)$ is defined as $\mathtt{T}_i(u)$, but with $\check{y}_i$ instead of y_i, and similarly for $\check{\mathtt{S}}_i(-u)$.

As vector spaces, $P_l \otimes_{\mathbb{C}} (\mathbb{C}^n)^{\otimes l} \cong \bigotimes_{i=1}^l \left(\mathbb{C}[z_i^{\pm 1}]\otimes_{\mathbb{C}}\mathbb{C}^n\right)$. Let e_η $(\eta = 1,\ldots,n)$ be the standard basis for the vector space $\mathbb{C}^n$ and set $e_{\eta,k} = z^{-k}e_\eta$ for $k\in\mathbb{Z}$. Then $\{e_{\eta,k} | 1\le\eta\le n,\ k\in\mathbb{Z}\}$ forms a basis of $\mathbb{C}[z^{\pm 1}]\otimes_{\mathbb{C}}\mathbb{C}^n$ and $\{e_{\eta_1,k_1}\otimes e_{\eta_2,k_2}\otimes\cdots\otimes e_{\eta_l,k_l} | 1\le\eta_i\le n,\ k_i\in\mathbb{Z}\ \forall i=1,\ldots,l\}$ forms a basis of $P_l\otimes_{\mathbb{C}}(\mathbb{C}^n)^{\otimes l}$. The action of W_l on $P_l\otimes_{\mathbb{C}}(\mathbb{C}^n)^{\otimes l}$ can be written as

$$\sigma : e_{\eta_1,k_1}\otimes e_{\eta_2,k_2}\otimes\cdots\otimes e_{\eta_l,k_l} \mapsto e_{\eta_{\sigma(1)},k_{\sigma(1)}}\otimes e_{\eta_{\sigma(2)},k_{\sigma(2)}}\otimes\cdots\otimes e_{\eta_{\sigma(l)},k_{\sigma(l)}} \text{ for } \sigma\in\mathfrak{S}_l,$$

$$\gamma_l : e_{\eta_1,k_1}\otimes e_{\eta_2,k_2}\otimes\cdots\otimes e_{\eta_l,k_l} \mapsto e_{\eta_1,k_1}\otimes e_{\eta_2,k_2}\otimes\cdots\otimes \epsilon_{\eta_l}e_{\eta_l,-k_l}.$$

Let $V_{\mathrm{aff}}^{\wedge l}$ be the subspace of $\bigwedge_{i=1}^l\left(\mathbb{C}[z_i^{\pm 1}]\otimes_{\mathbb{C}}\mathbb{C}^n\right)$ spanned by $\{e_{\eta_1,k_1}\wedge e_{\eta_2,k_2}\wedge\cdots\wedge e_{\eta_l,k_l} | k_i\in\mathbb{Z}_{\ge 0}, \eta_i = 1,\ldots,n\}$ where $\bigwedge$ is the usual wedge product. (If $V^l = (\mathbb{C}^n)^{\otimes l}$, then V_{aff}^l is its affinization $P_l\otimes_{\mathbb{C}}(\mathbb{C}^n)^{\otimes l}$, but we will not use this notation here.) The quotient map $P_l\otimes_{\mathbb{C}}(\mathbb{C}^n)^{\otimes l}\twoheadrightarrow D^{\mathrm{BC}_l,-1}(P_l)$ induces a vector space isomorphism between $V_{\mathrm{aff}}^{\wedge l}$ and $D^{\mathrm{BC}_l,-1}(P_l)$. This allows us to view $V_{\mathrm{aff}}^{\wedge l}$ as a representation of $\check{B}_\beta(n,p)$. We also use $\rho_{l,-1}$ to denote the algebra homomorphism $\check{B}_\beta(n,p)\to\mathrm{End}_{\mathbb{C}}(V_{\mathrm{aff}}^{\wedge l})$ corresponding to this representation.

We introduce a function ϕ on the set $\{(\eta,k) | \eta = 1,\ldots,n, k\in\mathbb{Z}_{\ge 0}\}$ by $\phi(\eta,k) := \eta - n(k+1)$. Then ϕ defines a one-to-one correspondence between the index set in question and the set of non-positive integers; it induces an order on the index set by $(\eta_1,k_1) > (\eta_2,k_2)$ if and only if $\phi(\eta_1,k_1) > \phi(\eta_2,k_2)$. In the future, when we write a wedge product (either finite or infinite), we will always use a decreasing order on each monomial part.

5.2. **Fock space.** Let us recall the Fock space $\mathcal{F}$ considered in [Ug]. It is defined as the vector space spanned by semi-infinite wedge products $e_{\eta_1,k_1} \wedge e_{\eta_2,k_2} \wedge \cdots$ with the asymptotic condition that $\phi(e_{\eta_i,k_i}) = \phi(e_{\eta_{i+1},k_{i+1}}) + 1$ for all but a finite number of $i \in \mathbb{N}$. For any non-positive integer M, define $\mathcal{F}_M$ as the subspace of $\mathcal{F}$ spanned by $e_{\eta_1,k_1} \wedge e_{\eta_2,k_2} \wedge \cdots$ with $\phi(\eta_1,k_1) > \phi(\eta_2,k_2) > \cdots$ and the asymptotic condition that $\phi(e_{\eta_i,k_i}) = M - i + 1$ for all but a finite number of $i \in \mathbb{N}$. We call M the charge of the wedge product; observe that $\mathcal{F} = \bigoplus_{M\in\mathbb{Z}} \mathcal{F}_M$. In each space $\mathcal{F}_M$, there is a special element $|M\rangle$ called the vacuum vector of charge M defined by

$$|M\rangle = e_{\eta_1,k_1} \wedge e_{\eta_2,k_2} \wedge \cdots, \text{ where } \phi(e_{\eta_i,k_i}) = M - i + 1 \text{ for all } i.$$

From now on, we fix a charge $M = s - n(k+1)$ for some integers $k \in \mathbb{Z}_{\geq 0}$ and $1 \leq s \leq n$. Set $\vartheta(d) = s + nd$ for $d \in \mathbb{Z}_{\geq 0}$. By definition, we have $|M\rangle = e_{s,k} \wedge e_{s-1,k} \wedge \cdots \wedge e_{1,k} \wedge e_{n,k+1} \wedge e_{n-1,k+1} \wedge \cdots$. For any vector $\mathbf{v} = e_{\eta_1,k_1} \wedge e_{\eta_2,k_2} \wedge \cdots$ in $\mathcal{F}_M$, define its M-degree to be

$$\deg^M(\mathbf{v}) := \sum_{j=1}^{s}(k - k_j) + \sum_{m=0}^{\infty}\sum_{j=1}^{n}(k + m + 1 - k_{s+mn+j}).$$

From the asymptotic condition for the vectors in $\mathcal{F}_M$, we can see that the M-degree is well-defined and $\deg^{\mathrm{M}}(|M\rangle) = 0$. Denote by $\mathcal{F}_M^d$ the subspace spanned by the homogeneous elements of degree $\vartheta(d)$ in $\mathcal{F}_M$ and by $\mathcal{F}_{M,-}$ the subspace of $\mathcal{F}_M$ formed by vectors $e_{i_1,k_1} \wedge e_{i_2,k_2} \wedge \cdots$ where all $k_i \geq 0$. Thus $\mathcal{F}_{M,-} = \bigoplus_d \mathcal{F}_{M,-}^d = \bigoplus_d (\mathcal{F}_M^d \cap \mathcal{F}_{M,-}^d)$.

We denote $V_{\mathrm{aff}}^{\wedge\vartheta(h)}$ by V_{aff}^h. It has a basis formed by ordered wedges: $\{e_{\eta_1,k_1} \wedge e_{\eta_2,k_2} \wedge \cdots \wedge e_{\eta_l,k_{s+nh}} | 1 \leq \eta_i \leq n,\ k_i \in \mathbb{Z}_{\geq 0}\}$. Define the M-degree for these vectors by

$$\deg^M(e_{\eta_1,k_1} \wedge e_{\eta_2,k_2} \wedge \cdots \wedge e_{\eta_l,k_{s+nh}}) = \sum_{j=1}^{s}(k - k_j) + \sum_{m=0}^{h-1}\sum_{j=1}^{n}(k + m + 1 - k_{s+mn+j}).$$

Let $V_{\mathrm{aff}}^h(d)$ be the subspace of V_{aff}^h spanned by wedge products with M-degree equal to $\vartheta(d)$. It is a $\check{B}_\beta(n,p)$-module, i.e., the twisted Yangian action preserves the M-degree.

Define $\iota_M^{d,h} : V_{\mathrm{aff}}^h(d) \to \mathcal{F}_M$ by sending any vector $\mathbf{v} \in V_{\mathrm{aff}}^h(d)$ to the vector $\mathbf{v} \wedge |-n(k+h+1)\rangle$. From the definition, we can see that $\deg^M(\mathbf{v}) = \deg^M(\mathbf{v} \wedge |-n(k+h+1)\rangle)$. Thus we have a map $\iota_M^{d,h} : V_{\mathrm{aff}}^h(d) \to \mathcal{F}_{M,-}^d$ and the following result analogous to proposition 3.3 and corollary 3.4 in [Ug].

Lemma 5.2. *For $0 \leq d \leq h$, $\iota_M^{d,h}$ is an isomorphism of vector spaces. Moreover, if $d \leq h_1 \leq h_2$, then the map $\iota_M^{d,h_1,h_2} = (\iota_M^{d,h_2})^{-1} \circ \iota_M^{d,h_1} : V_{\mathrm{aff}}^{h_1}(d) \to V_{\mathrm{aff}}^{h_2}(d)$ is an isomorphism.*

5.3. **Twisted Yangian action on a Fock space.** For any positive integer m, define a subspace in $\mathbb{C}[z_1^{-1}, \ldots, z_{l+n}^{-1}]$:

$$\mathcal{L}_{l,n,m} := \mathrm{Span}_{\mathbb{C}}\{z_1^{-m_1} z_2^{-m_2} \cdots z_{l+n}^{-m_{l+n}} | 0 \leq |m_i| \leq m\ \forall i \text{ and } \sharp\{i : |m_i| = m\} < n\}.$$

Let

$$f = z_1^{-k_1} \cdots z_l^{-k_l} (z_{l+1} \cdots z_{l+n})^{-m}, \text{ where } 1 \leq k_i < m \text{ for all } i = 1, \ldots, l. \tag{19}$$

The following lemma can be seen from direct calculations.

Lemma 5.3. *Let σ_{ij} and γ_i be the usual reflections in W_l. For $s, t \in \mathbb{Z}_{\geq 0}$ and $s < t$, we have the following identities:*

$$\begin{aligned}
\frac{1}{1 - z_i^{-1} z_j}(1 - \sigma_{ij})(z_i^{-s} z_j^{-t}) &= z_i^{-s} z_j^{-t} + z_i^{-s-1} z_j^{-t+1} + \cdots + z_i^{-t+1} z_j^{-s-1};\\
\frac{1}{1 - z_i z_j}(1 - \sigma_{ij}\gamma_i\gamma_j)(z_i^{-s} z_j^{-t}) &= z_i^{-s} z_j^{-t} + z_i^{-s+1} z_j^{-t+1} + \cdots + z_i^{t-1} z_j^{s-1};\\
\frac{1}{1 - z_i^2}(1 - \gamma_i)(z_i^{-s}) &= z_i^{-s} + z_i^{-s+2} + \cdots + z_i^{s-2}.
\end{aligned}$$

From Lemma 5.3 and the definition of the trigonometric Dunkl operators, we have

$$D_{i,l+n}(f) \equiv D_{i,l}(f) \bmod \mathcal{L}_{l,n,m} \text{ for } i = 1, \ldots, l; \tag{20}$$

$$D_{i,l+n}(f) \equiv \left(-m+(2l+n-i)\kappa_1 - \frac{\kappa_2}{2}\right) f \bmod \mathcal{L}_{l,n,m} \text{ for } i=l+1,\dots,l+n. \tag{21}$$

Set $C(i,l,n) = -m+(2l+n-i) - \frac{\kappa_2}{2}$. (Recall that we assume $\kappa_1 = 1$.)

Besides the previous two equivalences, we will also need a corollary of Lemma 5.4 below.

Let

$$\mathsf{T}_{ij}(u) = \sum_{1\le k_1,\dots,k_{n-1}\le n} \mathsf{T}^{(1)}_{i,k_1}(u)\mathsf{T}^{(2)}_{k_1,k_2}(u)\cdots\mathsf{T}^{(n)}_{k_{n-1},j}(u), \quad \mathsf{T}^{(k)}_{ab}(u) = \delta_{ab} + \frac{E^{(k)}_{ab}}{u+c_k},$$

where the c_k's are constant such that $c_{k+1} = c_k - 1$. Define also $\mathsf{S}_{ij}(u)$ and $\mathsf{S}^{(k)}_{ij}$ by

$$\mathsf{S}_{ij}(u) = \sum_{1\le k_1,\dots,k_{n-1}\le n} \mathsf{S}^{(n)}_{i,k_1}(u)\mathsf{S}^{(n-1)}_{k_1,k_2}(u)\cdots\mathsf{S}^{(1)}_{k_{n-1},j}(u), \quad \mathsf{S}^{(k)}_{ab}(u) = \delta_{ab} + \frac{E^{(k)}_{ab}}{u+d_k},$$

where the d_k's are also constant such that $d_{k+1} = d_k + 1$.

We view the coefficients of $\mathsf{T}^{(k)}_{ij}(u)$ and $\mathsf{S}^{(k)}_{ij}(u)$ as linear endomorphisms of $(\mathbb{C}^n)^{\otimes n}$. The coefficients of $\mathsf{T}_{ij}(u)$ and $\mathsf{S}_{ij}(u)$ are also endomorphisms of $(\mathbb{C}^n)^{\otimes n}$ and they descend to the quotient $\Lambda^n\mathbb{C}^n$. (Note that $\Lambda^n\mathbb{C}^n$ can be identified with $\mathsf{D}^{A,-1}(\text{triv})$, where triv is the trivial representation of the degenerate affine Hecke algebra.) Set $\omega_n = e_n \otimes e_{n-1}\otimes\cdots\otimes e_1$, where $\{e_1, e_2, \dots, e_n\}$ is the standard basis of $\mathbb{C}^n$. Let L_n be the subspace of $(\mathbb{C}^n)^{\otimes n}$ spanned by elements of the form $v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_n}$ with $v_{i_{j_1}} = v_{i_{j_2}}$ for at least one pair of distinct indices i_{j_1}, i_{j_2}, so that $\Lambda^n\mathbb{C}^n = (\mathbb{C}^n)^{\otimes n}/L_n$.

Lemma 5.4.

$$\mathsf{T}_{ii}(u)(\omega_n) \equiv \left(1+\frac{1}{u+c_1}\right)\omega_n \bmod L_n, \quad \mathsf{T}_{ij}(u)(\omega_n) \equiv 0 \bmod L_n, \quad 1\le i\ne j\le n,$$

$$\mathsf{S}_{ii}(u)(\omega_n) \equiv \left(1+\frac{1}{u+d_n}\right)\omega_n \bmod L_n, \quad \mathsf{S}_{ij}(u)(\omega_n) \equiv 0 \bmod L_n, \quad 1\le i\ne j\le n.$$

Proof. The proof is by induction on n. Let's assume first that $n=2$. By direct calculations, we have

$$\mathsf{T}^{(1)}_{11}(u)\mathsf{T}^{(2)}_{11}(u)(e_2\otimes e_1) = \left(1+\frac{1}{u+c_2}\right)e_2\otimes e_1, \quad \mathsf{T}^{(1)}_{12}(u)\mathsf{T}^{(2)}_{21}(u)(e_2\otimes e_1) = \frac{1}{(u+c_1)(u+c_2)}e_1\otimes e_2,$$

so $\mathsf{T}_{11}(u)(\omega_2) \equiv \left(1+\frac{1}{u+c_2} - \frac{1}{(u+c_1)(u+c_2)}\right)\omega_2 = \left(1+\frac{1}{u+c_1}\right)\omega_2 \bmod L_n$, where the last equality was obtained using the assumption that $c_2 = c_1 - 1$.

We have

$$\mathsf{T}^{(1)}_{12}(u)\mathsf{T}^{(2)}_{22}(u)(e_2\otimes e_1) = \frac{1}{u+c_1}e_1\otimes e_1, \quad \mathsf{T}^{(1)}_{11}(u)\mathsf{T}^{(2)}_{12}(u)(e_2\otimes e_1) = 0,$$

so $\mathsf{T}_{12}(u)(\omega_2) = \frac{1}{u+c_1}e_1\otimes e_1 \equiv 0 \bmod L_n$. Similarly, $\mathsf{T}_{22}(u)(\omega_2) \equiv \left(1+\frac{1}{u+c_1}\right)\omega_2 \bmod L_n$ and $\mathsf{T}_{21}(\omega_2) \equiv 0 \bmod L_n$.

Let's now consider the induction step. We consider a few subcases.

Suppose that $i=j\ne 1$. Then

$$\begin{aligned}
\mathsf{T}_{ij}(u)(\omega_n) &= \sum_{2\le k_1,\dots,k_{n-1}\le n} \mathsf{T}^{(1)}_{i,k_1}(u)\mathsf{T}^{(2)}_{k_1,k_2}(u)\cdots\mathsf{T}^{(n)}_{k_{n-1},i}(u)(\omega_n) \\
&\equiv \sum_{2\le k_1,\dots,k_{n-2}\le n} \mathsf{T}^{(1)}_{i,k_1}(u)\mathsf{T}^{(2)}_{k_1,k_2}(u)\cdots\mathsf{T}^{(n-1)}_{k_{n-2},i}(u)\mathsf{T}^{(n)}_{ii}(u)(\omega_n) \\
&\equiv \left(1+\frac{1}{u+c_1}\right)\omega_n \text{ by induction.}
\end{aligned}$$

Suppose that $i = j = 1$ and let $\omega_n' = e_n \otimes e_{n-1} \otimes \cdots \otimes e_3 \otimes e_1 \otimes e_2$. Then

$$\begin{aligned}
\mathsf{T}_{11}(u)(\omega_n') &= \sum_{\substack{1\le k_1,\dots,k_{n-1}\le n\\ k_1,\dots,k_{n-1}\neq 2}} \mathsf{T}^{(1)}_{1,k_1}(u)\mathsf{T}^{(2)}_{k_1,k_2}(u)\cdots \mathsf{T}^{(n)}_{k_{n-1},1}(u)(\omega_n') \\
&= \sum_{\substack{1\le k_1,\dots,k_{n-2}\le n\\ k_1,\dots,k_{n-2}\neq 2}} \mathsf{T}^{(1)}_{1,k_1}(u)\mathsf{T}^{(2)}_{k_1,k_2}(u)\cdots \mathsf{T}^{(n-1)}_{k_{n-2},1}(u)\mathsf{T}^{(n)}_{11}(u)(\omega_n') \\
&\equiv \left(1+\frac{1}{u+c_1}\right)\omega_n' \text{ by induction.}
\end{aligned}$$

It follows that $\mathsf{T}_{11}(u)(\omega_n) \equiv \left(1+\frac{1}{u+c_1}\right)\omega_n$.

Suppose now that $i \neq j$ and $j \neq 1$. Then

$$\begin{aligned}
\mathsf{T}_{ij}(u)(\omega_n) &\equiv \sum_{2\le k_1,\dots,k_{n-1}\le n} \mathsf{T}^{(1)}_{i,k_1}(u)\mathsf{T}^{(2)}_{k_1,k_2}(u)\cdots \mathsf{T}^{(n)}_{k_{n-1},j}(u)(\omega_n) \\
&\equiv \sum_{2\le k_1,\dots,k_{n-2}\le n} \mathsf{T}^{(1)}_{i,k_1}(u)\mathsf{T}^{(2)}_{k_1,k_2}(u)\cdots \mathsf{T}^{(n-1)}_{k_{n-2},j}(u)\mathsf{T}^{(n)}_{jj}(u)(\omega_n) \\
&\equiv 0 \text{ by induction.}
\end{aligned}$$

Finally, let's consider the case $i \neq j$ and $j = 1$. Then

$$\begin{aligned}
\mathsf{T}_{ij}(u)(\omega_n') &= \sum_{\substack{1\le k_1,\dots,k_{n-1}\le n\\ k_1,\dots,k_{n-1}\neq 2}} \mathsf{T}^{(1)}_{i,k_1}(u)\mathsf{T}^{(2)}_{k_1,k_2}(u)\cdots \mathsf{T}^{(n)}_{k_{n-1},1}(u)(\omega_n') \\
&\equiv \sum_{\substack{1\le k_1,\dots,k_{n-2}\le n\\ k_1,\dots,k_{n-2}\neq 2}} \mathsf{T}^{(1)}_{i,k_1}(u)\mathsf{T}^{(2)}_{k_1,k_2}(u)\cdots \mathsf{T}^{(n-1)}_{k_{n-2},1}(u)\mathsf{T}^{(n)}_{11}(u)(\omega_n') \\
&\equiv 0 \text{ by induction.}
\end{aligned}$$

It follows that $\mathsf{T}_{i1}(u)(\omega_n) \equiv 0$.

The proof is the same for S instead of T. □

Corollary 5.1. *For $\omega_n^\wedge = e_n \wedge e_{n-1} \wedge \cdots \wedge e_1$, the following equalities hold in $\Lambda^n\mathbb{C}^n$:*

$$\mathsf{T}_{ii}(u)(\omega_n^\wedge) = \left(1+\frac{1}{u+c_1}\right)\omega_n^\wedge, \quad \mathsf{T}_{ij}(u)(\omega_n^\wedge) = 0, \quad 1 \le i \neq j \le n.$$

$$\mathsf{S}_{ii}(u)(\omega_n^\wedge) = \left(1+\frac{1}{u+d_n}\right)\omega_n^\wedge, \quad \mathsf{S}_{ij}(u)(\omega_n^\wedge) = 0, \quad 1 \le i \neq j \le n.$$

Now take $l = \vartheta(h)$, so $l + n = \vartheta(h+1)$. Let $w \in V^h_{\mathrm{aff}}(d)$ and let $\equiv$ be the equivalence modulo $\bigoplus_{d'>d} V^{h+1}_{\mathrm{aff}}(d')$. From the definition of $\iota^{d,h,h+1}_M$,

$$\iota^{d,h,h+1}_M(w) = w \wedge e_{n,(h+k+1)} \wedge \cdots \wedge e_{1,(h+k+1)} \in V^{h+1}_{\mathrm{aff}}(d).$$

Let $\check{b}^{(r)}_{ij}$ be the generators of the twisted reflection algebra $\check{B}_\beta(n,p)$ and $\check{b}_{ij}(u)$ be its generating series. We denote the polynomial generators of $\check{\mathbb{H}}^\ell_{\kappa_1,\kappa_2}$ by $\check{y}_{k,\ell}$ since we will need to consider different values of ℓ. In this section, we set

$$\mathsf{T}^{\ell,(k)}_{ij}(u) = \delta_{i,j} + \frac{E^{(k)}_{ij}}{u - \frac{n}{2} + \check{y}_{k,\ell}},\ \mathsf{S}^{\ell,(k)}_{ij}(u) = \delta_{i,j} + \frac{E^{(k)}_{ij}}{u - \frac{n}{2} - \check{y}_{k,\ell}},\ \beta = \frac{\kappa_2}{2} + \frac{n-2p}{2},\ a_s(u) = \epsilon_s - \beta u^{-1},$$

$$\xi^\pm_l(u) = \left(1 + \frac{2}{2u - n \pm 2C(l+1,l,n)}\right), \quad \xi_l(u) = \xi^+_l(u)\xi^-_l(u).$$

Thus we have:

$$\begin{aligned}
&\rho_{l+n,-1}(\check{b}_{ij}(u))\big(\iota_M^{d,h,h+1}(w)\big)\\
&= \sum_{\substack{s\\ i_1,i_2,\dots,i_{l+n-1}\\ j_1,j_2,\dots,j_{l+n-1}}} a_s(u)\Big(\mathtt{S}^{l+n,(l+n)}_{i,i_{l+n-1}}(u)\mathtt{S}^{l+n,(l+n-1)}_{i_{l+n-1},i_{l+n-2}}(u)\cdots\mathtt{S}^{l+n,(l+1)}_{i_{l+1},i_l}(u)\mathtt{S}^{l+n,(l)}_{i_l,i_{l-1}}(u)\cdots\mathtt{S}^{l+n,(1)}_{i_1,s}(u)\\
&\qquad\cdot\mathtt{T}^{l+n,(1)}_{s,j_1}(u)\cdots\mathtt{T}^{l+n,(l)}_{j_{l-1},j_l}(u)\mathtt{T}^{l+n,(l+1)}_{j_l,j_{l+1}}(u)\cdots\mathtt{T}^{l+n,(l+n)}_{j_{l+n-1},j}(u)\Big)\big(\iota_M^{d,h,h+1}(w)\big)\\
&\equiv \sum_{\substack{s\\ i_1,i_2,\dots,i_{l+n-1}\\ j_1,j_2,\dots,j_{l+n-1}}} a_s(u)\Big(\mathtt{S}^{l+n,(l+n)}_{i,i_{l+n-1}}(u)\mathtt{S}^{l+n,(l+n-1)}_{i_{l+n-1},i_{l+n-2}}(u)\cdots\mathtt{S}^{l+n,(l+1)}_{i_{l+1},i_l}(u)\mathtt{S}^{l+n,(l)}_{i_l,i_{l-1}}(u)\cdots\mathtt{S}^{l+n,(1)}_{i_1,s}(u)\\
&\qquad\cdot\mathtt{T}^{l+n,(1)}_{s,j_1}(u)\cdots\mathtt{T}^{l+n,(l)}_{j_{l-1},j_l}(u)\widetilde{\mathtt{T}}^{l+n,(l+1)}_{j_l,j_{l+1}}(u)\cdots\widetilde{\mathtt{T}}^{l+n,(l+n)}_{j_{l+n-1},j}(u)\Big)\big(\iota_M^{d,h,h+1}(w)\big)
\end{aligned}$$

by (21) where $\widetilde{\mathtt{T}}^{l+n,(k)}_{ab}(u)=\delta_{ab}+\frac{E^{(k)}_{ab}}{u-n/2+C(k,l,n)}$. It is possible to simplify the expression on the last line using Corollary 5.1: we get

$$\begin{aligned}
&\rho_{l+n,-1}(\check{b}_{ij}(u))\big(\iota_M^{d,h,h+1}(w)\big)\\
&\equiv \xi_l^+(u)\sum_{\substack{s\\ i_1,i_2,\dots,i_{l+n-1}\\ j_1,j_2,\dots,j_{l-1}}} a_s(u)\Big(\mathtt{S}^{l+n,(l+n)}_{i,i_{l+n-1}}(u)\mathtt{S}^{l+n,(l+n-1)}_{i_{l+n-1},i_{l+n-2}}(u)\cdots\mathtt{S}^{l+n,(l+1)}_{i_{l+1},i_l}(u)\\
&\qquad\cdot\mathtt{S}^{l+n,(l)}_{i_l,i_{l-1}}(u)\cdots\mathtt{S}^{l+n,(1)}_{i_1,s}(u)\cdot\mathtt{T}^{l+n,(1)}_{s,j_1}(u)\cdots\mathtt{T}^{l+n,(l)}_{j_{l-1},j}(u)\Big)\big(\iota_M^{d,h,h+1}(w)\big)\\
&\equiv \xi_l^+(u)\sum_{\substack{s\\ i_1,i_2,\dots,i_{l+n-1}\\ j_1,j_2,\dots,j_{l-1}}} a_s(u)\Big(\mathtt{S}^{l+n,(l+n)}_{i,i_{l+n-1}}(u)\mathtt{S}^{l+n,(l+n-1)}_{i_{l+n-1},i_{l+n-2}}(u)\cdots\mathtt{S}^{l+n,(l+1)}_{i_{l+1},i_l}(u)\\
&\qquad\cdot\mathtt{S}^{l,(l)}_{i_l,i_{l-1}}(u)\cdots\mathtt{S}^{l,(1)}_{i_1,s}(u)\cdot\mathtt{T}^{l,(1)}_{s,j_1}(u)\cdots\mathtt{T}^{l,(l)}_{j_{l-1},j}(u)\Big)\big(\iota_M^{d,h,h+1}(w)\big)\text{ by (20)}\\
&= \xi_l^+(u)\sum_{i_l,i_{l+1},\dots,i_{l+n-1}}\Big(\mathtt{S}^{l+n,(l+n)}_{i,i_{l+n-1}}(u)\mathtt{S}^{l+n,(l+n-1)}_{i_{l+n-1},i_{l+n-2}}(u)\cdots\mathtt{S}^{l+n,(l+1)}_{i_{l+1},i_l}(u)\Big)\big(\iota_M^{d,h,h+1}(\rho_{l,-1}(\check{b}_{i_l,j}(u))(w))\big)\\
&\equiv \xi_l^+(u)\sum_{i_l,i_{l+1},\dots,i_{l+n-1}}\Big(\widetilde{\mathtt{S}}^{l+n,(l+n)}_{i,i_{l+n-1}}(u)\widetilde{\mathtt{S}}^{l+n,(l+n-1)}_{i_{l+n-1},i_{l+n-2}}(u)\cdots\widetilde{\mathtt{S}}^{l+n,(l+1)}_{i_{l+1},i_l}(u)\Big)\big(\iota_M^{d,h,h+1}(\rho_{l,-1}(\check{b}_{i_l,j}(u))(w))\big)
\end{aligned}$$

by (21), where $\widetilde{\mathtt{S}}^{l+n,(k)}_{ab}(u)=\delta_{ab}+\frac{E^{(k)}_{ab}}{u-n/2-C(k,l,n)}$. Using Corollary 5.1 again, we see that this last expression is congruent to $\xi_l^+(u)\xi_l^-(u)\iota_M^{d,h,h+1}\big(\rho_{l,-1}(\check{b}_{ij}(u))(w)\big)$.

The previous computations lead to the following proposition:

Proposition 5.1. *Assume $d\le h$. For any $w\in V^h_{\mathrm{aff}}(d)$ and $1\le i,j\le n$,*

$$\rho_{s+(h+1)n,-1}(\check{b}_{ij}(u))\big(\iota_M^{d,h,h+1}(w)\big)=\xi_{\vartheta(h)}(u)\iota_M^{d,h,h+1}\big(\rho_{s+hn,-1}(\check{b}_{ij}(u))(w)\big).$$

Set $\Xi_h(u)=\prod_{i=0}^{h-1}\xi_{\vartheta(i)}(u)$ and define the following renormalized action of the twisted Yangian on the space $V^h_{\mathrm{aff}}(d)$:

$$\bar{\rho}_h=\frac{1}{\Xi_h(u)}\rho_{\vartheta(h),-1}:\check{B}_\beta(n,p)\to\mathrm{End}_{\mathbb{C}}(V^h_{\mathrm{aff}}(d)).$$

From Proposition 5.1, we can get the following conclusion.

Proposition 5.2. *For $d\le h_1\le h_2$, $\iota_M^{d,h_1,h_2}\circ\bar{\rho}_{h_1}=\bar{\rho}_{h_2}\circ\iota_M^{d,h_1,h_2}$. Moreover, ι_M^{d,h_1,h_2} induces an isomorphism between the $\check{B}_\beta(n,p)$-modules $V^{h_1}_{\mathrm{aff}}(d)$ and $V^{h_2}_{\mathrm{aff}}(d)$ with renormalized actions.*

The following theorem is the main conclusion of this section and is a corollary of the previous proposition.

Theorem 5.1. *For $0 \le d \le h$, set $\bar{\rho}_d(\check{b}_{ij}^{(r)}) = \iota_M^{d,h} \circ \bar{\rho}_h(\check{b}_{ij}^{(r)}) \circ (\iota_M^{d,h})^{-1} \in \mathrm{End}_{\mathbb{C}}(\mathcal{F}_{M,-}^d)$. Then $\bar{\rho}_d(\check{b}_{ij}^{(r)})$ does not depend on h, so we have a well-defined (independent of h) action of $\check{B}_\beta(n,p)$ on each degree d piece of the Fock space $\mathcal{F}_{M,-}$, and hence on all of $\mathcal{F}_{M,-}$.*

6. Twisted quantum loop algebra of type AIII

6.1. Twisted loop algebra and Onsager algebra.

Definition 6.1. *The twisted loop algebra $\mathfrak{gl}_n(\mathbb{C}[s,s^{-1}])^\theta$ is defined as $\{f(s) \in \mathfrak{gl}_n(\mathbb{C}[s,s^{-1}]) \mid \theta(f(s)) = f(s^{-1})\}$.*

The twisted quantum loop algebra of type AIII to be introduced later is a quantization of the enveloping algebra of $\mathfrak{gl}_n(\mathbb{C}[s,s^{-1}])^\theta$. The twisted loop algebra $\mathfrak{gl}_n(\mathbb{C}[s,s^{-1}])^\theta$ (or rather $\mathfrak{sl}_n(\mathbb{C}[s,s^{-1}])^\theta$) can be viewed as a generalized Onsager algebra as suggested in example 3.10 in [NSS]. Note that $\mathfrak{gl}_n(\mathbb{C}[s,s^{-1}])^\theta = \mathfrak{sl}_n(\mathbb{C}[s,s^{-1}])^\theta \oplus \mathbb{C}I_n \otimes_{\mathbb{C}} \mathbb{C}[s,s^{-1}]^\Gamma$ where Γ acts on $\mathbb{C}[s,s^{-1}]$ by $\gamma(p(s,s^{-1})) = p(s^{-1},s)$, so that $\mathbb{C}[s,s^{-1}]^\Gamma$ is a polynomial ring in the variable $s+s^{-1}$.

Affine Kac-Moody algebras (without the derivation) are universal central extensions of loop algebras. It is thus natural to wonder if the twisted loop algebra $\mathfrak{sl}_n(\mathbb{C}[s,s^{-1}])^\theta$ admits a non-trivial central extension. At least when $n=2p$, we can show that the answer is negative: see Proposition 6.1 below and the paragraph just above it.

Lemma 6.1. *If $n=2p$, then $\mathfrak{gl}_n(\mathbb{C}[s,s^{-1}])^\theta$ is isomorphic to $\mathfrak{gl}_p(\Lambda)$, where $\Lambda = \mathbb{C}\langle t,t^{-1},\gamma\rangle/(\gamma^2-1, \gamma t - t^{-1}\gamma)$.*

Proof. Define a linear map $\psi : \mathfrak{gl}_p(\Lambda) \longrightarrow \mathfrak{gl}_n(\mathbb{C}[s,s^{-1}])^\theta$ by

$$\psi\Big(E_{ij}\big((t^k + (-1)^a t^{-k})\mathbf{e}_b\big)\Big) = E_{i+ap+bp,j+bp}\big(s^k + (-1)^a s^{-k}\big),$$

where $a,b = 0,1$ and the indices of $E_{i+ap+bp,j+bp}$ should be taken modulo n.

This is a linear isomorphism and is a homomorphism of Lie algebras since $\psi\Big(\big[E_{ij}\big((t^{r_1}+(-1)^a t^{-r_1})\mathbf{e}_b\big), E_{kl}\big((t^{r_2} + (-1)^c t^{-r_2})\mathbf{e}_d\big)\big]\Big)$ equals

$$\begin{aligned}
&\psi\Big(\delta_{jk}\delta_{b+c,d}E_{il}\big((t^{r_1}+(-1)^a t^{-r_1})(t^{r_2}+(-1)^c t^{-r_2})\mathbf{e}_d\big) - \delta_{il}\delta_{d+a,b}E_{kj}\big((t^{r_2}+(-1)^c t^{-r_2})(t^{r_1}+(-1)^a t^{-r_1})\mathbf{e}_b\big)\Big)\\
&= \delta_{jk}\delta_{b+c,d}\psi\Big(E_{il}\big((t^{r_1+r_2}+(-1)^{a+c}t^{-r_1-r_2})\mathbf{e}_d\big)\Big) + \delta_{jk}\delta_{b+c,d}\psi\Big(E_{il}\big(((-1)^c t^{r_1-r_2}+(-1)^a t^{r_2-r_1})\mathbf{e}_d\big)\Big)\\
&\quad - \delta_{il}\delta_{d+a,b}\psi\Big(E_{kj}\big((t^{r_2+r_1}+(-1)^{a+c}t^{-r_2-r_1})\mathbf{e}_b\big)\Big) - \delta_{il}\delta_{d+a,b}\psi\Big(E_{kj}\big(((-1)^a t^{r_2-r_1}+(-1)^c t^{r_1-r_2})\mathbf{e}_b\big)\Big)\\
&= \delta_{jk}\delta_{b+c,d}E_{i+(a+c)p+dp,l+dp}\big(s^{r_1+r_2}+(-1)^{a+c}s^{-r_1-r_2}+(-1)^c s^{r_1-r_2}+(-1)^a s^{r_2-r_1}\big)\\
&\quad - \delta_{il}\delta_{d+a,b}E_{k+(a+c)p+bp,j+bp}\big(s^{r_2+r_1}+(-1)^{a+c}s^{-r_2-r_1}+(-1)^a s^{r_2-r_1}+(-1)^c s^{r_1-r_2}\big)
\end{aligned} \tag{22}$$

whereas $\Big[\psi\big(E_{ij}\big((t^{r_1}+(-1)^a t^{-r_1})\mathbf{e}_b\big)\big), \psi\big(E_{kl}\big((t^{r_2}+(-1)^c t^{-r_2})\mathbf{e}_d\big)\big)\Big]$ equals

$$\begin{aligned}
&\Big[E_{i+ap+bp,j+pb}(s^{r_1}+(-1)^a s^{-r_1}), E_{k+cp+dp,l+dp}(s^{r_2}+(-1)^c s^{-r_2})\Big]\\
&\qquad = \delta_{jk}\delta_{b,c+d}E_{i+ap+bp,l+dp}\big(s^{r_1+r_2}+(-1)^{a+c}s^{-r_1-r_2}+(-1)^a s^{r_2-r_1}+(-1)^c s^{r_1-r_2}\big)\\
&\qquad\quad - \delta_{il}\delta_{d,a+b}E_{k+cp+dp,j+bp}\big(s^{r_1+r_2}+(-1)^{a+c}s^{-r_1-r_2}+(-1)^a s^{r_2-r_1}+(-1)^c s^{r_1-r_2}\big),
\end{aligned} \tag{23}$$

and (22) is the same as (23) (a,b,c,d should be viewed modulo 2). ☐

It is known from the work of C. Kassel and J.-L. Loday [KaLo] that the center of the universal central extension of $\mathfrak{sl}_p(\Lambda)$ is isomorphic to the first cyclic homology group $HC_1(\Lambda)$. That $\mathfrak{sl}_p(\Lambda)$ does not admit a non-trivial central extension is thus a corollary of the next proposition.

Proposition 6.1. $HC_1(\Lambda) = \{0\}$.

Proof. $HC_1(\Lambda)$ is the kernel of the map $\langle\Lambda,\Lambda\rangle \longrightarrow [\Lambda,\Lambda]$, where $\langle\Lambda,\Lambda\rangle$ is the quotient of the space $\Lambda\otimes_{\mathbb{C}}\Lambda$ by the subspace spanned by $a\otimes b+b\otimes a$ and by $ab\otimes c+bc\otimes a+ca\otimes b\;\forall a,b,c\in\Lambda$.

A spanning set for $\langle\Lambda,\Lambda\rangle$ is $\{t^i\gamma^j\otimes t, t^i\gamma^j\otimes t^{-1}, t^i\gamma^j\otimes\gamma\}$ for $i\in\mathbb{Z}$, $j=0,1$; let's find a smaller spanning set. $t^i\otimes t=0\;\forall\, i\in\mathbb{Z}\setminus\{-1\}$ since $HC_1(\mathbb{C}[t,t^{-1}])=\mathbb{C}t^{-1}\otimes t$. Moreover,

$$t^i\gamma\otimes t^{-1}=\gamma t^{-i}\otimes t^{-1}=t^{-i}\otimes t^{-1}\gamma+\gamma\otimes t^{-i-1}=t^{-i}\otimes\gamma t+\gamma\otimes t^{-i-1}=t^{-i}\gamma\otimes t+t^{-i+1}\otimes\gamma+\gamma\otimes t^{-i-1},$$

so $t^i\gamma\otimes t^{-1}$ can be removed from the spanning set. The elements $(t^i+t^{-i})\gamma\otimes\gamma$ and $(t^i+t^{-i})\otimes\gamma$ are in the kernel, but actually they equal 0 in $\langle\Lambda,\Lambda\rangle$ since

$$0=(t^i+t^{-i})\otimes 1=(t^i+t^{-i})\otimes\gamma^2=(t^i+t^{-i})\gamma\otimes\gamma+\gamma(t^i+t^{-i})\otimes\gamma=2(t^i+t^{-i})\gamma\otimes\gamma$$

and similarly with $(t^i+t^{-i})\gamma$ instead of (t^i+t^{-i}).

What is more surprising is that $t\otimes t^{-1}=0$:

$$t\otimes t^{-1}=t\otimes\gamma^2t^{-1}=t\otimes\gamma t\gamma=t\gamma\otimes t\gamma+t\gamma t\otimes\gamma=0+\gamma\otimes\gamma=0.$$

The conclusion of all these computations is that $\langle\Lambda,\Lambda\rangle$ is spanned by $t^i\gamma\otimes t,(t^i-t^{-i})\otimes\gamma$ and $(t^i-t^{-i})\gamma\otimes\gamma$ for $i\in\mathbb{Z}$. It is even possible to restrict i to $i\in\mathbb{Z}_{\geq 1}$ and still obtain a spanning set because

$$\begin{aligned} t^i\gamma\otimes t &= \gamma t^{-i}\otimes t=t^{-i}\otimes t\gamma+\gamma\otimes t^{1-i}=t^{-i-1}\otimes t\gamma t+t\otimes t^{-i}\gamma-t^{1-i}\otimes\gamma\\ &= t^{-i-1}\otimes\gamma-t^{-i}\gamma\otimes t-t^{1-i}\otimes\gamma. \end{aligned}$$

It follows that the kernel of $\langle\Lambda,\Lambda\rangle\longrightarrow[\Lambda,\Lambda]$ is trivial. ☐

Remark 6.1. *Λ can be given a $\mathbb{Z}/2\mathbb{Z}$-grading with $\deg(t)=0$ and $\deg(\gamma)=1$. Denote by Λ^{gr} the resulting super algebra. The Lie superalgebra $\mathfrak{sl}_n(\Lambda^{gr})$ was studied in* [ChGu1] *and it was determined that $HC_1^{\mathbb{Z}/2\mathbb{Z}}(\Lambda^{gr})$ is one dimensional, where $HC_1^{\mathbb{Z}/2\mathbb{Z}}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded version of cyclic homology. It follows from* [ChGu2] *that $\mathfrak{sl}_n(\Lambda^{gr})$ possesses a universal central extension with a one dimensional center.*

$\mathfrak{sl}_n(\mathbb{C}[s,s^{-1}])^\theta$ is a generalized Onsager algebra in the sense of [NSS], so proposition 6.2 of loc.cit. can be applied to it. Set $\mathcal{R}_{\mathbb{C}^\times}=\bigcup_{x\in\mathbb{C}^\times}\mathcal{R}_x$, where $\mathcal{R}_x$ is the set of isomorphism classes of irreducible finite dimensional representations of $\mathfrak{sl}_n^x$ and $\mathfrak{sl}_n^x=\mathfrak{sl}_n$ if $x\neq\pm1$ whereas $\mathfrak{sl}_n^{\pm1}=\mathfrak{sl}_n^\theta$.

Theorem 6.1 ([NSS] proposition 6.2)**.** *Any finite dimensional irreducible representation of $\mathfrak{sl}_n(\mathbb{C}[s,s^{-1}])^\theta$ is an evaluation representation (in the terminology of* [NSS]*; in more standard terminology, it is a tensor product of evaluation representations: see remark 4.6 in loc. cit.) and irreducible finite dimensional representations of $\mathfrak{sl}_n(\mathbb{C}[s,s^{-1}])^\theta$ are parametrized by the set of finitely supported Γ-equivariant functions $\Psi:\mathbb{C}^\times\longrightarrow\mathcal{R}_{\mathbb{C}^\times}$ such that $\Psi(x)\in\mathcal{R}_x$.*

6.2. **Quantum loop algebra.** Let P be the permutation operator $P(v_1\otimes v_2)=v_2\otimes v_1$. Let us recall the RTT-presentation of the quantum group $\mathfrak{U}_q(\mathfrak{gl}_n)$. We will view q as a variable and work over $\mathbb{C}(q)$, unless stated otherwise. Set $\mathbb{C}_q^n=\mathbb{C}^n\otimes_{\mathbb{C}}\mathbb{C}(q)$.

Definition 6.2. *The quantum R-matrix of finite type, which is an element of $\mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q^n)^{\otimes 2}$, is given by*

$$R=\sum_{i,j=1}^n q^{\delta_{ij}}E_{ii}\otimes E_{jj}+(q-q^{-1})\sum_{\substack{i,j=1\\ i>j}}^n E_{ij}\otimes E_{ji}. \tag{24}$$

Set $\widetilde{R}=R-(q-q^{-1})P=PR^{-1}P$.

Definition 6.3. *The quantum group $\mathfrak{U}_q(\mathfrak{gl}_n)$ is the associative $\mathbb{C}(q)$-algebra generated by $t_{ij},\bar{t}_{ij}, i,j=1,\dots,n$, with relations:*

$$RT_2T_1=T_1T_2R,\quad R\overline{T}_2\overline{T}_1=\overline{T}_1\overline{T}_2R,\quad RT_2\overline{T}_1=\overline{T}_1T_2R;$$
$$t_{ij}=\bar{t}_{ji}=0 \text{ if } 1\leq j<i\leq n,\quad t_{ii}\bar{t}_{ii}=\bar{t}_{ii}t_{ii}=1, i=1,\dots,n.$$

Here $T=\sum_{i,j=1}^n t_{ij}\otimes E_{ij}$ and $\overline{T}=\sum_{i,j=1}^n\bar{t}_{ij}\otimes E_{ij}$ belong to $\mathfrak{U}_q(\mathfrak{gl}_n)\otimes_{\mathbb{C}(q)}\mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q^n)$.

Remark 6.2. *Our R-matrix is the conjugate by P of the quantum R-matrix of finite type considered in* [MRS]*. We will stick with the definition used in* [JoMa]*.*

Definition 6.4. *The affine quantum R-matrix is the element of* $\mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q^n)^{\otimes 2} \otimes_{\mathbb{C}} \mathbb{C}[u,v]$ *given by (see* [MRS]*)*

$$R(u,v) = u\widetilde{R} - vR. \tag{25}$$

We will need the RTT-presentation of the quantized enveloping algebra of $\mathfrak{gl}_n(\mathbb{C}[s,s^{-1}])$, so we recall its definition. Set $\mathcal{L}\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C}[s,s^{-1}])$.

Definition 6.5. $\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n)$ *is the* $\mathbb{C}(q)$*-algebra generated by* $t_{ij}^{(r)}, \bar{t}_{ij}^{(r)}$ *with* $1 \le i,j \le n, r \in \mathbb{Z}_{\ge 0}$ *such that*

$$t_{ij}^{(0)} = \bar{t}_{ji}^{(0)} = 0, \text{ if } 1 \le j < i \le n, \quad t_{ii}^{(0)}\bar{t}_{ii}^{(0)} = \bar{t}_{ii}^{(0)}t_{ii}^{(0)} = 1, \ \ 1 \le i \le n, \tag{26}$$

$$R(u,v)T_2(v)T_1(u) = T_1(u)T_2(v)R(u,v), \tag{27}$$

$$R(u,v)\overline{T}_2(v)\overline{T}_1(u) = \overline{T}_1(u)\overline{T}_2(v)R(u,v), \tag{28}$$

$$R(u,v)T_2(v)\overline{T}_1(u) = \overline{T}_1(u)T_2(v)R(u,v), \tag{29}$$

where we have set $T(u) = \sum_{i,j=1}^n t_{ij}(u) \otimes E_{ij}, \overline{T}(u) = \sum_{i,j=1}^n \bar{t}_{ij}(u) \otimes E_{ij}$ *and* $t_{ij}(u) = \sum_{r=0}^\infty t_{ij}^{(r)} u^{-r}$, $\bar{t}_{ij}(u) = \sum_{r=0}^\infty \bar{t}_{ij}^{(r)} u^r$.

$\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n)$ is a Hopf algebra with coproduct given by

$$\Delta(t_{ij}(u)) = \sum_{k=1}^n t_{ik}(u) \otimes t_{kj}(u), \ \ \Delta(\bar{t}_{ij}(u)) = \sum_{k=1}^n \bar{t}_{ik}(u) \otimes \bar{t}_{kj}(u).$$

Later, in order to understand that the twisted quantum loop algebras of type AIII provide a quantization of the enveloping algebra of a twisted loop algebra, we will need to know how $\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n)$ specializes to $\mathfrak{U}(\mathcal{L}\mathfrak{gl}_n)$. We follow the explanations given in [MRS]. Let $\mathcal{A}$ be the localization of $\mathbb{C}[q,q^{-1}]$ at the ideal $(q-1)$. Let $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_n)$ be the $\mathcal{A}$-subalgebra of $\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n)$ generated by the elements $\tau_{ij}^{(r)}, \overline{\tau}_{ij}^{(r)}$ given by

$$\tau_{ij}^{(r)} = \frac{t_{ij}^{(r)}}{q-q^{-1}}, \ \ \overline{\tau}_{ij}^{(r)} = \frac{\bar{t}_{ij}^{(r)}}{q-q^{-1}} \text{ for } r \ge 0, 1 \le i,j \le n,$$

except that, when $r = 0$ and $i = j$, we set

$$\tau_{ii}^{(0)} = \frac{t_{ii}^{(0)} - 1}{q-q^{-1}}, \ \ \overline{\tau}_{ii}^{(0)} = \frac{\bar{t}_{ii}^{(0)} - 1}{q-q^{-1}}.$$

Theorem 6.2 (Section 3 of [MRS])**.** *The assignment* $E_{ji}s^r \mapsto \tau_{ij}^{(r)}$, $-E_{ji}s^{-r} \mapsto \overline{\tau}_{ij}^{(r)}$ $\forall\, r \ge 0, 1 \le i,j \le n$ *except if* $r = 0$ *and* $1 \le j < i \le n$ *induces an isomorphism* $\mathfrak{U}(\mathcal{L}\mathfrak{gl}_n) \xrightarrow{\sim} \mathfrak{U}_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_n) \otimes_{\mathcal{A}} \mathbb{C}$, *where* $\mathbb{C}$ *is viewed as an* $\mathcal{A}$*-module via* $\mathcal{A}/(q-1) \xrightarrow{\sim} \mathbb{C}$.

6.3. **Twisted quantum loop algebra of type AIII.** The twisted quantum loop algebra of type AIII is a quantization of the twisted loop algebra $\mathfrak{gl}_n(\mathbb{C}[s,s^{-1}])^\theta$.

We will need to consider another involution θ' obtained as θ, but from the matrix Θ_p' given by (see [DiSt, JoMa])

$$\Theta_p' = g\Theta_p g^{-1} = -\sum_{k=p+1}^{n-p} E_{kk} + \sum_{k=1}^p E_{k,n-k+1} + \sum_{k=1}^p E_{n-k+1,k},$$

where

$$g = \sum_{k=1}^p E_{kk} - \sum_{k=p+1}^n E_{kk} + \sum_{k=1}^p E_{n-k+1,k} + \sum_{k=1}^p E_{k,n-k+1}.$$

We need a deformation of the matrix Θ'_p. As suggested in [NoSu, DiSt, JoMa], for a new variable ξ, set

$$J^\xi = (\xi - \xi^{-1}) \sum_{k=1}^{p} E_{kk} - \xi^{-1} \sum_{k=p+1}^{n-p} E_{kk} + \sum_{k=1}^{p} E_{k,n-k+1} + \sum_{k=1}^{p} E_{n-k+1,k}. \tag{30}$$

Two useful properties of J^ξ are that it satisfies the Hecke relation $(J^\xi - \xi)(J^\xi + \xi^{-1}) = 0$ and is a solution of the reflection equation (see [DiSt, JoMa]). Set

$$G^\xi(u) = \frac{uJ^\xi - u^{-1}(J^\xi)^{-1}}{u - u^{-1}}. \tag{31}$$

Lemma 6.2. *$G^\xi(u)$ satisfies the reflection equation with parameters:*

$$R_{21}(v,u)G^\xi_1(u)R(u^{-1},v)G^\xi_2(v) = G^\xi_2(v)R_{21}(u^{-1},v)G^\xi_1(u)R(v,u). \tag{32}$$

Proof. Since $R(u,v) = u\widetilde{R} - vR$, (32) is equivalent to

$$(v\widetilde{R}_{21} - uR_{21})G^\xi_1(u)(u^{-1}\widetilde{R} - vR)G^\xi_2(v) = G^\xi_2(v)(u^{-1}\widetilde{R}_{21} - vR_{21})G^\xi_1(u)(v\widetilde{R} - uR).$$

We thus have to check the following relations:

$$R_{21}J^\xi_1 R J^\xi_2 = J^\xi_2 R_{21} J^\xi_1 R, \tag{33}$$

$$\widetilde{R}_{21}(J^\xi_1)^{-1}\widetilde{R}(J^\xi_2)^{-1} = (J^\xi_2)^{-1}\widetilde{R}_{21}(J^\xi_1)^{-1}\widetilde{R},\ \ R_{21}(J^\xi_1)^{-1}\widetilde{R}(J^\xi_2)^{-1} = (J^\xi_2)^{-1}\widetilde{R}_{21}(J^\xi_1)^{-1}R, \tag{34}$$

$$\widetilde{R}_{21}(J^\xi_1)^{-1}RJ^\xi_2 = J^\xi_2 R_{21}(J^\xi_1)^{-1}\widetilde{R},\ \ R_{21}J^\xi_1 R(J^\xi_2)^{-1} = (J^\xi_2)^{-1}R_{21}J^\xi_1 R,\ \ \widetilde{R}_{21}(J^\xi_1)^{-1}\widetilde{R}J^\xi_2 = J^\xi_2\widetilde{R}_{21}(J^\xi_1)^{-1}\widetilde{R}, \tag{35}$$

$$\widetilde{R}_{21}J^\xi_1 R J^\xi_2 = J^\xi_2 R_{21} J^\xi_1 \widetilde{R},\ \ R_{21}J^\xi_1\widetilde{R}(J^\xi_2)^{-1} = (J^\xi_2)^{-1}\widetilde{R}_{21}J^\xi_1 R, \tag{36}$$

$$-\widetilde{R}_{21}(J^\xi_1)^{-1}R(J^\xi_2)^{-1} + R_{21}(J^\xi_1)^{-1}\widetilde{R}J^\xi_2 = -(J^\xi_2)^{-1}R_{21}(J^\xi_1)^{-1}\widetilde{R} + J^\xi_2\widetilde{R}_{21}(J^\xi_1)^{-1}R, \tag{37}$$

$$-\widetilde{R}_{21}J^\xi_1\widetilde{R}(J^\xi_2)^{-1} + R_{21}(J^\xi_1)^{-1}R(J^\xi_2)^{-1} = -(J^\xi_2)^{-1}\widetilde{R}_{21}J^\xi_1\widetilde{R} + (J^\xi_2)^{-1}R_{21}(J^\xi_1)^{-1}R, \tag{38}$$

$$-R_{21}J^\xi_1\widetilde{R}J^\xi_2 + \widetilde{R}_{21}J^\xi_1 R(J^\xi_2)^{-1} = -J^\xi_2\widetilde{R}_{21}J^\xi_1 R + (J^\xi_2)^{-1}R_{21}J^\xi_1\widetilde{R}, \tag{39}$$

$$-R_{21}(J^\xi_1)^{-1}RJ^\xi_2 + \widetilde{R}_{21}J^\xi_1\widetilde{R}J^\xi_2 = -J^\xi_2 R_{21}(J^\xi_1)^{-1}R + J^\xi_2\widetilde{R}_{21}J^\xi_1\widetilde{R}. \tag{40}$$

Identity (33) holds according to [JoMa]. Using $\widetilde{R} = PR^{-1}P = R - (q - q^{-1})P$ and the Hecke relation for J^ξ, it is possible to get the first equation in each of (34), (35) and (36) from (33). The other equations on each of these three lines follow from the first one on the same line.

Using $\widetilde{R} = R - (q - q^{-1})P$, the Hecke relation for J^ξ and (33), (35), one can show that (37) holds. Similarly one can prove (38) using (34) and (35), and (39), (40) using (33) and (35). ☐

Definition 6.6. *The twisted quantum loop algebra $\mathfrak{U}^p_q(\mathcal{L}\mathfrak{gl}_n)$ is the associative $\mathbb{C}(q,\xi)$-algebra generated by elements $s^{(r)}_{ij}, 1 \le i,j \le n, r \in \mathbb{Z}$, such that the matrix $S(u) = \sum_{i,j=1}^n s_{ij}(u) \otimes E_{ij}$, where $s_{ij}(u) = \sum_{r=0}^\infty s^{(r)}_{ij}u^{-r}$, satisfies the reflection equation*

$$R_{21}(v,u)S_1(u)R(u^{-1},v)S_2(v) = S_2(v)R_{21}(u^{-1},v)S_1(u)R(v,u). \tag{41}$$

Moreover, we require that $s^{(0)}_{ij} = \begin{pmatrix} * & * & X \\ * & * & 0 \\ Y & 0 & 0 \end{pmatrix}_{ij}$*, where the blocks are of size $(p, n-2p, p) \times (p, n-2p, p)$, X and Y are upper triangular with respect to their second diagonal and $X_{i,p-i} = Y_{p-i,i}$. (See [JoMa], proposition 7.6.)*

We will later deduce that $\mathfrak{U}^p_q(\mathcal{L}\mathfrak{gl}_n)$ has a PBW-type basis. For the moment, we state one half of this fact. Introduce on the generators $s^{(r)}_{ij}$ a total order $\prec$ via $s^{(r)}_{ij} \prec s^{(p)}_{kl}$ if and only if $r < p$ or $r = p$ but $i < k$ or $r = p, i = k$ but $j < l$.

Proposition 6.2. *The set of monomials in the generators $s^{(r)}_{ij}$ ordered with respect to $\preceq$ is a spanning set for $\mathfrak{U}^p_q(\mathcal{L}\mathfrak{gl}_n)$.*

Proof. Set $\zeta = q - q^{-1}$. We need to start by writing the defining relation (41) in terms of the generating series $s_{ij}(u)$:

$$\begin{aligned}
&(u^{-1}q^{-\delta_{ik}} - v^{-1}q^{\delta_{ik}})\left(q^{-\delta_{jk}}s_{ij}(u)s_{kl}(v) - \zeta\delta_{jk}\sum_{e<j}s_{ie}(u)s_{el}(v)\right)\\
&\qquad +(uq^{\delta_{ik}} - vq^{-\delta_{ik}})\left(q^{\delta_{jk}}s_{ij}(u)s_{kl}(v) + \delta_{jk}\zeta\sum_{e>j}s_{ie}(u)s_{el}(v)\right)\\
&-\zeta(u^{-1}\delta_{i>k} + v^{-1}\delta_{i<k})\left(q^{-\delta_{ij}}s_{kj}(u)s_{il}(v) - \zeta\delta_{ij}\sum_{e<j}s_{ke}(u)s_{el}(v)\right)\\
&\qquad +\zeta(u\delta_{i<k} + v\delta_{i>k})\left(q^{\delta_{ij}}s_{kj}(u)s_{il}(v) + \zeta\delta_{ij}\sum_{e>j}s_{ke}(u)s_{el}(v)\right)\\
&= (u^{-1}q^{-\delta_{jl}} - v^{-1}q^{\delta_{jl}})\left(q^{-\delta_{il}}s_{kl}(v)s_{ij}(u) - \delta_{il}\zeta\sum_{i>e}s_{ke}(v)s_{ej}(u)\right)\\
&\qquad +(uq^{\delta_{jl}} - vq^{-\delta_{jl}})\left(q^{\delta_{il}}s_{kl}(v)s_{ij}(u) + \delta_{il}\zeta\sum_{i<e}s_{ke}(v)s_{ej}(u)\right)\\
&-\zeta(u^{-1}\delta_{l<j} + v^{-1}\delta_{l>j})\left(q^{-\delta_{ij}}s_{kj}(v)s_{il}(u) - \delta_{ij}\zeta\sum_{i>e}s_{ke}(v)s_{el}(u)\right)\\
&\qquad +\zeta(u\delta_{l>j} + v\delta_{l<j})\left(q^{\delta_{ij}}s_{kj}(v)s_{il}(u) + \delta_{ij}\zeta\sum_{i<e}s_{ke}(v)s_{el}(u)\right).
\end{aligned}$$

Set $f_{ab}(u,v) = (uq^{\delta_{ab}} - vq^{-\delta_{ab}})$, $g_{ab}(u,v) = u\delta_{b>a} + v\delta_{b<a}$ and

$$H^{\pm}_{abcd}(u,v) = q^{\pm\delta_{bc}}s_{ab}(u)s_{cd}(v) \pm \delta_{bc}\zeta\sum_{\substack{e=1\\ \pm e>\pm b}}^{n} s_{ae}(u)s_{ed}(v).$$

The defining relation (41) can be rewritten as

$$\begin{aligned}
&f_{ik}(u,v)H^{+}_{ijkl}(u,v) - f_{ik}(v^{-1},u^{-1})H^{-}_{ijkl}(u,v) + \zeta g_{ik}(u,v)H^{+}_{kjil}(u,v) - \zeta g_{ik}(v^{-1},u^{-1})H^{-}_{kjil}(u,v)\\
&= f_{jl}(u,v)H^{+}_{klij}(v,u) - f_{jl}(v^{-1},u^{-1})H^{-}_{klij}(v,u) + \zeta g_{jl}(u,v)H^{+}_{kjil}(v,u) - \zeta g_{jl}(v^{-1},u^{-1})H^{-}_{kjil}(v,u)
\end{aligned} \tag{42}$$

for $1 \le i,j,k,l \le n$.

All these relations give us straightening rules to express any monomial in the generators $s_{ij}^{(r)}$ into a sum of monomials ordered with respect to $\preceq$. Considering the coefficient of uv^{-r} in (42), we deduce that, for any $r \ge 0$,

$$\begin{aligned}
&q^{\delta_{ik}}\left(q^{\delta_{jk}}s_{ij}^{(0)}s_{kl}^{(r)} + \delta_{jk}\zeta\sum_{e>j}s_{ie}^{(0)}s_{el}^{(r)}\right) + \zeta\delta_{i<k}\left(q^{\delta_{ij}}s_{kj}^{(0)}s_{il}^{(r)} + \zeta\delta_{ij}\sum_{e>j}s_{ke}^{(0)}s_{el}^{(r)}\right)\\
&= q^{\delta_{jl}}\left(q^{\delta_{il}}s_{kl}^{(r)}s_{ij}^{(0)} + \delta_{il}\zeta\sum_{i<e}s_{ke}^{(r)}s_{ej}^{(0)}\right) + \zeta\delta_{l>j}\left(q^{\delta_{ij}}s_{kj}^{(r)}s_{il}^{(0)} + \delta_{ij}\zeta\sum_{i<e}s_{ke}^{(r)}s_{el}^{(0)}.\right)
\end{aligned} \tag{43}$$

Suppose that $s_{ij}^{(0)} \prec s_{kl}^{(r)}$ and $r \ge 1$. We see that $s_{kl}^{(r)}s_{ij}^{(0)}$ can be written as the sum of a scalar multiple of $s_{ij}^{(0)}s_{kl}^{(r)}$ and scalar multiples of monomials of the form $s_{ab}^{(0)}s_{cd}^{(r)}$ and of the form $s_{ab}^{(r)}s_{cd}^{(0)}$ with $c > i$, so that $s_{ij}^{(0)} \prec s_{cd}^{(0)}$. (Note that such terms do not occur if $i = n$.) By repeatedly applying this relation, we can eventually write $s_{ij}^{(0)}s_{kl}^{(r)}$ as the sum of a scalar multiple of $s_{ij}^{(0)}s_{kl}^{(r)}$ and scalar multiples of properly ordered monomials of the form $s_{ab}^{(0)}s_{cd}^{(r)}$.

Assume now that $r=0$ and $s_{kl}^{(0)} \prec s_{ij}^{(0)}$, so that $k \le i$. (Because of our choice for the relation (43), it is preferable to consider $s_{kl}^{(0)} \prec s_{ij}^{(0)}$ instead of $s_{ij}^{(0)} \prec s_{kl}^{(0)}$.) If $i=k$ and $l<j$, relation (43) shows that $s_{ij}^{(0)} s_{kl}^{(0)}$ can be written as the sum of a scalar multiple of $s_{kl}^{(0)} s_{ij}^{(0)}$ and scalar multiples of monomials of the form $s_{ab}^{(0)} s_{cd}^{(0)}$ with $a=k$ and $c>i$. If $k<i$, relation (43) shows that $s_{ij}^{(0)} s_{kl}^{(0)}$ can be written as the sum of a scalar multiple of $s_{kl}^{(0)} s_{ij}^{(0)}$, of $s_{kj}^{(0)} s_{il}^{(0)}$ and scalar multiples of monomials of the form $s_{ab}^{(0)} s_{cd}^{(0)}$ with either $k=a<c$ or $i=a, c>k$. In this last case (which does not occur if $k=n$), it is possible to reuse relation (43) finitely many times to be able to express $s_{ij}^{(0)} s_{kl}^{(0)}$ as a sum of properly ordered monomials.

The rest of the proof proceeds by induction. We have to show that $s_{kl}^{(r_1)} s_{ij}^{(r_2)}$ with $r_1 \ge r_2$ can always be written as a sum of properly ordered monomials. Induction is on r_2, the case $r_2=0$ having been dealt with already. Suppose that $r_1>r_2$; considering the coefficient of $v^{-r_1}u^{-r_2+1}$ in (42) and removing the monomials of the form $s_{ab}^{(m_1)} s_{cd}^{(m_2)}$ with $\min\{m_1,m_2\}<r_2$ (since the inductive assumption can be applied to them) yields the following relation:

$$q^{\delta_{ik}}\left(q^{\delta_{jk}} s_{ij}^{(r_2)} s_{kl}^{(r_1)} + \delta_{jk}\zeta \sum_{e>j} s_{ie}^{(r_2)} s_{el}^{(r_1)}\right) + \zeta\delta_{i<k}\left(q^{\delta_{ij}} s_{kj}^{(r_2)} s_{il}^{(r_1)} + \zeta\delta_{ij}\sum_{e>j} s_{ke}^{(r_2)} s_{el}^{(r_1)}\right)$$
$$\equiv q^{\delta_{jl}}\left(q^{\delta_{il}} s_{kl}^{(r_1)} s_{ij}^{(r_2)} + \delta_{il}\zeta\sum_{i<e} s_{ke}^{(r_1)} s_{ej}^{(r_2)}\right) + \zeta\delta_{l>j}\left(q^{\delta_{ij}} s_{kj}^{(r_1)} s_{il}^{(r_2)} + \delta_{ij}\zeta\sum_{i<e} s_{ke}^{(r_1)} s_{el}^{(r_2)}\right).$$

We see from this that if $l \le j$, then $s_{kl}^{(r_1)} s_{ij}^{(r_2)}$ can be expressed as a sum of a scalar multiple of $s_{ij}^{(r_2)} s_{kl}^{(r_1)}$ and as a sum of scalar multiples of monomials of the form $s_{ab}^{(m_1)} s_{cd}^{(m_2)}$ with $m_1=r_2<r_1=m_2$, or $\min\{m_1,m_2\}<r_2$ or with $m_1=r_1>m_2=r_2, c>i$. The monomials of the latter type can be shown, by induction on c, to be sums of properly ordered monomials. (Note that this latter case does not occur if $i=n$, so induction is on decreasing values of c from n to 1.) If $l>j$, then $s_{kl}^{(r_1)} s_{ij}^{(r_2)}$ can also be expressed as a sum of similar monomials to which a scalar multiple of $s_{kj}^{(r_1)} s_{il}^{(r_2)}$ must be added: this last monomial is not properly ordered, by since $j<l$, it falls into the previous case just considered. Therefore, when $r_1>r_2$, $s_{kl}^{(r_1)} s_{ij}^{(r_2)}$ can be expressed as a sum of ordered monomials.

Suppose now that $r_1=r_2$ and $s_{kl}^{(r_1)} \prec s_{ij}^{(r_2)}$ (so either $k<i$ or $k=i, l<j$). For this last case, we switch the roles of i,j,r_2 and k,l,r_1. We want to see that $s_{ij}^{(r_2)} s_{kl}^{(r_1)}$ can be expressed as a sum of properly ordered monomials. Considering the coefficient of $u^{-r_2}v^{-r_1+1}$ in (42) and removing the monomials of the form $s_{ab}^{(m_1)} s_{cd}^{(m_2)}$ with $\min\{m_1,m_2\}<r_2$ (since the inductive assumption can be applied to them) yields the following relation:

$$-q^{-\delta_{ik}}\left(q^{\delta_{jk}} s_{ij}^{(r_2)} s_{kl}^{(r_1)} + \delta_{jk}\zeta \sum_{e>j} s_{ie}^{(r_2)} s_{el}^{(r_1)}\right) + \zeta\delta_{i>k}\left(q^{\delta_{ij}} s_{kj}^{(r_2)} s_{il}^{(r_1)} + \zeta\delta_{ij}\sum_{e>j} s_{ke}^{(r_2)} s_{el}^{(r_1)}\right)$$
$$\equiv -q^{-\delta_{jl}}\left(q^{\delta_{il}} s_{kl}^{(r_1)} s_{ij}^{(r_2)} + \delta_{il}\zeta\sum_{i<e} s_{ke}^{(r_1)} s_{ej}^{(r_2)}\right) + \zeta\delta_{l<j}\left(q^{\delta_{ij}} s_{kj}^{(r_1)} s_{il}^{(r_2)} + \delta_{ij}\zeta\sum_{i<e} s_{ke}^{(r_1)} s_{el}^{(r_2)}\right).$$

We see from this that if $k \le i$, then $s_{ij}^{(r_2)} s_{kl}^{(r_1)}$ can be expressed as a sum of a scalar multiple of $s_{kl}^{(r_1)} s_{ij}^{(r_2)}$ and as a sum of scalar multiples of monomials of the form $s_{ab}^{(m_1)} s_{cd}^{(m_2)}$ with $\min\{m_1,m_2\}<r_2$ or with $m_1=r_1=r_2=m_2$ and $s_{ab}^{(m_1)} \prec s_{cd}^{(m_2)}$, or with $c>k$. The monomials with $c>k$ can be shown, by induction on c, to be sums of properly ordered monomials. (Note that this case does not occur if $k=n$, so induction is on decreasing values of c from n to 1.) A similar argument works when $i=k, l<j$.

Having proved that the product of any two generators of $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$ can be expressed as a sum of properly ordered monomials in two generators, it follows immediately that the same is true of the product of any number of generators. □

6.4. **Embedding in the quantum loop algebra.** In [MRS], q-twisted Yangians were realized as subalgebras of the quantum loop algebra $\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n)$. This is also possible for $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$.

Theorem 6.3. *The twisted quantum loop algebra $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$ can be realized as a subalgebra of $\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n)\otimes_{\mathbb{C}(q)}\mathbb{C}(q,\xi)$ via the embedding $\iota : S(u)\mapsto T(u)G^{\xi}(u)\overline{T}^{-1}(u^{-1})$.*

Proof. We have to check that ι gives a well-defined homomorphism $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)\longrightarrow\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n)\otimes_{\mathbb{C}(q)}\mathbb{C}(q,\xi)$, that is, it respects the relation (41). (As for the vanishing condition on $s_{ij}^{(0)}$, see proposition 7.6 in [JoMa].) The proof eventually relies on the fact that $G^{\xi}(u)$ satisfies the reflection equation. We have to substitute $T(u)G^{\xi}(u)\overline{T}^{-1}(u^{-1})$ into (41) and check that it is satisfied.

$$
\begin{aligned}
&R_{21}(v,u)T_1(u)G_1^{\xi}(u)\overline{T}_1^{-1}(u^{-1})R(u^{-1},v)T_2(v)G_2^{\xi}(v)\overline{T}_2^{-1}(v^{-1})\\
&\quad= R_{21}(v,u)T_1(u)G_1^{\xi}(u)T_2(v)R(u^{-1},v)\overline{T}_1^{-1}(u^{-1})G_2^{\xi}(v)\overline{T}_2^{-1}(v^{-1}) \text{ by (29)}\\
&\quad= R_{21}(v,u)T_1(u)T_2(v)G_1^{\xi}(u)R(u^{-1},v)G_2^{\xi}(v)\overline{T}_1^{-1}(u^{-1})\overline{T}_2^{-1}(v^{-1})\\
&\quad= T_2(v)T_1(u)R_{21}(v,u)G_1^{\xi}(u)R(u^{-1},v)G_2^{\xi}(v)\overline{T}_1^{-1}(u^{-1})\overline{T}_2^{-1}(v^{-1}) \text{ by (27)}\\
&\quad= T_2(v)T_1(u)G_2^{\xi}(v)R_{21}(u^{-1},v)G_1^{\xi}(u)R(v,u)\overline{T}_1^{-1}(u^{-1})\overline{T}_2^{-1}(v^{-1}) \text{ by (32)}\\
&\qquad\text{(using } R(u,v)=uvR(v^{-1},u^{-1})\text{)}\\
&\quad= u^{-1}vuvT_2(v)T_1(u)G_2^{\xi}(v)R_{21}(v^{-1},u)G_1^{\xi}(u)R(u^{-1},v^{-1})\overline{T}_1^{-1}(u^{-1})\overline{T}_2^{-1}(v^{-1})\\
&\quad= v^2T_2(v)G_2^{\xi}(v)T_1(u)R_{21}(v^{-1},u)G_1^{\xi}(u)\overline{T}_2^{-1}(v^{-1})\overline{T}_1^{-1}(u^{-1})R(u^{-1},v^{-1}) \text{ by (28)}\\
&\quad= v^2T_2(v)G_2^{\xi}(v)T_1(u)R_{21}(v^{-1},u)\overline{T}_2^{-1}(v^{-1})G_1^{\xi}(u)\overline{T}_1^{-1}(u^{-1})R(u^{-1},v^{-1})\\
&\quad= u^{-1}vuvT_2(v)G_2^{\xi}(v)\overline{T}_2^{-1}(v^{-1})R_{21}(v^{-1},u)T_1(u)G_1^{\xi}(u)\overline{T}_1^{-1}(u^{-1})R(u^{-1},v^{-1}) \text{ by (29)}\\
&\qquad\text{(using } R(u,v)=uvR(v^{-1},u^{-1})\text{)}\\
&\quad= T_2(v)G_2^{\xi}(v)\overline{T}_2^{-1}(v^{-1})R_{21}(u^{-1},v)T_1(u)G_1^{\xi}(u)\overline{T}_1^{-1}(u^{-1})R(v,u).
\end{aligned}
$$

This proves that ι is a homomorphism of algebras. We have to see why it is injective. We can argue as in [MRS] by passing to the limit $q\mapsto 1$. (See [JoMa] for the finite case.) Recall that $\mathcal{A}$ is the localization of $\mathbb{C}[q,q^{-1}]$ at the prime ideal $(q-1)$.

We can view $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$ as an algebra over $\mathbb{C}(q)$ if we set $\xi=q^{\ell}$ for some $\ell\in\mathbb{Z}$. Set $\sigma_{ij}^{(r)}=\frac{s_{ij}^{(r)}}{q-q^{-1}}$ if $r>0$ or $i\neq j,n-j+1$ or if $r=0, 1\leq i=j\leq p$; set $\sigma_{i,n-i+1}^{(0)}=\frac{s_{i,n-i+1}^{(0)}-1}{q-q^{-1}}$ if $1\leq i\leq p$ or $n-p+1\leq i\leq n$ and $\sigma_{ii}^{(0)}=\frac{\xi^{-1}+s_{ii}^{(0)}}{q-q^{-1}}$ if $p+1\leq i\leq n-p$. (See the proof of claim 10.5 in [JoMa].)

Let $\mathfrak{U}_{\mathcal{A}}^p(\mathcal{L}\mathfrak{gl}_n)$ be the $\mathcal{A}$-subalgebra of $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$ generated by the elements $\sigma_{ij}^{(r)}$. Let $\iota_{\mathcal{A}}$ be defined as ι, but from $\mathfrak{U}_{\mathcal{A}}^p(\mathcal{L}\mathfrak{gl}_n)$ to $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_n)$. Since $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_n)/(q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_n)\cong\mathfrak{U}(\mathcal{L}\mathfrak{gl}_n)$, Image$(\iota_{\mathcal{A}})/(q-1)$Image$(\iota_{\mathcal{A}})$ can be mapped to a subalgebra of the enveloping algebra of $\mathcal{L}\mathfrak{gl}_n$: we want to see that this subalgebra is the enveloping algebra of $\mathfrak{gl}_n(\mathbb{C}[s,s^{-1}])^{\theta}$. Let $\mathsf{L}:\mathfrak{U}_{\mathcal{A}}^p(\mathcal{L}\mathfrak{gl}_n)\longrightarrow\mathfrak{U}(\mathcal{L}\mathfrak{gl}_n)$ be the map which is the composite of $\iota_{\mathcal{A}}$ with $\widetilde{\mathsf{L}}:\mathfrak{U}_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_n)\longrightarrow\mathfrak{U}_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_n)/(q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_n)\cong\mathfrak{U}(\mathcal{L}\mathfrak{gl}_n)$.

It is instructive to compute the limit of $\iota_{\mathcal{A}}(\sigma_{ij}^{(r)})$ when $q\mapsto 1$, that is, to identify $\mathsf{L}(\sigma_{ij}^{(r)})$ as an element of $\mathfrak{U}(\mathcal{L}\mathfrak{gl}_n)$. When $r=0$, this was done in Section 10.4 in [JoMa]. We could remove the assumption that $\xi=q^{\ell}$ if we followed the approach in [JoMa] and worked over $\mathbb{C}[[h]]$, in which case q and ξ should be of the form e^h, e^{ch} for some constant $c\in\mathbb{C}$. Set $\widetilde{T}(u)=\overline{T}(u)^{-1}$ and denote its matrix entries by $\sum_{r=0}^{\infty}\widetilde{t}_{ij}^{(r)}u^{-r}$. Denote by j_{km} and $\widetilde{j}_{km}$ the entries of J^{ξ} and $(J^{\xi})^{-1}$. When $r>0$,

$$\iota(s_{ij}^{(r)})=\sum_{k,m=1}^{n}\left(\sum_{d=0}^{\lfloor r/2\rfloor}\sum_{s=0}^{r-2d}t_{ik}^{(s)}j_{km}\widetilde{t}_{mj}^{(r-2d-s)}-\sum_{d=0}^{\lfloor (r-2)/2\rfloor}\sum_{s=0}^{r-2d-2}t_{ik}^{(s)}\widetilde{j}_{km}\widetilde{t}_{mj}^{(r-2d-2-s)}\right),$$

so, for $r \geq 1$, $\mathsf{L}(\sigma_{ij}^{(r)})$ equals

$$\widetilde{\mathsf{L}}\left(\sum_{k,m=1}^{n}\left(\sum_{d=0}^{\lfloor r/2\rfloor}\left(\frac{t_{ik}^{(0)}j_{km}\widetilde{t}_{mj}^{(r-2d)}}{q-q^{-1}}+\frac{t_{ik}^{(r-2d)}j_{km}\widetilde{t}_{mj}^{(0)}}{q-q^{-1}}\right)-\sum_{d=0}^{\lfloor (r-2)/2\rfloor}\left(\frac{t_{ik}^{(0)}\widetilde{j}_{km}\widetilde{t}_{mj}^{(r-2d-2)}}{q-q^{-1}}+\frac{t_{ik}^{(r-2d-2)}\widetilde{j}_{km}\widetilde{t}_{mj}^{(0)}}{q-q^{-1}}\right)\right)\right).$$

Therefore, to compute $\mathsf{L}(\sigma_{ij}^{(r)})$, it is enough to determine $\widetilde{\mathsf{L}}\left(\sum_{k,m=1}^{n}\left(\frac{t_{ik}^{(0)}j_{km}\widetilde{t}_{mj}^{(r)}}{q-q^{-1}}+\frac{t_{ik}^{(r)}j_{km}\widetilde{t}_{mj}^{(0)}}{q-q^{-1}}\right)\right)$ when $r\geq 1$. Moreover, since $r \geq 1$, j_{km} and $\widetilde{j}_{km}$ can be replaced by the entry Θ'_{km} of Θ'_p. (Note that $\Theta'_p = (\Theta'_p)^{-1}$.) It is necessary to consider several separate cases as in the proof of claim 10.5 in [JoMa]. Set

$$L = \widetilde{\mathsf{L}}\left(\sum_{k,m=1}^{n}\left(\frac{t_{ik}^{(0)}\Theta'_{km}\widetilde{t}_{mj}^{(r)}}{q-q^{-1}}+\frac{t_{ik}^{(r)}\Theta'_{km}\widetilde{t}_{mj}^{(0)}}{q-q^{-1}}\right)\right).$$

In the computations below, we also give gLg^{-1} since this is useful in understanding how the image of L is isomorphic to $\mathfrak{Ugl}_n(\mathbb{C}[s,s^{-1}])^\theta$. Note that

$$g^{-1} = \frac{1}{2}\sum_{k=1}^{p}E_{kk} - \sum_{k=p+1}^{n-p}E_{kk} - \frac{1}{2}\sum_{k=n-p+1}^{n}E_{kk} + \frac{1}{2}\sum_{k=1}^{p}E_{n-k+1,k} + \frac{1}{2}\sum_{k=1}^{p}E_{k,n-k+1}.$$

Case 1. $1 \leq i,j \leq p$.

$$L_1 = \mathsf{L}\left(\frac{t_{ii}^{(0)}\Theta'_{i,n-i+1}\widetilde{t}_{n-i+1,j}^{(r)}}{q-q^{-1}}+\frac{t_{i,n-j+1}^{(r)}\Theta'_{n-j+1,j}\widetilde{t}_{jj}^{(0)}}{q-q^{-1}}\right) = E_{j,n-i+1}s^{-r} + E_{n-j+1,i}s^{r},$$

$$gL_1g^{-1} = \frac{1}{2}E_{ji}(s^r+s^{-r}) - \frac{1}{2}E_{n-j+1,n-i+1}(s^r+s^{-r}) - \frac{1}{2}E_{n-j+1,i}(s^r-s^{-r}) + \frac{1}{2}E_{j,n-i+1}(s^r-s^{-r}).$$

Case 2. $1 \leq i \leq p$, $p+1 \leq j \leq n-p$.

$$L_2 = \mathsf{L}\left(\frac{t_{ii}^{(0)}\Theta'_{i,n-i+1}\widetilde{t}_{n-i+1,j}^{(r)}}{q-q^{-1}}+\frac{t_{ij}^{(r)}\Theta'_{jj}\widetilde{t}_{jj}^{(0)}}{q-q^{-1}}\right) = E_{j,n-i+1}s^{-r} - E_{ji}s^{r},$$

$$gL_2g^{-1} = \frac{1}{2}E_{ji}(s^r-s^{-r}) + \frac{1}{2}E_{j,n-i+1}(s^r+s^{-r}).$$

Case 3. $1 \leq i \leq p$, $n-p+1 \leq j \leq n$.

$$L_3 = \mathsf{L}\left(\frac{t_{ii}^{(0)}\Theta'_{i,n-i+1}\widetilde{t}_{n-i+1,j}^{(r)}}{q-q^{-1}}+\frac{t_{i,n-j+1}^{(r)}\Theta'_{n-j+1,j}\widetilde{t}_{jj}^{(0)}}{q-q^{-1}}\right) = E_{j,n-i+1}s^{-r} + E_{n-j+1,i}s^{r},$$

$$gL_3g^{-1} = \frac{1}{2}E_{ji}(s^r-s^{-r}) + \frac{1}{2}E_{n-j+1,n-i+1}(s^r-s^{-r}) + \frac{1}{2}E_{n-j+1,i}(s^r+s^{-r}) + \frac{1}{2}E_{j,n-i+1}(s^r+s^{-r}).$$

Case 4. $p+1 \leq i \leq n-p$, $1 \leq j \leq p$.

$$L_4 = \mathsf{L}\left(\frac{t_{ii}^{(0)}\Theta'_{ii}\widetilde{t}_{ij}^{(r)}}{q-q^{-1}}+\frac{t_{i,n-j+1}^{(r)}\Theta'_{n-j+1,j}\widetilde{t}_{jj}^{(0)}}{q-q^{-1}}\right) = -E_{ji}s^{-r} + E_{n-j+1,i}s^{r},$$

$$gL_4g^{-1} = -E_{ji}(s^r-s^{-r}) + E_{n-j+1,i}(s^r+s^{-r}).$$

Case 5. $p+1 \leq i,j \leq n-p$.

$$L_5 = \mathsf{L}\left(\frac{t_{ii}^{(0)}\Theta'_{ii}\widetilde{t}_{ij}^{(r)}}{q-q^{-1}} + \frac{t_{ij}^{(r)}\Theta'_{jj}\widetilde{t}_{jj}^{(0)}}{q-q^{-1}}\right) = -E_{ji}s^{-r} - E_{ji}s^{r},$$
$$gL_5g^{-1} = -E_{ji}(s^r + s^{-r}).$$

Case 6. $p+1 \le i \le n-p$, $n-p+1 \le j \le n$.

$$L_6 = \mathsf{L}\left(\frac{t_{ii}^{(0)}\Theta'_{ii}\widetilde{t}_{ij}^{(r)}}{q-q^{-1}} + \frac{t_{i,n-j+1}^{(r)}\Theta'_{n-j+1,j}\widetilde{t}_{jj}^{(0)}}{q-q^{-1}}\right) = -E_{ji}s^{-r} + E_{n-j+1,i}s^{r},$$
$$gL_6g^{-1} = -E_{ji}(s^r + s^{-r}) - E_{n-j+1,i}(s^r - s^{-r}).$$

Case 7. $n-p+1 \le i \le n$, $1 \le j \le p$.

$$L_7 = \mathsf{L}\left(\frac{t_{ii}^{(0)}\Theta'_{i,n-i+1}\widetilde{t}_{n-i+1,j}^{(r)}}{q-q^{-1}} + \frac{t_{i,n-j+1}^{(r)}\Theta'_{n-j+1,j}\widetilde{t}_{jj}^{(0)}}{q-q^{-1}}\right) = E_{j,n-i+1}s^{-r} + E_{n-j+1,i}s^{r},$$
$$gL_7g^{-1} = -\frac{1}{2}E_{ji}(s^r - s^{-r}) - \frac{1}{2}E_{n-j+1,n-i+1}(s^r - s^{-r}) + \frac{1}{2}E_{n-j+1,i}(s^r + s^{-r}) + \frac{1}{2}E_{j,n-i+1}(s^r + s^{-r}).$$

Case 8. $n-p+1 \le i \le n$, $p+1 \le j \le n-p$.

$$L_8 = \mathsf{L}\left(\frac{t_{ii}^{(0)}\Theta'_{i,n-i+1}\widetilde{t}_{n-i+1,j}^{(r)}}{q-q^{-1}} + \frac{t_{ij}^{(r)}\Theta'_{jj}\widetilde{t}_{jj}^{(0)}}{q-q^{-1}}\right) = E_{j,n-i+1}s^{-r} - E_{ji}s^{r},$$
$$gL_8g^{-1} = -\frac{1}{2}E_{ji}(s^r + s^{-r}) + \frac{1}{2}E_{j,n-i+1}(s^r - s^{-r}).$$

Case 9. $n-p+1 \le i,j \le n$.

$$L_9 = \mathsf{L}\left(\frac{t_{ii}^{(0)}\Theta'_{i,n-i+1}\widetilde{t}_{n-i+1,j}^{(r)}}{q-q^{-1}} + \frac{t_{i,n-j+1}^{(r)}\Theta'_{n-j+1,j}\widetilde{t}_{jj}^{(0)}}{q-q^{-1}}\right) = E_{j,n-i+1}s^{-r} + E_{n-j+1,i}s^{r},$$
$$gL_9g^{-1} = -\frac{1}{2}E_{ji}(s^r + s^{-r}) + \frac{1}{2}E_{n-j+1,n-i+1}(s^r + s^{-r}) + \frac{1}{2}E_{j,n-i+1}(s^r - s^{-r}) - \frac{1}{2}E_{n-j+1,i}(s^r - s^{-r}).$$

It follows from all this that the quasi-classical limit of the image of $\iota_{\mathcal{A}}$ is the enveloping algebra of the Lie algebra $\mathfrak{gl}_n(\mathbb{C}[s,s^{-1}])^{\theta'}$, which is isomorphic to $\mathfrak{gl}_n(\mathbb{C}[s,s^{-1}])^{\theta}$ via conjugation by the matrix g. For instance, if we denote by L_k the limit computed in case k ($1 \le k \le 9$), and if $1 \le i,j \le p$, then we can write (after relabeling the indices in the case of gL_3g^{-1} and gL_7g^{-1})

$$\begin{pmatrix} gL_1g^{-1} \\ gL_3g^{-1} \\ gL_7g^{-1} \\ gL_9g^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} E_{ji}(s^r + s^{-r}) \\ E_{n-j+1,i}(s^r - s^{-r}) \\ E_{j,n-i+1}(s^r - s^{-r}) \\ E_{n-j+1,n-i+1}(s^r + s^{-r}) \end{pmatrix}.$$

Since the matrix is invertible, we see that we can express $E_{ji}(s^r + s^{-r}), E_{n-j+1,i}(s^r - s^{-r}), E_{j,n-i+1}(s^r - s^{-r})$ and $E_{n-j+1,n-i+1}(s^r + s^{-r})$ in terms of $gL_1g^{-1}, gL_3g^{-1}, gL_7g^{-1}$ and gL_9g^{-1}.

Under the specialization $q \mapsto 1$, the spanning monomials provided by Proposition 6.2 are mapped by L to a PBW basis of the enveloping algebra of $\mathfrak{gl}_n(\mathbb{C}[s,s^{-1}])^{\theta'}$. Therefore, they must be linearly independent. It follows that ι is injective and we can conclude that it is an isomorphism. (See [MRS] for the analogous result for orthogonal and symplectic twisted q-Yangians.) In more detail, if $\sum_{i\in I} c_{M_i} M_i = 0$ is a relation of linear dependence where M_i is one of the monomials in Proposition 6.2 and $c_{M_i} \in \mathbb{C}(q,\xi)$, then we can clear denominators and assume that $c_{M_i} \in \mathbb{C}(q)[\xi,\xi^{-1}]$. We can find $\ell \in \mathbb{Z}$ such that some c_{M_i} does not belong to the ideal of $\mathbb{C}(q)[\xi,\xi^{-1}]$ generated by $\xi - q^\ell$. Passing to the quotient $\mathbb{C}(q)[\xi,\xi^{-1}]/(\xi - q^\ell)$ and replacing $s_{ij}^{(r)}$ by $\sigma_{ij}^{(r)}$, we can obtain a relation for monomials in $\mathfrak{U}^p_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_n)$ with coefficients in $\mathcal{A}$ and we can assume that not all the coefficients belong to the ideal $(q-1)$. Applying $\iota_{\mathcal{A}}$ and passing to $\mathfrak{U}^p_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_n)/(q-1)\mathfrak{U}^p_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_n)$, we obtain a contradiction because of the linear independence of the images of the monomials M_i in $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_n)/(q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_n)$. Therefore, all the coefficients c_{M_i} must vanish. □

Let us collect the last part of the previous proof inside a corollary.

Corollary 6.1. *The $\mathcal{A}$-subalgebra of $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}\mathfrak{gl}_n)$ generated by the coefficients of the entries of $T(u)G^{\xi}(u)\overline{T}^{-1}(u^{-1})$ specializes to the enveloping algebra $\mathfrak{U}(\mathfrak{g}_n(\mathbb{C}[s,s^{-1}])^{\theta})$ as $q \mapsto 1$ when $\xi = q^\ell, \ell \in \mathbb{Z}$.*

Corollary 6.2. *The ordered monomials from Proposition 6.2 constitute a vector space basis of $\mathfrak{U}^p_q(\mathcal{L}\mathfrak{gl}_n)$ over $\mathbb{C}(q,\xi)$.*

Corollary 6.3. *$\mathfrak{U}^p_q(\mathcal{L}\mathfrak{gl}_n)$ is a coideal subalgebra of $\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n) \otimes_{\mathbb{C}(q)} \mathbb{C}(q,\xi)$ with coproduct given by*

$$\Delta(s_{ij}(u)) = \sum_{k,l=1}^{n} t_{ik}(u)\widetilde{t}_{lj}(u^{-1}) \otimes s_{kl}(u).$$

7. Drinfeld functor for twisted quantum loop algebras of type AIII

7.1. **From affine Hecke algebra modules to representations of quantum loop algebras: the $\mathfrak{gl}_n$ case.** In this section, we present the construction of a functor between categories of modules over the affine Hecke algebra of type A and over $\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n)$. This is essentially the functor studied in [ChPr2], although we are using a different set of generators for $\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n)$. Similar constructions can be found in [Ch1] and the current section should not be considered original work, but we have decided to keep it since it makes the relevant construction in type A more understandable for our purposes.

Let $V = \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}(q,\xi)$ be the vector representation of $\mathfrak{U}_q(\mathfrak{gl}_n)$ used in [JoMa], section 7.2 and extended to $\mathbb{C}(q,\xi)$. Consider the tensor product $V^{\otimes l}$. There is an $\mathcal{H}^l_{q^{-1}}$-module structure on it given by

$$\sigma_i \mapsto q^{-1} R^{-1}_{i,i+1} P_{i,i+1} \in \mathrm{End}_{\mathbb{C}(q,\xi)}(V^{\otimes l}), i = 1,\dots,l-1, \tag{44}$$

where $P_{i,j} = \sum_{s,t=1}^n E^{(i)}_{st} E^{(j)}_{ts}$ is the permutation operator and

$$R_{i,j} = \sum_{\substack{s,t=1\\ s\neq t}}^{n} q^{\delta_{st}} E^{(i)}_{ss} E^{(j)}_{tt} + (q-q^{-1}) \sum_{\substack{s,t=1\\ s>t}}^{n} E^{(i)}_{st} E^{(j)}_{ts}.$$

Moreover, if we define an action of σ_l on $V^{\otimes l}$ by

$$\sigma_l(v_1 \otimes v_2 \otimes \cdots \otimes v_l) = v_1 \otimes v_2 \otimes \cdots \otimes (\xi^{-1}(J^{\xi})^{-1} v_l),$$

where J^{ξ} is the n by n matrix defined in (30), we obtained an $\mathcal{H}^l_{q^{-1},\xi^{-1}}$-module structure on $V^{\otimes l}$. For more details, see [JoMa].

Remark 7.1. *In the future, we will use the braid group module structure where the braid group action is obtained from the natural quotient map of the group algebra of the braid group onto the finite Hecke algebra.*

From lemma 2.1 in [FrMu], we deduce that, for any commutative $\mathbb{C}(q)$-algebra $\mathbf{A}$ and any invertible element $a \in \mathbf{A}$, the following formula defines a homomorphism $\mathrm{ev}_a : \mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n) \to \mathbf{A} \otimes_{\mathbb{C}(q)} \mathfrak{U}_q(\mathfrak{gl}_n)$, given by

$$\mathrm{ev}_a(t_{ij}(u)) = \frac{a t_{ij} - u^{-1}\bar{t}_{ij}}{a - u^{-1}}, \quad \mathrm{ev}_a(\bar{t}_{ij}(u)) = \frac{a^{-1}\bar{t}_{ij} - u t_{ij}}{a^{-1} - u}. \tag{45}$$

Let $\rho : \mathfrak{U}_q(\mathfrak{gl}_n) \to \mathrm{End}_{\mathbb{C}(q)}(V)$ be the vector representation, and $\rho_k = \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \rho \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} : \mathfrak{U}_q(\mathfrak{gl}_n) \to \mathrm{End}_{\mathbb{C}(q)}(V^{\otimes l}) \cong \mathrm{End}_{\mathbb{C}(q)}(V)^{\otimes l}$ with ρ as the k-th tensor factor. Then for $k = 1, \ldots, l$, set

$$I_k^+ = \sum_{i,j=1}^{n} (\rho_k \otimes \rho)\big((t_{ij})_k \otimes E_{ij}\big),\ I_k^- = \sum_{i,j=1}^{n} (\rho_k \otimes \rho)\big((\bar{t}_{ij})_k \otimes E_{ij}\big) \in \mathrm{End}_{\mathbb{C}(q)}(V^{\otimes l}) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n).$$

We index the last tensor factor in $\mathrm{End}_{\mathbb{C}(q)}(V)^{\otimes l} \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ by $l+1$, so that $I_k^+ = (\rho_k \otimes \rho)(R_{k,l+1})$ and $I_k^- = (\rho_k \otimes \rho)(P_{k,l+1} R_{k,l+1}^{-1} P_{k,l+1})$ (where the last ρ denotes the natural representation of $\mathfrak{gl}_n$).

For any left $\mathbf{H}_{q^{-1}}^l$-module M, consider the tensor product $M \otimes_{\mathbb{C}(q)} V^{\otimes l}$. It has a $\mathtt{B}_l^A$-module structure through the diagonal action, where the $\mathtt{B}_l^A$-module structures on M and $V^{\otimes l}$ are given by the natural projection $\mathbb{C}[\mathtt{B}_l^A] \twoheadrightarrow \mathcal{H}_{q^{-1}}^l$. Define the quotient space $\mathcal{D}_A(M)$ as:

$$\mathcal{D}_A(M) = M \otimes_{\mathbb{C}(q)} V^{\otimes l} / \sum_{i=1}^{l-1} \mathrm{Im}(\sigma_i + q^{-2}). \tag{46}$$

Now for $k = 1, \ldots, l$, introduce the following elements in $\mathbf{H}_{q^{-1}}^l[[u^{-1}]] \otimes_{\mathbb{C}(q)} \mathrm{End}_{\mathbb{C}(q)}(V^{\otimes l}) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ and in $\mathbf{H}_{q^{-1}}^l[[u]] \otimes_{\mathbb{C}(q)} \mathrm{End}_{\mathbb{C}(q)}(V^{\otimes l}) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ respectively:

$$\mathbf{T}_k^+(u) = \frac{q^{-k} Y_k^{-1}}{q^{-k} Y_k^{-1} - u^{-1}} \otimes I_k^+ - \frac{u^{-1}}{q^{-k} Y_k^{-1} - u^{-1}} \otimes I_k^-, \tag{47}$$

$$\mathbf{T}_k^-(u) = \frac{(q^k Y_k - u) q^k Y_k}{(q^k Y_k)^2 - (q^2 + q^{-2}) u q^k Y_k + u^2} \otimes I_k^- - \frac{(q^k Y_k - u) u}{(q^k Y_k)^2 - (q^2 + q^{-2}) q^k u Y_k + u^2} \otimes I_k^+. \tag{48}$$

Theorem 7.1. *The map*

$$T(u) \mapsto \mathbf{T}_1^+(u) \mathbf{T}_2^+(u) \cdots \mathbf{T}_l^+(u), \qquad \overline{T}(u) \mapsto \mathbf{T}_1^-(u) \mathbf{T}_2^-(u) \cdots \mathbf{T}_l^-(u), \tag{49}$$

defines a $\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n)$-module structure on $\mathcal{D}_A(M)$. Thus we have a functor $\mathcal{D}_A$ from the category of left $\mathbf{H}_{q^{-1}}^l$-modules to the category of left $\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n)$-modules.

The proof of this theorem requires the next lemma.

Lemma 7.1. *Let V be the vector representation of $\mathfrak{U}_q(\mathfrak{gl}_n)$ and let $\mathcal{H}_{q^{-1}}^l$ act on $V^{\otimes l}$ by (44). Then we have the following identities for $k = 1, \ldots, l-1$, the first two holding in $\mathrm{End}_{\mathbb{C}(q)}(V^{\otimes l}) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ and the last one in $\mathbf{H}_{q^{-1}}^l$:*

(1) $I_k^{\pm} I_{k+1}^{\pm} \sigma_k = \sigma_k I_k^{\pm} I_{k+1}^{\pm}$;
(2) $I_k^+ I_{k+1}^- \sigma_k = \sigma_k I_k^- I_{k+1}^+, \quad I_k^- I_{k+1}^+ \sigma_k = \sigma_k I_k^+ I_{k+1}^- + (1 - q^{-2})(I_k^+ I_{k+1}^- - I_k^- I_{k+1}^+)$;
(3) $Y_k \sigma_k = (q\sigma_k + q - q^{-1}) Y_{k+1},\ Y_{k+1} \sigma_k = (q^{-2} - 1) Y_{k+1} + q^{-1} \sigma_k Y_k,\ Y_k^{\pm 1} Y_{k+1}^{\pm 1} \sigma_k = \sigma_k Y_k^{\pm 1} Y_{k+1}^{\pm 1},\ Y_{k+1}^{-1} \sigma_k = (q\sigma_k + q - q^{-1}) Y_k^{-1},\ Y_k^{-1} \sigma_k = (q^{-2} - 1) Y_k^{-1} + q^{-1} \sigma_k Y_{k+1}^{-1}$.

Proof. From the definition,

$$\begin{aligned} q[I_k^+ I_{k+1}^+, \sigma_k] &= R_{k,l+1} R_{k+1,l+1} R_{k,k+1}^{-1} P_{k,k+1} - R_{k,k+1}^{-1} P_{k,k+1} R_{k,l+1} R_{k+1,l+1} \\ &= (R_{k,l+1} R_{k+1,l+1} R_{k,k+1}^{-1} - R_{k,k+1}^{-1} R_{k+1,l+1} R_{k,l+1}) P_{k,k+1}, \end{aligned}$$

which is 0 since R is a solution of the Yang-Baxter equation. Similarly,

$$\begin{aligned} q[I_k^- I_{k+1}^-, \sigma_k] &= R_{l+1,k}^{-1} R_{l+1,k+1}^{-1} R_{k,k+1}^{-1} P_{k,k+1} - R_{k,k+1}^{-1} P_{k,k+1} R_{l+1,k}^{-1} R_{l+1,k+1}^{-1} \\ &= P_{k,k+1} (R_{l+1,k+1}^{-1} R_{l+1,k}^{-1} R_{k+1,k}^{-1} - R_{k+1,k}^{-1} R_{l+1,k}^{-1} R_{l+1,k+1}^{-1}) = 0. \end{aligned}$$

So (1) holds, and the proof that $I_k^+ I_{k+1}^- \sigma_k = \sigma_k I_k^- I_{k+1}^+$, $I_k^- I_{k+1}^+ \sigma_k^{-1} = \sigma_k^{-1} I_k^+ I_{k+1}^-$ is analogous. Using $\sigma_k^{-1} = q^2 \sigma_k + q^2 - 1$, we can prove (2). (3) can be obtained directly from the defining relations in the Hecke algebra $\mathbf{H}_{q^{-1}}^l$. ◻

Proof of Theorem 7.1. From the commutativity of $Y_i^{\pm}$ and the evaluation homomorphism (45), it is enough to show that the map (49) factors through the quotient (46). From Lemma 2.4, the coefficients of u^{-1} and of u, respectively, in $\prod_{k=1}^{l}(q^{-k}Y_k^{-1}-u^{-1})$ and in $\prod_{k=1}^{l}\frac{(q^kY_k)^2-(q^2+q^{-2})uq^kY_k+u^2}{(q^kY_k-u)}$, are in the center of $\mathbf{H}^l_{q^{-1}}$. So we only need to show that $\widetilde{\mathbf{T}}_1^+(u)\widetilde{\mathbf{T}}_2^+(u)\cdots\widetilde{\mathbf{T}}_l^+(u)$ and $\widetilde{\mathbf{T}}_1^-(u)\widetilde{\mathbf{T}}_2^-(u)\cdots\widetilde{\mathbf{T}}_l^-(u)$ factor through the quotient $\mathcal{D}_A(M)$, where $\widetilde{\mathbf{T}}_k^{\pm}(u)=(q^kY_k)^{\mp 1}\otimes I_k^{\pm}-u^{\mp 1}\otimes I_k^{\mp}$.

Using Lemma 7.1, we have

$$\begin{aligned}
&\widetilde{\mathbf{T}}_k^+(u)\widetilde{\mathbf{T}}_{k+1}^+(u)\circ\sigma_k-\sigma_k\circ\widetilde{\mathbf{T}}_k^+(u)\widetilde{\mathbf{T}}_{k+1}^+(u)\\
&\quad=-u^{-1}q^{-k-1}Y_{k+1}^{-1}\sigma_k\otimes I_k^-I_{k+1}^+\sigma_k-u^{-1}q^{-k}Y_k^{-1}\sigma_k\otimes I_k^+I_{k+1}^-\sigma_k\\
&\qquad+u^{-1}q^{-k-1}\sigma_kY_{k+1}^{-1}\otimes\sigma_kI_k^-I_{k+1}^++u^{-1}q^{-k}\sigma_kY_k^{-1}\otimes\sigma_kI_k^+I_{k+1}^-\\
&\quad=-u^{-1}q^{-k}(1-q^{-2})(q^2\sigma_k+1)\Big((\sigma_k+1-q^{-2})Y_k^{-1}\otimes(I_k^+I_{k+1}^--I_k^-I_{k+1}^+)\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{\mathbf{T}}_k^-(u)\widetilde{\mathbf{T}}_{k+1}^-(u)\circ\sigma_k-\sigma_k\circ\widetilde{\mathbf{T}}_k^-(u)\widetilde{\mathbf{T}}_{k+1}^-(u)\\
&\quad=-uq^{k+1}Y_{k+1}\sigma_k\otimes I_k^+I_{k+1}^-\sigma_k-uq^kY_k\sigma_k\otimes I_k^-I_{k+1}^+\sigma_k\\
&\qquad+uq^{k+1}\sigma_kY_{k+1}\otimes\sigma_kI_k^+I_{k+1}^-+uq^k\sigma_kY_k\otimes\sigma_kI_k^-I_{k+1}^+\\
&\quad=-uq^{k+1}(1-q^{-2})(\sigma_k+q^{-2})\Big((q^2\sigma_k+q^2-1)Y_{k+1}\otimes(I_k^+I_{k+1}^--I_k^-I_{k+1}^+)\Big).
\end{aligned}$$

So the image of $u^{-1}(\widetilde{\mathbf{T}}_k^{\pm}(u)\widetilde{\mathbf{T}}_{k+1}^{\pm}(u)\circ\sigma_k-\sigma_k\circ\widetilde{\mathbf{T}}_k^{\pm}(u)\widetilde{\mathbf{T}}_{k+1}^{\pm}(u))$ (as an element in $\mathrm{End}_{\mathbb{C}(q)}(M\otimes_{\mathbb{C}(q)}V^{\otimes l})$) belongs to the image of σ_k+q^{-2}, which implies that the action defined by (49) induces an action on the quotient (46).

Finally, any homomorphism $f: M_1\longrightarrow M_2$ between two $\mathbf{H}^l_{q^{-1}}$-modules induces a homomorphism $f\otimes\mathrm{id}:\mathcal{D}_A(M_1)\longrightarrow\mathcal{D}_A(M_2)$. □

7.2. **From affine Hecke algebra modules to representations of twisted quantum loop algebras: The type BC case.** Let M be a left $\mathbf{H}^l_{q^{-1},\xi^{-1}}$-module and let V be the vector representation of $\mathfrak{U}_q(\mathfrak{gl}_n)$. Then we know that $M\otimes_{\mathbb{C}(q)}V^{\otimes l}$ is a $\mathtt{B}_l^B$-module. Define

$$\mathcal{D}_B(M)=M\otimes_{\mathbb{C}(q)}V^{\otimes l}/\left(\sum_{i=1}^{l-1}\mathrm{Im}(\sigma_i+q^{-2})+\mathrm{Im}(\sigma_l+\xi^{-2})\right).$$

Let $\mathbf{T}_i^{\pm}(u)$ be the elements defined in (47), (48). Let $\mathbf{G}^{\xi}(u)=\mathrm{Id}\otimes\mathrm{Id}\otimes G^{\xi}(u)\in\mathbf{H}^l_{q^{-1},\xi^{-1}}\otimes_{\mathbb{C}(q)}\mathrm{End}_{\mathbb{C}(q)}(V^{\otimes l})\otimes_{\mathbb{C}(q)}\mathrm{End}_{\mathbb{C}(q)}V[[u^{-1}]]$, where $G^{\xi}(u)$ is the matrix defined in (31). Define

$$\mathbf{S}^{\xi}(u)=\mathbf{T}_1^+(u)\mathbf{T}_2^+(u)\cdots\mathbf{T}_l^+(u)\mathbf{G}^{\xi}(u)(\mathbf{T}_l^-(u^{-1}))^{-1}(\mathbf{T}_{l-1}^-(u^{-1}))^{-1}\cdots(\mathbf{T}_1^-(u^{-1}))^{-1}.$$

Theorem 7.2. *For any $\mathbf{H}^l_{q^{-1},\xi^{-1}}$-module M, the map $S(u)\to\mathbf{S}^{\xi}(u)$ defines a $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$-module structure on $\mathcal{D}_B(M)$. Thus we get a functor $\mathcal{D}_B$ from the category of $\mathbf{H}^l_{q^{-1},\xi^{-1}}$-modules to the category of $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$-modules.*

In order to prove the theorem, we need the following lemmas.

Lemma 7.2. *In $\mathrm{End}_{\mathbb{C}(q)}(M\otimes_{\mathbb{C}(q)}V^{\otimes l})[[u^{-1}]]$, we have*

$$(\mathbf{T}_k^-(u^{-1}))^{-1}=\frac{q^kY_k}{q^kY_k-u^{-1}}\otimes R_{l+1,k}-\frac{u^{-1}}{q^kY_k-u^{-1}}\otimes R_{k,l+1}^{-1}.$$

Proof. This lemma can be proved by direct calculations. □

Lemma 7.3. *Denote by J_k^{ξ} the operator in $\mathrm{End}_{\mathbb{C}(q)}(V^{\otimes l})\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ defined by applying J^{ξ} to the k-th tensor factor in $V^{\otimes l}$. We have the following identities:*

(1) $J_l^{\xi}R_{l,l+1}J_{l+1}^{\xi}R_{l+1,l}=R_{l,l+1}J_{l+1}^{\xi}R_{l+1,l}J_l^{\xi}$;

(2) $(J_l^\xi)^{-1}R_{l+1,l}^{-1}(J_{l+1}^\xi)^{-1}R_{l+1,l} = R_{l,l+1}(J_{l+1}^\xi)^{-1}R_{l,l+1}^{-1}(J_l^\xi)^{-1}$;

(3) $R_{l+1,l}^{-1}J_{l+1}^\xi R_{l+1,l}(J_l^\xi)^{-1} = (J_l^\xi)^{-1}R_{l,l+1}J_{l+1}^\xi R_{l,l+1}^{-1} + (\xi-\xi^{-1})\Phi$ *where*

$$\Phi = R_{l,l+1}J_{l+1}^\xi R_{l,l+1}^{-1} - R_{l+1,l}^{-1}J_{l+1}^\xi R_{l+1,l};$$

(4) $R_{l+1,l}^{-1}(J_{l+1}^\xi)^{-1}R_{l+1,l}(J_l^\xi)^{-1} = (J_l^\xi)^{-1}R_{l,l+1}(J_{l+1}^\xi)^{-1}R_{l,l+1}^{-1} + (\xi-\xi^{-1})\Psi$ *where*

$$\Psi = R_{l,l+1}(J_{l+1}^\xi)^{-1}R_{l,l+1}^{-1} - R_{l+1,l}^{-1}(J_{l+1}^\xi)^{-1}R_{l+1,l};$$

(5) *in* $\mathbf{H}^l_{q^{-1},\xi^{-1}}$, *we have* $Y_l\sigma_l = (1-\xi^{-2})q^{-2l}Y_l^{-1} + q^{-2l}\sigma_l Y_l^{-1}$ *and* $Y_l^{-1}\sigma_l = (\xi^{-2}-1)Y_l^{-1} + q^{2l}\sigma_l Y_l$.

Proof. From [JoMa], lemma 7.4, we know that (1) holds. (2) can be obtained from (1).

Using $J^\xi - (J^\xi)^{-1} = \xi - \xi^{-1}$ and (1), we can get (3). Similarly, we have (4). (5) can be checked directly from the definition. □

Proof of Theorem 7.2. From Theorem 6.3 and Theorem 7.1, we deduce that it is enough to show that the map $S(u) \to \mathbf{S}^\xi(u)$ factors through the quotient by the image of $\sigma_l + \xi^{-2}$. From Lemma 7.2, we have

$$\begin{aligned}\mathbf{S}^\xi(u) = &\left(\frac{q^{-1}Y_1^{-1}}{q^{-1}Y_1^{-1}-u^{-1}} \otimes R_{1,l+1} - \frac{u^{-1}}{q^{-1}Y_1^{-1}-u^{-1}} \otimes R_{l+1,1}^{-1}\right)\\ &\cdots\left(\frac{q^{-l}Y_l^{-1}}{q^{-l}Y_l^{-1}-u^{-1}} \otimes R_{l,l+1} - \frac{u^{-1}}{q^{-l}Y_l^{-1}-u^{-1}} \otimes R_{l+1,l}^{-1}\right)\\ &\cdot G^\xi(u)\cdot\left(\frac{q^lY_l}{q^lY_l-u^{-1}} \otimes R_{l+1,l} - \frac{u^{-1}}{q^lY_l-u^{-1}} \otimes R_{l,l+1}^{-1}\right)\\ &\cdots\left(\frac{qY_1}{qY_1-u^{-1}} \otimes R_{l+1,1} - \frac{u^{-1}}{qY_1-u^{-1}} \otimes R_{1,l+1}^{-1}\right).\end{aligned}$$

From the definition of the σ_l and the fact that all the coefficients of powers of u^{-1} in the expansion of $\prod_{i=1}^l(q^iY_i - u^{-1})(q^{-i}Y_i^{-1}-u^{-1})$ lie in the center of the Hecke algebra $\mathbf{H}^l_{q^{-1},\xi^{-1}}$ by Lemma 2.4, it is enough to show that the coefficients of the powers of u^{-1} in

$$\begin{aligned}(50)\qquad &(q^{-l}Y_l^{-1}\otimes R_{l,l+1} - u^{-1}\otimes R_{l+1,l}^{-1})G^\xi(u)(q^lY_l\otimes R_{l+1,l} - u^{-1}\otimes R_{l,l+1}^{-1})\sigma_l\\ &-\sigma_l(q^{-l}Y_l^{-1}\otimes R_{l,l+1} - u^{-1}\otimes R_{l+1,l}^{-1})G^\xi(u)(q^lY_l\otimes R_{l+1,l} - u^{-1}\otimes R_{l,l+1}^{-1})\end{aligned}$$

belong to the right ideal generated by $\sigma_l + \xi^{-2}$ in $\mathrm{End}_{\mathbb{C}(q)}(M\otimes_{\mathbb{C}(q)} V^{\otimes l})$. This is equivalent to the following congruences modulo this ideal:

(i) $\sigma_l\otimes R_{l,l+1}J_{l+1}^\xi R_{l+1,l}(J_l^\xi)^{-1} \equiv \sigma_l\otimes(J_l^\xi)^{-1}R_{l,l+1}J_{l+1}^\xi R_{l+1,l}$;

(i') $\sigma_l\otimes R_{l+1,l}^{-1}J_{l+1}^\xi R_{l,l+1}^{-1}(J_l^\xi)^{-1} - \sigma_l\otimes R_{l,l+1}(J_{l+1}^\xi)^{-1}R_{l+1,l}(J_l^\xi)^{-1}$
$\equiv \sigma_l\otimes(J_l^\xi)^{-1}R_{l+1,l}^{-1}J_{l+1}^\xi R_{l,l+1}^{-1} - \sigma_l\otimes(J_l^\xi)^{-1}R_{l,l+1}(J_{l+1}^\xi)^{-1}R_{l+1,l}$;

(ii) $\sigma_l\otimes R_{l+1,l}^{-1}(J_{l+1}^\xi)^{-1}R_{l,l+1}^{-1}(J_l^\xi)^{-1} \equiv \sigma_l\otimes(J_l^\xi)^{-1}R_{l+1,l}^{-1}(J_{l+1}^\xi)^{-1}R_{l,l+1}^{-1}$;

(iii) $q^lY_l\sigma_l\otimes R_{l+1,l}^{-1}J_{l+1}^\xi R_{l+1,l}(J_l^\xi)^{-1} + q^{-l}Y_l^{-1}\sigma_l\otimes R_{l,l+1}J_{l+1}^\xi R_{l,l+1}^{-1}(J_l^\xi)^{-1}$
$\equiv q^l\sigma_lY_l\otimes(J_l^\xi)^{-1}R_{l+1,l}^{-1}J_{l+1}^\xi R_{l+1,l} + q^{-l}\sigma_lY_l^{-1}\otimes(J_l^\xi)^{-1}R_{l,l+1}J_{l+1}^\xi R_{l,l+1}^{-1}$;

(iv) $q^lY_l\sigma_l\otimes R_{l+1,l}^{-1}(J_{l+1}^\xi)^{-1}R_{l+1,l}(J_l^\xi)^{-1} + q^{-l}Y_l^{-1}\sigma_l\otimes R_{l,l+1}(J_{l+1}^\xi)^{-1}R_{l,l+1}^{-1}(J_l^\xi)^{-1}$
$\equiv q^l\sigma_lY_l\otimes(J_l^\xi)^{-1}R_{l+1,l}^{-1}(J_{l+1}^\xi)^{-1}R_{l+1,l} + q^{-l}\sigma_lY_l^{-1}\otimes(J_l^\xi)^{-1}R_{l,l+1}(J_{l+1}^\xi)^{-1}R_{l,l+1}^{-1}$.

(i), (i') and (ii) can be easily obtained from Lemma 7.3. For (iii), we have

$$\begin{aligned}&q^lY_l\sigma_l\otimes R_{l+1,l}^{-1}J_{l+1}^\xi R_{l+1,l}(J_l^\xi)^{-1} + q^{-l}Y_l^{-1}\sigma_l\otimes R_{l,l+1}J_{l+1}^\xi R_{l,l+1}^{-1}(J_l^\xi)^{-1}\\ &\quad - q^l\sigma_lY_l\otimes(J_l^\xi)^{-1}R_{l+1,l}^{-1}J_{l+1}^\xi R_{l+1,l} - q^{-l}\sigma_lY_l^{-1}\otimes(J_l^\xi)^{-1}R_{l,l+1}J_{l+1}^\xi R_{l,l+1}^{-1}\qquad\text{(by Lemma 7.3 (3))}\end{aligned}$$

$$= (q^l Y_l \sigma_l - q^{-l}\sigma_l Y_l^{-1}) \otimes (J_l^\xi)^{-1} R_{l,l+1} J_{l+1}^\xi R_{l,l+1}^{-1} + (\xi - \xi^{-1}) q^l Y_l \sigma_l \otimes \Phi$$
$$+ (q^{-l} Y_l^{-1}\sigma_l - q^l \sigma_l Y_l) \otimes (J_l^\xi)^{-1} R_{l+1,l}^{-1} J_{l+1}^\xi R_{l+1,l}$$
(by Lemma 7.3 (5))
$$= q^{-l}(1-\xi^{-2}) Y_l^{-1} \otimes (J_l^\xi)^{-1}\Phi + (\xi - \xi^{-1}) q^l Y_l \sigma_l \otimes \Phi$$
$$= (\sigma_l + \xi^{-2})(\xi - \xi^{-1})\big(q^{-l}\xi^2(\sigma_l Y_l^{-1} \otimes \Phi) + q^{-l}(\xi^2 - 1)(Y_l^{-1} \otimes \Phi)\big).$$

Similarly, for (iv), we have

$$q^l Y_l \sigma_l \otimes R_{l+1,l}^{-1}(J_{l+1}^\xi)^{-1} R_{l+1,l}(J_l^\xi)^{-1} + q^{-l} Y_l^{-1}\sigma_l \otimes R_{l,l+1}(J_{l+1}^\xi)^{-1} R_{l,l+1}^{-1}(J_l^\xi)^{-1}$$
$$- q^l \sigma_l Y_l \otimes (J_l^\xi)^{-1} R_{l+1,l}^{-1}(J_{l+1}^\xi)^{-1} R_{l+1,l} - q^{-l}\sigma_l Y_l^{-1} \otimes (J_l^\xi)^{-1} R_{l,l+1}(J_{l+1}^\xi)^{-1} R_{l,l+1}^{-1} \text{ (by Lemma 7.3 (4))}$$
$$= (q^l Y_l \sigma_l - q^{-l}\sigma_l Y_l^{-1}) \otimes (J_l^\xi)^{-1} R_{l,l+1}(J_{l+1}^\xi)^{-1} R_{l,l+1}^{-1} + q^l(\xi - \xi^{-1}) Y_l \sigma_l \otimes \Psi$$
$$+ (q^{-l} Y_l^{-1}\sigma_l - q^l \sigma_l Y_l) \otimes (J_l^\xi)^{-1} R_{l+1,l}^{-1}(J_{l+1}^\xi)^{-1} R_{l+1,l} \text{ (by Lemma 7.3 (5))}$$
$$= q^{-l}(1-\xi^{-2}) Y_l^{-1} \otimes (J_l^\xi)^{-1}\Psi + q^l(\xi - \xi^{-1}) Y_l \sigma_l \otimes \Psi$$
$$= (\sigma_l + \xi^{-2})(\xi - \xi^{-1})\big(q^{-l}\xi^2(\sigma_l Y_l^{-1} \otimes \Psi) + q^{-l}(\xi^2 - 1)(Y_l^{-1} \otimes \Psi)\big).$$

Therefore, (iii) and (iv) hold. Finally, any homomorphism $f : M_1 \longrightarrow M_2$ between two $\mathbf{H}^l_{q^{-1},q^{2l}}$-modules induces a homomorphism $f \otimes \mathrm{id} : \mathcal{D}_B(M_1) \longrightarrow \mathcal{D}_B(M_2)$. □

8. Center of the twisted quantum loop algebra $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$

The method that we use to determine the center of $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$ is similar to the one used in [MRS] via the Sklyanin determinant.

For any choice of indices i, j, k with $1 \le i < j < k \le n$, the affine quantum R-matrix $R(u,v)$ satisfies the Yang-Baxter equation

$$R_{ij}(u,v)R_{ik}(u,w)R_{jk}(v,w) = R_{jk}(v,w)R_{ik}(u,w)R_{ij}(u,v).$$

From the definition of $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$, we have

$$R_{ji}(u_j,u_i)S_i(u_i)R_{ij}(u_i^{-1},u_j)S_j(u_j) = S_j(u_j)R_{ji}(u_i^{-1},u_j)S_i(u_i)R_{ij}(u_j,u_i).$$

Using these identities, we derive the following relation for $S(u)$:

$$[R_{n,n-1}(u_n,u_{n-1})\cdots R_{21}(u_2,u_1)]S_1(u_1)\big(R_{12}(u_1^{-1},u_2)\cdots R_{1n}(u_1^{-1},u_n)\big)S_2(u_2)\cdots R_{n-1,n}(u_{n-1}^{-1},u_n)S_n(u_n)$$
$$= S_n(u_n)R_{n,n-1}(u_{n-1}^{-1},u_n)\cdots(R_{n1}(u_1^{-1},u_n)\cdots R_{21}(u_1^{-1},u_2))S_1(u_1)[R_{n-1,n}(u_n,u_{n-1})\cdots R_{12}(u_2,u_1)].$$

We set

$$R(u_n,\ldots,u_1) = R_{n,n-1}(u_n,u_{n-1})\Big(R_{n,n-2}(u_n,u_{n-2})R_{n-1,n-2}(u_{n-1},u_{n-2})\Big)\cdots\Big(R_{n1}(u_n,u_1)\cdots R_{21}(u_2,u_1)\Big)$$

and

$$S(u_n,\ldots,u_1) = R_{n-1,n}(u_n,u_{n-1})\Big(R_{n-2,n}(u_n,u_{n-2})R_{n-2,n-1}(u_{n-1},u_{n-2})\Big)\cdots\Big(R_{1n}(u_n,u_1)\cdots R_{12}(u_2,u_1)\Big).$$

As in [MRS], consider the q-permutation operator $P^q \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^n)$ defined by

$$P^q = \sum_{i=1}^n E_{ii} \otimes E_{ii} + q \sum_{n \ge i > j \ge 1} E_{ij} \otimes E_{ji} + q^{-1} \sum_{1 \le i < j \le n} E_{ij} \otimes E_{ji}.$$

An action of the symmetric group $\mathfrak{S}_n$ on the space $(\mathbb{C}^n)^n$ can be defined by setting $s_i \mapsto P^q_{i,i+1}$ for $i = 1, \ldots, n-1$, where s_i denotes the transposition $(i, i+1)$. If $\sigma = s_{i_1}\cdots s_{i_l}$ is a reduced decomposition of an element $\sigma \in \mathfrak{S}_n$, we set $P^q_\sigma = P^q_{s_{i_1}}\cdots P^q_{s_{i_l}}$ where $P^q_{s_i} = P^q_{i,i+1}$. We denote by A^q_n the q-antisymmetrizer

$$A^q_n = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \cdot P^q_\sigma.$$

Proposition 8.1. *The relations*

$$R(q^{2n-2},\ldots,q^2,1)=\left(\prod_{0\le i<j\le n-1}(q^{2j}-q^{2i})\right)\cdot A_n^q$$

and

$$S(q^{2n-2},\ldots,q^2,1)=\left(\prod_{0\le i<j\le n-1}(q^{2j}-q^{2i})\right)\cdot A_n^{q^{-1}}$$

hold in $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)^{\otimes n}$.

Proof. This can be deduced from proposition 4.1 in [MRS]. □

For $1\le i,j\le n$, we set $u_i=uq^{2i-2}$ and

$$R_{ij}=R_{ij}(u_i,u_j);\quad R_{ij}^{\dagger}=R_{ij}(u_i^{-1},u_j);\quad S_{ij}=R_{ij}(u_j,u_i);\quad S_{ij}^{\dagger}=R_{ij}(u_j^{-1},u_i);\quad S_i=S_i(u_i).$$

Now the above proposition implies

$$A_n^qS_1(R_{12}^{\dagger}\cdots R_{1n}^{\dagger})S_2(R_{23}^{\dagger}\cdots R_{2n}^{\dagger})S_3\cdots S_{n-1}R_{n-1,n}^{\dagger}S_n=S_n(S_{n,n-1}^{\dagger})S_{n-1}\cdots S_2(S_{n1}^{\dagger}\cdots S_{21}^{\dagger})S_1A_n^{q^{-1}}. \tag{51}$$

Since the q-antisymmetrizer A_n^q is proportional to an idempotent (indeed $(A_n^q)^2=n!A_n^q$) and maps the space $(\mathbb{C}^n)^{\otimes n}$ into a one dimensional subspace, both sides of (53) must be equal to A_n^q times a series $\mathrm{sdet}\,S(u)$ in u^{-1} with coefficients in $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$, i.e.,

$$A_n^qS_1(R_{12}^{\dagger}\cdots R_{1n}^{\dagger})S_2(R_{23}^{\dagger}\cdots R_{2n}^{\dagger})S_3\cdots S_{n-1}R_{n-1,n}^{\dagger}S_n=A_n^q\mathrm{sdet}\,S(u).$$

We call this series the Sklyanin determinant of $S(u)$.

For $\pi\in\mathfrak{S}_n$, set $w_\pi=e_{\pi(1)}\otimes e_{\pi(2)}\otimes\cdots e_{\pi(n)}$ and let $l(w)$ be the length of the permutation w. We have

$$A_n^qw_\pi=(-q)^{-l(\pi)}A_n^qw_{\mathrm{id}}\ \text{ and }\ A_n^{q^{-1}}w_\pi=(-q)^{l(\pi)}A_n^{q^{-1}}w_{\mathrm{id}}.$$

Hence

$$\begin{aligned}\mathrm{sdet}\,S(u)A_n^qw_\pi=&(-q)^{-l(\pi)}\mathrm{sdet}\,S(u)A_n^qw_{\mathrm{id}}\\=&(-q)^{-l(\pi)}A_n^qS_1(R_{12}^{\dagger}\cdots R_{1n}^{\dagger})S_2(R_{23}^{\dagger}\cdots R_{2n}^{\dagger})S_3\cdots S_{n-1}R_{n-1,n}^{\dagger}S_nw_{\mathrm{id}}\\=&(-q)^{-l(\pi)}S_n(S_{n,n-1}^{\dagger})S_{n-1}\cdots S_2(S_{n1}^{\dagger}\cdots S_{21}^{\dagger})S_1A_n^{q^{-1}}w_{\mathrm{id}}\\=&(-q)^{-2l(\pi)}S_n(S_{n,n-1}^{\dagger})S_{n-1}\cdots S_2(S_{n1}^{\dagger}\cdots S_{21}^{\dagger})S_1A_n^{q^{-1}}w_\pi\\=&q^{-2l(\pi)}A_n^qS_1(R_{12}^{\dagger}\cdots R_{1n}^{\dagger})S_2(R_{23}^{\dagger}\cdots R_{2n}^{\dagger})S_3\cdots S_{n-1}R_{n-1,n}^{\dagger}S_nw_\pi.\end{aligned}$$

The following theorem provides an expression of $\mathrm{sdet}\,S(u)$ in terms of quantum determinants.

Theorem 8.1. *We have*

$$\mathrm{sdet}S(u)=\theta_{n,\xi}(u)\mathrm{qdet}T(uq^{2n-2})(\mathrm{qdet}\overline{T}(u^{-1}))^{-1}, \tag{52}$$

where $\theta_{n,\xi}(u)$ *is the Sklyanin determinant of* $G^\xi(u)$ *and the quantum determinant is defined by*

$$\mathrm{qdet}T(u)=\sum_{\sigma\in\mathfrak{S}_n}(-q)^{-l(\sigma)}t_{\sigma(1)1}(uq^{-2n+2})t_{\sigma(2)2}(uq^{-2n+4})\cdots t_{\sigma(n)n}(u).$$

Proof. We follow the arguments of [MNO]. We regard $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$ as a subalgebra of $\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n)$; see their theorem 6.3. We substitute $S(u)=T(u)G^\xi(u)\overline{T}^{-1}(u^{-1})$ into the identity (53) and transform the left hand side using the relations

$$\overline{T}_i^{-1}(u_i^{-1})R_{ij}(u_i^{-1},u_j)T_j(u_j)=T_j(u_j)R_{ij}(u_i^{-1},u_j)\overline{T}_i^{-1}(u_i^{-1}),$$

which is equivalent to (29). Then the left hand side of (53) becomes

$$A_n^qT_1(u)T_2(uq^2)\cdots T_n(uq^{2n-2})\tilde{R}(u)\overline{T}_1^{-1}(u^{-1})\overline{T}_2^{-1}(u^{-1}q^{-2})\cdots\overline{T}_n^{-1}(u^{-1}q^{-2n+2}), \tag{53}$$

where

$$\tilde{R}(u) = G_1^{\xi}(u)R_{12}^{\dagger}\cdots R_{1n}^{\dagger}G_2^{\xi}(uq^2)\cdots G_{n-1}^{\xi}(uq^{2n-4})R_{n-1,n}^{\dagger}G_n^{\xi}(uq^{2n-2}).$$

By the definition of the quantum determinant $\mathrm{qdet}\,T(u)$, we have

$$A_n^q T_1(u)T_2(uq^2)\cdots T_n(uq^{2n-2}) = A_n^q \mathrm{qdet}T(q^{2n-2}u). \tag{54}$$

Therefore, we can bring (55) to the form

$$\mathrm{qdet}T(q^{2n-2}u)A_n^q\tilde{R}(u)\overline{T}_1^{-1}(u^{-1})\overline{T}_2^{-1}(u^{-1}q^{-2})\cdots\overline{T}_n^{-1}(u^{-1}q^{-2n+2}).$$

By Lemma 6.2, the mapping $S(u)\mapsto G^{\xi}(u)$ defines a representation of the twisted quantum loop algebra $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$. Therefore, (53) gives

$$A_n^q\tilde{R}(u) = \tilde{S}(u)A_n^{q^{-1}} = A_n^q\mathrm{sdet}\,G^{\xi}(u) = A_n^q\theta_{n,\xi}(u),$$

where

$$\tilde{S}(u) = G_n^{\xi}(uq^{2n-2})(S_{n,n-1}^{\dagger})G_{n-1}^{\xi}(uq^{2n-4})\cdots G_2^{\xi}(uq^2)(S_{n1}^{\dagger}\cdots S_{21}^{\dagger})G_1^{\xi}(u).$$

Now we write (55) as

$$\mathrm{qdet}\,T(q^{2n-2}u)\tilde{S}(u)\Big(A_n^{q^{-1}}\overline{T}_1^{-1}(u^{-1})\overline{T}_2^{-1}(u^{-1}q^{-2})\cdots\overline{T}_n^{-1}(u^{-1}q^{-2n+2})\Big).$$

Furthermore, we have

$$A_n^{q^{-1}}\overline{T}_1^{-1}(u^{-1})\overline{T}_2^{-1}(u^{-1}q^{-2})\cdots\overline{T}_n^{-1}(u^{-1}q^{-2n+2}) = A_n^{q^{-1}}(\mathrm{qdet}\overline{T}(u^{-1}))^{-1}. \tag{55}$$

This follows from (56) (with $\overline{T}$ instead of T) if we multiply both sides by $\overline{T}_n^{-1}(uq^{2n-2})\overline{T}_{n-1}^{-1}(uq^{2n-4})\cdots\overline{T}_1^{-1}(u)$ from the right, replace u with $u^{-1}q^{-2n+2}$ and then conjugate the two sides by the permutation of the indices $1,\ldots,n$ which sends i to $n-i+1$. (Notice that A_n^q becomes $A_n^{q^{-1}}$ after the conjugation). Now (55) becomes

$$\mathrm{qdet}T(q^{2n-2}u)\Big(\tilde{S}(u)A_n^{q^{-1}}\Big)(\mathrm{qdet}\overline{T}(u^{-1}))^{-1} = \mathrm{qdet}T(q^{2n-2}u)(\mathrm{qdet}\overline{T}(u^{-1}))^{-1}A_n^q\theta_{n,\xi}(u).$$

□

Corollary 8.1. *The coefficients of the series* $\mathrm{sdet}\,S(u)$ *belong to the center of the algebra* $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$.

Proof. This is an immediate consequence of the previous theorem and of the centrality of $\mathrm{qdet}\,T(u)$ and $\mathrm{qdet}\,\bar{T}(u)$ in $\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n)$. □

Introduce the series $c(u)$ and the elements c_k of the center of the algebra $\mathfrak{U}_q^p(\mathcal{L}\mathfrak{gl}_n)$ by the formula

$$c(u) = \theta_{n,\xi}(u)^{-1}\mathrm{sdet}\,S(u) = 1 + \sum_{k=1}^{\infty}c_ku^{-k}. \tag{56}$$

Proposition 8.2. *The coefficients* c_k, $k\geq 1$, *are algebraically independent.*

Proof. Use an argument similar to the one in proposition 4.4 in [MRS]. □

Now we try to find an explicit expression for the scalar function $\theta_{n,\xi}(u)$. We have $(n-1)!A_n^q = A_n^q\tilde{A}_{n-1}^q$, where $\tilde{A}_{n-1}^q$ is the quantum antisymmetrizer in the tensor product of the copies of $\mathrm{End}_{\mathbb{C}}\mathbb{C}^n$ corresponding to the indices $2,\ldots,n$. Note that $\tilde{A}_{n-1}^q$ commutes with $G_1^{\xi}(u)$. Furthermore, we have the identity

$$\tilde{A}_{n-1}^qR_{12}^{\dagger}\cdots R_{1n}^{\dagger} = R_{1n}^{\dagger}\cdots R_{12}^{\dagger}\tilde{A}_{n-1}^q.$$

This follows from the Yang-Baxter relation and Proposition 8.1. Similarly we define the operators $\tilde{A}_i^q$ for $i = 1, 2, \ldots, n-2$. Now we have:

$$\begin{aligned}\left(\prod_{i=1}^{n-1} i!\right) A_n^q \theta_{n,\xi}(u) &= \left(\prod_{i=1}^{n-2} i!\right) A_n^q \tilde{A}_{n-1}^q G_1^\xi(u) R_{12}^\dagger \cdots R_{1n}^\dagger G_2^\xi(uq^2) \cdots G_{n-1}^\xi(uq^{2n-4}) R_{n-1,n}^\dagger G_n^\xi(uq^{2n-2}) \\ &= \left(\prod_{i=1}^{n-2} i!\right) A_n^q G_1^\xi(u) R_{1n}^\dagger \cdots R_{12}^\dagger \Big(\tilde{A}_{n-1}^q G_2^\xi(uq^2) \cdots G_{n-1}^\xi(uq^{2n-4}) R_{n-1,n}^\dagger G_n^\xi(uq^{2n-2}) \Big) \\ &= \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ &= A_n^q G_1^\xi(u) \cdots \Big(\tilde{A}_{n-1}^q G_2^\xi(uq^2) \cdots \Big(\tilde{A}_2^q G_{n-1}^\xi(uq^{2n-4}) R_{n-1,n}^\dagger \big(\tilde{A}_1^q G_n^\xi(uq^{2n-2}) \big) \Big) \Big).\end{aligned}$$

Now consider the action of the operators on $w = e_{p+1} \otimes \cdots \otimes e_{n-p} \otimes (e_p \otimes e_{n-p+1}) \otimes \cdots \otimes (e_1 \otimes e_n)$, where $\{e_i\}_{i=1}^n$ denotes the canonical basis of $\mathbb{C}^n$. We obtain

$$\begin{aligned} q^{2p(n-p-1)} \left(\prod_{i=1}^{n-1} i!\right) A_n^q \theta_{n,\xi}(u) w &= \Big(A_n^q G_1^\xi(u) \cdots \Big(\tilde{A}_{n-1}^q G_2^\xi(uq^2) \cdots \Big(\tilde{A}_2^q G_{n-1}^\xi(uq^{2n-4}) R_{n-1,n}^\dagger \big(\tilde{A}_1^q G_n^\xi(uq^{2n-2}) \big) \Big) \Big) \Big) w \\ &= \Big(A_n^q G_1^\xi(u) \cdots \Big(\tilde{A}_{n-1}^q G_2^\xi(uq^2) \cdots \Big(\tilde{A}_2^q \tilde{A}_1^q G_{n-1}^\xi(uq^{2n-4}) R_{n-1,n}^\dagger G_n^\xi(uq^{2n-2}) \Big) \Big) \Big) w \\ &= A_n^q G_1^\xi(u) \cdots \Big(\tilde{A}_4^q G_{n-3}^\xi(uq^{2n-8}) \cdots \Big(\tilde{A}_3^q G_{n-2}^\xi(uq^{2n-6}) \cdots \big(\tilde{A}_2^q \theta'_{2,\xi}(u) w \big) \Big) \Big) \\ &= \overline{\theta}_{2,\xi}(u) \Big(A_n^q G_1^\xi(u) \cdots \Big(\tilde{A}_4^q \tilde{A}_3^q \tilde{A}_2^q G_{n-3}^\xi(uq^{2n-8}) \cdots G_{n-2}^\xi(uq^{2n-6}) \cdots \Big) w \Big) \\ &= 2!3! \theta'_{2,\xi}(u) \Big(A_n^q G_1^\xi(u) \cdots \Big(\tilde{A}_4^q G_{n-3}^\xi(uq^{2n-8}) \cdots G_{n-2}^\xi(uq^{2n-6}) \cdots \Big) w \Big) \\ &= \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ &= \left(\prod_{i=1}^{2p} i!\right) \left(\prod_{j=1}^{p} \overline{\theta}_{2j,\xi}(u)\right) \Big(A_n^q G_1^\xi(u) \cdots \Big(\tilde{A}_{2p+1}^q G_{n-2p}^\xi(uq^{2n-4p-2}) \cdots \Big) w \Big) \\ &= \left(\prod_{i=1}^{2p} i!\right) \left(\prod_{j=1}^{p} \overline{\theta}_{2j,\xi}(u)\right) \Big(A_n^q G_1^\xi(u) \cdots \tilde{A}_{2p+1}^q \theta'_{2p+1,\xi}(u) w \Big) \\ &= \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ &= \left(\prod_{i=1}^{n-1} i!\right) \left(\prod_{j=1}^{p} \overline{\theta}_{2j,\xi}(u)\right) \left(\prod_{k=2p+1}^{n} \theta'_{k,\xi}(u)\right) A_n^q w. \end{aligned}$$

This implies that

$$\theta_{n,\xi}(u) = \left(\prod_{i=1}^{p} \overline{\theta}_{2i,\xi}(u)\right) \left(\prod_{j=2p+1}^{n} \theta'_{j,\xi}(u)\right),$$

where $\overline{\theta}_{2i,\xi}(u)$ satisfies

$$\tilde{A}_{2i}^q \overline{\theta}_{2i,\xi}(u) w = \tilde{A}_{2i}^q G_{n-2i+1}^\xi(uq^{2n-4i}) R_{n-2i+1,n-2i+2}^\dagger \cdots G_{n-2i+2}^\xi(uq^{2n-4i+2}) R_{n-2i+2,n-2i+3}^\dagger \cdots R_{n-2i+2,n}^\dagger w$$

for $1 \le i \le p$, and $\theta'_{j,\xi}(u)$ satisfies

$$\tilde{A}_j^q \theta'_{j,\xi}(u) w_p = \tilde{A}_j^q G_{n-j+1}^\xi(uq^{2n-2j}) R_{n-j+1,n-j+2}^\dagger \cdots R_{n-j+1,n}^\dagger w_p$$

for $2p+1 \le j \le n$. It suffices to find $\overline{\theta}_{2,\xi}(u), \overline{\theta}_{4,\xi}(u), \ldots, \overline{\theta}_{2p,\xi}(u), \theta'_{2p+1,\xi}(u), \theta'_{2p+2,\xi}(u), \ldots, \theta'_{n,\xi}(u)$. We will give details of one case below and simply state the formulae for the others because computations are quite long and tedious.

If $p=0$ and $1\le j\le n$, we have

$$\theta'_{j,\xi}(u)=\frac{\xi u_{n-j+1}^{-1}-\xi^{-1}u_{n-j+1}}{u_{n-j+1}-u_{n-j+1}^{-1}}\cdot\left(\prod_{k=1}^{j-1}(u_{n-j+1}^{-1}-u_{n-k})\right)$$

and

$$\theta_{n,\xi}(u)=\prod_{j=1}^{n}\theta'_{j,\xi}(u)=\prod_{j=1}^{n}\left(\frac{\xi u_{n-j+1}^{-1}-\xi^{-1}u_{n-j+1}}{u_{n-j+1}-u_{n-j+1}^{-1}}\cdot\left(\prod_{k=1}^{j-1}(u_{n-j+1}^{-1}-u_{n-k})\right)\right).$$

If $p\ge 1$, then

$$\bar\theta_{2,\xi}(u)=u_n-u_n^{-1}-\frac{(\xi-\xi^{-1})^2}{u_n-u_n^{-1}}.$$

We also need to consider the tensor product

$$w_i=e_{p+1}\otimes\cdots\otimes e_{n-p}\otimes(e_p\otimes e_{n-p+1})\otimes\cdots\otimes(e_i\otimes e_{n-i+1})\otimes e_1\otimes\cdots e_{i-1}\otimes e_{n-i+2}\otimes\cdots\otimes e_n.$$

Then

$$\begin{aligned}(2i-2)!\tilde A^q_{2i}\bar\theta_{2i,\xi}(u)w_i&=(-q)^{(i-1)(i-2)}\tilde A^q_{2i}\bar\theta_{2i,\xi}(u)w\\&=(-q)^{(i-1)(i-2)}\tilde A^q_{2i}G^\xi_{n-2i+1}(uq^{2n-4i})R^\dagger_{n-2i+1,n}\cdots G^\xi_{n-2i+2}(uq^{2n-4i+2})R^\dagger_{n-2i+2,n}\cdots R^\dagger_{n-2i+2,n-2i+3}\tilde A^q_{2i-2}w\\&=(2i-2)!\tilde A^q_{2i}G^\xi_{n-2i+1}(uq^{2n-4i})R^\dagger_{n-2i+1,n-2i+2}\cdots G^\xi_{n-2i+2}(uq^{2n-4i+2})R^\dagger_{n-2i+2,n-2i+3}\cdots R^\dagger_{n-2i+2,n}w_i,\end{aligned}$$

i.e.,

$$\tilde A^q_{2i}\bar\theta_{2i,\xi}(u)w_i=\tilde A^q_{2i}G^\xi_{n-2i+1}(uq^{2n-4i})R^\dagger_{n-2i+1,n-2i+2}\cdots G^\xi_{n-2i+2}(uq^{2n-4i+2})R^\dagger_{n-2i+2,n-2i+3}\cdots R^\dagger_{n-2i+2,n}w_i.$$

Now assume that $2\le i\le p$. By the property of $\tilde A^q_{2i-1}$, we have

$$\tilde A^q_{2i-1}G^\xi_{n-2i+2}(uq^{2n-4i+2})R^\dagger_{n-2i+2,n-2i+3}\cdots R^\dagger_{n-2i+2,n}w_i=A_i\cdot\left(\tilde A^q_{2i-1}(w_i)\right)+B_i\cdot\left(\tilde A^q_{2i-1}(w_i')\right),$$

where

$$w_i'=e_{p+1}\otimes\cdots\otimes e_{n-p}\otimes(e_p\otimes e_{n-p+1})\otimes\cdots\otimes(e_i\otimes e_i)\otimes e_1\otimes\cdots e_{i-1}\otimes e_{n-i+2}\otimes\cdots\otimes e_n,$$

$$B_i=\prod_{k=0}^{2i-3}(u_{n-2i+2}^{-1}-u_{n-k})\text{ and }A_i=-\left(\prod_{k=1}^{2i-3}(u_{n-2i+2}^{-1}-u_{n-k})\right)u_{n-2i+2}^{-1}(\xi-\xi^{-1}).$$

Let

$$\overline{w}_i=e_{p+1}\otimes\cdots\otimes e_{n-p}\otimes(e_p\otimes e_{n-p+1})\otimes\cdots\otimes e_i\otimes e_1\otimes\cdots e_{i-1}\otimes e_{n-i+1}\otimes e_{n-i+2}\otimes\cdots\otimes e_n$$

and

$$\tilde w_i=e_{p+1}\otimes\cdots\otimes e_{n-p}\otimes(e_p\otimes e_{n-p+1})\otimes\cdots\otimes(e_{i+1}\otimes e_{n-i})\otimes e_1\otimes\cdots e_i\otimes e_{n-i+1}\otimes\cdots\otimes e_n.$$

Now we have

$$\begin{aligned}(2i-1)!\tilde A^q_{2i}\bar\theta_{2i,\xi}(u)w_i=&\tilde A^q_{2i}G^\xi_{n-2i+1}(uq^{2n-4i})R^\dagger_{n-2i+1,n}\cdots R^\dagger_{n-2i+1,n-2i+2}\Big(A_i\cdot\tilde A^q_{2i-1}(w_i)+B_i\cdot\tilde A^q_{2i-1}(w_i')\Big)\\=&(-q)^{1-i}(2i-1)!A_i\cdot\tilde A^q_{2i}G^\xi_{n-2i+1}(uq^{2n-4i})R^\dagger_{n-2i+1,n-2i+2}\cdots R^\dagger_{n-2i+1,n}\overline{w}_i\\&+(2i-1)!B_i\cdot\tilde A^q_{2i}G^\xi_{n-2i+1}(uq^{2n-4i})R^\dagger_{n-2i+1,n-2i+2}\cdots R^\dagger_{n-2i+1,n}w_i'\\=&(2i-1)!\Big((-q)^{1-i}A_i\cdot C_i+B_i\cdot D_i\Big)\tilde A^q_{2i}\tilde w_i,\end{aligned}$$

where

$$C_i\tilde A^q_{2i}\tilde w_i=\tilde A^q_{2i}G^\xi_{n-2i+1}(uq^{2n-4i})R^\dagger_{n-2i+1,n-2i+2}\cdots R^\dagger_{n-2i+1,n}\overline{w}_i$$

and

$$D_i\tilde A^q_{2i}\tilde w_i=\tilde A^q_{2i}G^\xi_{n-2i+1}(uq^{2n-4i})R^\dagger_{n-2i+1,n-2i+2}\cdots R^\dagger_{n-2i+1,n}w_i'.$$

Rather long computations lead to

$$C_i=(-q)^{1-i}u_n(\xi^{-1}-\xi)\left(\prod_{k=1}^{2i-2}(u_{n-2i+1}^{-1}-u_{n-k})\right)$$

and

$$D_i = (u_{n-i+1} - u_{n-i+1}^{-1})\Big(\prod_{k=1}^{2i-2}(u_{n-2i+1}^{-1} - u_{n-k})\Big).$$

Therefore

$$\begin{aligned}\bar{\theta}_{2i,\xi}(u) =& q^{2i-2}\Big((-q)^{1-i}A_i \cdot C_i + B_i \cdot D_i\Big)\\ =& \Bigg(\prod_{r=2}^{3}\prod_{k=3-r}^{2i-r}(u_{n-2i+r-1}^{-1} - u_{n-k})\Bigg) \cdot \Bigg(\Big(u_n - u_{n-2i+2}^{-1}\Big) - \frac{q^{4i-4}(\xi-\xi^{-1})^2}{u_{n-2i+2}^{-1} - u_n}\Bigg).\end{aligned}$$

For $2p+1 \le j \le n$, we replace w by

$$\overline{w}_j = e_{p+1} \otimes \cdots \otimes e_{n-j+p+1} \otimes e_1 \otimes \cdots e_p \otimes e_{n-j+p+2} \otimes \cdots \otimes e_n$$

and let $\widetilde{w}_j$ be

$$\widetilde{w}_j = e_{p+1} \otimes \cdots \otimes e_{n-j+p} \otimes e_1 \otimes \cdots e_p \otimes e_{n-j+p+1} \otimes \cdots \otimes e_n.$$

We have

$$(-q)^{-p}\tilde{A}_j^q \theta'_{j,\xi}(u)\widetilde{w}_j = \tilde{A}_j^q \theta'_{j,\xi}(u)\overline{w}_j = \tilde{A}_j^q G_{n-j+1}^{\xi}(uq^{2n-2j})R_{n-j+1,n-j+2}^{\dagger} \cdots R_{n-j+1,n}^{\dagger}\overline{w}_j.$$

Set

$$J^{(r)} = \{r+1, r+2, \dots, j-2\}, \quad J_1^{(r)} = \{r+1, r+2, \dots, p-1\}, \quad J_2^{(r)} = \{p, p+1, \dots, j-2\},$$

$$J_3^{(r)} = \{j-p-1, j-p, \dots, j-2\}, \quad J_4^{(r)} = \{r+1, r+2, \dots, j-p-2\}$$

and

$$F_j(1) = (-q)^{-p}\left(\prod_{k=0}^{j-2}(u_{n-j+1}^{-1} - u_{n-k})\right) \cdot \frac{u_{n-j+1}^{-1}\xi - u_{n-j+1}\xi^{-1}}{u_{n-j+1} - u_{n-j+1}^{-1}},$$

$$\begin{aligned}F_j(2) =& \sum_{r=0}^{p-1}\left(\prod_{k=0}^{r-1}(u_{n-j+1}^{-1} - u_{n-k})\right)\Big(-u_{n-r}(q-q^{-1})\Big)\sum_{I \subseteq J_1^{(r)}}\Bigg(\Big(-u_{n-j+1}^{-1}(q-q^{-1})\Big)^{|I|} \cdot \\ & \Bigg(\prod_{k \in (J_1^{(r)} \setminus I) \cup J_2^{(r)}}(u_{n-j+1}^{-1} - u_{n-k})\Bigg)(-q^{-1})^{2(j-r)-p-3-|I|}\Bigg)\\ =& (-q)^p(q^{2p}-1)u_{n-j+1}\left(\prod_{k=1}^{j-2}(u_{n-j+1}^{-1} - u_{n-k})\right).\end{aligned}$$

Let's explain from where this formula for $F_j(2)$ comes, the other ones being obtained via similar considerations. The value of r indicates the first index in $R_{n-j+1,n-r}^{\dagger}$ where we consider the operator $\sum_{\substack{i,j=1\\i>j}}^n E_{ij} \otimes E_{ji}$. This explains the product $\Big(\prod_{k=0}^{r-1}(u_{n-j+1}^{-1} - u_{n-k})\Big)\Big(-u_{n-r}(q-q^{-1})\Big)$ since for $k < r$ only the operator $\sum_{i,j=1}^n E_{ii} \otimes E_{jj}$ in $R_{n-j+1,n-k}^{\dagger}$ is applied. The index set I indices the factors $R_{n-j+1,n-k}^{\dagger}$ where the operator $\sum_{\substack{i,j=1\\i<j}}^n E_{ij} \otimes E_{ji}$ is applied, so if $k \notin I$, then instead it is the operators $\sum_{i,j=1}^n q^{\delta_{ij}} E_{ii} \otimes E_{jj}$ and $\sum_{i,j=1}^n q^{-\delta_{ij}} E_{ii} \otimes E_{jj}$ which are applied. The factor $(-q^{-1})^{2(j-r)-p-3-|I|}$ comes by applying $\widetilde{A}_j^q$ to the resulting tensor to bring it back to $\overline{w}_j$ and by counting the number of inversions.

$$F_j(3) = \sum_{r=0}^{j-p-2}\left(\prod_{k=0}^{r-1}(u_{n-j+1}^{-1}-u_{n-k})\right)\Big(-u_{n-r}(q-q^{-1})\Big)\sum_{I\subseteq J_4^{(r)}}\left(\Big(-u_{n-j+1}^{-1}(q-q^{-1})\Big)^{|I|}\cdot\right.$$
$$\left.\left(\prod_{k\in J_4^{(r)}\setminus I\cup J_3^{(r)}}(u_{n-j+1}^{-1}-u_{n-k})\right)(-q^{-1})^{2j-2r-p-3-|I|}\right)$$
$$=(-q)^p(q^{2j-2p-2}-1)u_{n-j+1}\left(\prod_{k=1}^{j-2}(u_{n-j+1}^{-1}-u_{n-k})\right),$$

$$F_j(4) = \sum_{r=0}^{j-p-2}\left(\prod_{k=0}^{r-1}(u_{n-j+1}^{-1}-u_{n-k})\right)\Big(-u_{n-r}(q-q^{-1})\Big)\sum_{I\subseteq J^{(r)}}\left(\Big(-u_{n-j+1}^{-1}(q-q^{-1})\Big)^{|I|}\cdot\right.$$
$$\left.\left(\prod_{k\in J^{(r)}\setminus I}(u_{n-j+1}^{-1}-u_{n-k})\right)(-q^{-1})^{2j-2r-p-3-|I|}\right)$$
$$=\frac{(-q)^p(1-q^{2+2p-2j})u_n\left(\prod_{k=1}^{j-1}(u_{n-j+1}^{-1}-u_{n-k})\right)}{u_{n-j+1}^{-1}-u_{n-j+p+1}},$$

$$F_j(5) = \sum_{r=j-p-1}^{j-2}\left(\prod_{k=0}^{r-1}(u_{n-j+1}^{-1}-u_{n-k})\right)\Big(-u_{n-j+1}^{-1}(q-q^{-1})\Big)\sum_{I\subseteq J^{(r)}}\left(\Big(-u_{n-j+1}^{-1}(q-q^{-1})\Big)^{|I|}\cdot\right.$$
$$\left.\left(\prod_{k\in J^{(r)}\setminus I}(u_{n-j+1}^{-1}-u_{n-k})\right)(-q^{-1})^{p-1-|I|}\right)$$
$$=\frac{(-q)^{-p}(q^{2p}-1)u_{n-j+1}^{-1}\left(\prod_{k=0}^{j-2}(u_{n-j+1}^{-1}-u_{n-k})\right)}{u_{n-j+1}^{-1}-u_{n-j+p+1}}.$$

Using these, we can compute

$$(-q)^{-p}\theta'_{j,\xi}(u) =F_i(1)+F_j(2)\frac{u_{n-2i+1}^{-1}(\xi-\xi^{-1})}{u_{n-2i+1}-u_{n-2i+1}^{-1}}+(F_j(3)-F_j(2))\frac{u_{n-j+1}^{-1}\xi-u_{n-j+1}\xi^{-1}}{u_{n-j+1}-u_{n-j+1}^{-1}}$$
$$+(F_j(4)-F_j(3))\frac{u_{n-2i+1}(\xi-\xi^{-1})}{u_{n-2i+1}-u_{n-2i+1}^{-1}}+F_j(5)\frac{u_{n-2i+1}(\xi-\xi^{-1})}{u_{n-2i+1}-u_{n-2i+1}^{-1}}$$
$$=F_i(1)+F_j(2)\xi^{-1}-F_j(3)\xi+(F_j(4)+F_j(5))\frac{u_{n-2i+1}(\xi-\xi^{-1})}{u_{n-2i+1}-u_{n-2i+1}^{-1}}$$
$$=\Big((-q)^pu_{n-j+p+1}\xi^{-1}-(-q)^{-p}u_{n-j+1}^{-1}\xi\Big)\left(\prod_{k=1}^{j-2}(u_{n-j+1}^{-1}-u_{n-k})\right).$$

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