Quantum algebras and quivers

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Abstract

Given a finite quiver Q without loops, we introduce a new class of quantum algebras $\mathsf{D}(Q)$ which are deformations of the enveloping algebra of a Lie algebra which is a central extension of $\mathfrak{sl}_n(\Pi(Q))$ where $\Pi(Q)$ is the preprojective algebra of Q. When Q is an affine Dynkin quiver of type A, D or E, we can relate them to Γ-deformed double current algebras. We are able to construct functors between different categories of modules over $\mathsf{D}(Q)$. We also give some general results about $\widehat{\mathfrak{sl}}_n(A)$ for a quadratic algebra A and about $\widehat{\mathfrak{g}}(\mathbb{C}[u,v])$, which we use to introduce deformed double current algebras associated to a simple Lie algebra $\mathfrak{g}$.

1 Introduction

Quivers have been studied for a long time and the discovery of a geometric link between quiver varieties and Kac-Moody algebras by H. Nakajima [Na] around fifteen years ago rekindled the interest of representation theorists for that subject. To a quiver, one can associate its preprojective algebra $\Pi(Q)$ and its deformed versions $\Pi^{\lambda}(Q)$, whose representation theory is related to the geometry of a certain moment map. In [CBHo], W. Crawley-Boevey and M. Holland were able to connect deformed preprojective algebras of affine Dynkin quivers (of type A,D,E) to certain non-commutative deformations of Kleinian singularities. The theory of symplectic reflection algebras introduced by P. Etingof and V. Ginzburg in [EtGi] is a generalization of the Crawley-Boevey-Holland theory of non-commutative deformations. In [GaGi], symplectic reflection algebras for wreath products of finite subgroups Γ of $SL_2(\mathbb{C})$ were shown to be Morita equivalent to a new family of algebras $\Pi_l^{\lambda,\nu}(Q)$ which can be seen as deformed preprojective algebras for wreath products $S_l \wr \Gamma$; they are also called Gan-Ginzburg algebras in the literature. (We denote by S_l the symmetric group on l letters.)

In [Gu3], we introduced the quantum algebra analogs of symplectic reflection algebras for wreath products $S_l \wr \Gamma$, which we called Γ-deformed double current algebras (Γ-DDCA). In this paper, we want to construct the quantum analogs $\mathsf{D}_n^{\lambda,\nu}(Q)$ of the deformed preprojective algebras for wreath products. We start by proving general results for the Lie algebra $\mathfrak{sl}_n(A)$ when A is a quadratic algebra. We also extend some results from [Gu2] about $\mathfrak{sl}_n(\mathbb{C}[u,v])$ to any semisimple Lie algebra $\mathfrak{g}$ (of rank ≥ 3), we apply this to suggest a definition of deformed double current algebras for $\mathfrak{g}$ and justify why they are, conjecturally, limit forms of affine Yangians.

Afterwards, we define the deformed enveloping quiver algebras $\mathsf{D}_n^{\lambda,\nu}(Q)$ and explain how they are related to Gan-Ginzburg algebras via a functor of Schur-Weyl type. We are able to generalize to $\mathsf{D}_n^{\lambda,\nu}(Q)$ some of the main results of [CBHo, Ga, GaGi]. When the graph underlying Q is an affine Dynkin diagram of type A, D or E corresponding to a finite subgroup Γ of $SL_2(\mathbb{C})$ via the McKay correspondence, we connect a certain subalgebra of the Γ-DDCA $\mathsf{D}_n^{\beta,\mathbf{b}}(\Gamma)$ to a quotient of $\mathsf{D}_n^{\lambda,\nu}(Q)$. Furthermore, we are able to construct functors between categories of modules over $\mathsf{D}_n^{\lambda,\nu}(Q)$ for values of the deformation parameters which differ by a reflection of the Weyl group associated to Q. For $\Pi_l^{\lambda,\nu}(Q)$, this was achieved in [Ga], generalizing the results of [CBHo] for $\Pi^{\lambda}(Q)$. The Schur-Weyl functor intertwines the reflection functors for deformed enveloping quiver algebras and for Gan-Ginzburg algebras when $l+1 \leq n$.

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3 Universal central extensions of type A for quadratic algebras

We are interested in $\mathfrak{sl}_n(A)$ when A is a quadratic algebra as defined in [BrGa]. Let V be a finite dimensional bimodule over a semisimple finite dimensional $\mathbb{C}$-algebra B. Let $R_j \in V \otimes_B V, \alpha_j \in V, \beta_j \in B$ and set $P_j = R_j + \alpha_j + \beta_j \in V \otimes_B V \oplus V \oplus B$, with $j \in J$, J being some indexing set.

Definition 3.1. *A quadratic algebra A is an algebra of the form $A = T_B V/(P_j)_{j\in J}$ where $T_B V = \oplus_{k\geq 0} V \otimes_B V \otimes_B \cdots \otimes_B V$ (k times) is the tensor algebra and $(P_j)_{j\in J}$ is the ideal generated by the elements P_j. The algebra A is said to be homogeneous if $\alpha_j = 0 = \beta_j \, \forall j \in J$.*

Let $\pi_2, \pi_1, \pi_0 : V \otimes_B V \oplus V \oplus B \twoheadrightarrow V \otimes_B V, V, B$ be the projection maps. The Lie algebra $\mathfrak{sl}_n(A)$ is defined as the derived Lie algebra of $\mathfrak{gl}_n(A)$, that is, $\mathfrak{sl}_n(A) = [\mathfrak{gl}_n(A), \mathfrak{gl}_n(A)]$. Since it is a perfect Lie algebra, it possesses a universal central extension. The following theorem will be essential later in this section.

Theorem 3.1 ([KaLo])**.** *Let A be an associative algebra over $\mathbb{C}$ (not necessarily quadratic), $n \geq 3$. The universal central extension $\widehat{\mathfrak{sl}}_n(A)$ of $\mathfrak{sl}_n(A)$ is the Lie algebra generated by elements $E_{ab}(p), 1 \leq a \neq b \leq n, p \in A$, satisfying the following relations:*

$$E_{ab}(t_1 p_1 + t_2 p_2) = t_1 E_{ab}(p_1) + t_2 E_{ab}(p_2) \quad \forall\, t_1, t_2 \in \mathbb{C}, p_1, p_2 \in A \tag{1}$$

$$[E_{ab}(p_1), E_{bc}(p_2)] = E_{ac}(p_1 p_2) \text{ if } a \neq b \neq c \neq a \tag{2}$$

$$[E_{ab}(p_1), E_{cd}(p_2)] = 0 \text{ if } a \neq b \neq c \neq d \neq a \tag{3}$$

The main result of this section is the following proposition.

Proposition 3.1. *Suppose that $n \geq 5$ and let A be a quadratic algebra as in definition 3.1. The universal central extension $\widehat{\mathfrak{sl}}_n(A)$ of $\mathfrak{sl}_n(A)$ is the Lie algebra generated by the elements $\mathsf{E}_{ab}(v), \mathsf{E}_{ab}(e)$ for $1 \leq a \neq b \leq n, v \in V, e \in B$ satisfying $\mathsf{E}_{ab}(t_1 v_1 + t_2 v_2) = t_1 \mathsf{E}_{ab}(v_1) + t_2 \mathsf{E}_{ab}(v_2), \mathsf{E}_{ab}(t_1 e_1 + t_2 e_2) = t_1 \mathsf{E}_{ab}(e_1) + t_2 \mathsf{E}_{ab}(e_2)$ and the following relations:*

If $a \neq b \neq c \neq a \neq d \neq c$,

$$[\mathsf{E}_{ab}(v_1), \mathsf{E}_{bc}(v_2)] = [\mathsf{E}_{ad}(v_1), \mathsf{E}_{dc}(v_2)],\ [\mathsf{E}_{ab}(e), \mathsf{E}_{bc}(v)] = \mathsf{E}_{ac}(ev), [\mathsf{E}_{ab}(e_1), \mathsf{E}_{bc}(e_2)] = \mathsf{E}_{ac}(e_1 e_2) \tag{4}$$

$$\text{and } [\mathsf{E}_{ab}(v), \mathsf{E}_{bc}(e)] = \mathsf{E}_{ac}(ve) \text{ for } v, v_1, v_2 \in V, e \in B. \tag{5}$$

If a, b, c are all distinct, $P = R + \alpha + \beta$, $\alpha = \pi_1(P), \beta = \pi_0(P)$ and $R = \pi_2(P) = \sum_k v_k \otimes \widetilde{v}_k \in V \otimes_B V$ for some $v_k, \widetilde{v}_k \in V$, then

$$\sum_k [\mathsf{E}_{ab}(v_k), \mathsf{E}_{bc}(\widetilde{v}_k)] = -\mathsf{E}_{ac}(\alpha) - \mathsf{E}_{ac}(\beta). \tag{6}$$

If $a \neq b \neq c \neq d \neq a$ and $v_1, v_2 \in V, e_1, e_2 \in B$,

$$[\mathsf{E}_{ab}(v_1), \mathsf{E}_{cd}(v_2)] = 0 = [\mathsf{E}_{ab}(e), \mathsf{E}_{cd}(v)] = [\mathsf{E}_{ab}(e_1), \mathsf{E}_{cd}(e_2)]. \tag{7}$$

Remark 3.1. *The elements* $\mathsf{E}_{ab}(e)\ \forall\, 1 \le a \neq b \le n,\ \forall\, e \in B$*, generate a Lie subalgebra isomorphic to* $\mathfrak{sl}_n(B)$*. Moreover,* $\mathfrak{sl}_n(B)$ *is semisimple: since* B *is a finite dimensional semi-simple* $\mathbb{C}$*-algebra, it is isomorphic to a direct sum of matrix algebras* $M_k(\mathbb{C})$ *and* $\mathfrak{sl}_n\big(M_k(\mathbb{C})\big)$ *is simple because* $\mathfrak{sl}_n\big(M_k(\mathbb{C})\big) \cong \mathfrak{sl}_{nk}(\mathbb{C})$*.*

Proof. Let us assume that $n \geq 5$. Let $\mathfrak{f}$ be the Lie algebra defined by the same generators and only relations (4),(5),(7). We first show that $\mathfrak{f} \cong \widehat{\mathfrak{sl}}_n(T_B V)$. We would like to define $\mathsf{E}_{ab}(p)$ for all $p \in T_B V$ by induction via $\mathsf{E}_{ab}(p) = [\mathsf{E}_{ac}(v), \mathsf{E}_{cb}(\widetilde{p})]$ for some $c \neq a, b$ if $p = v \otimes \widetilde{p}$. Let us assume that $\mathsf{E}_{ab}(p)$ has been defined in this way if the degree of p is $\leq k-1$. By assumption, this is true if $k = 1, 2$ and does not depend on the choice of c.

Let $p_i \in V^{\otimes k_i}, i = 1, 2$. We want to prove by induction on $k_1 + k_2$ that $[\mathsf{E}_{ab}(p_1), \mathsf{E}_{bc}(p_2)] = \mathsf{E}_{ac}(p_1 p_2)$ if $a \neq b \neq c \neq a$ and $[\mathsf{E}_{ab}(p_1), \mathsf{E}_{cd}(p_2)] = 0$ if $a \neq b \neq c \neq d \neq a$, so that we can apply theorem 3.1. We know from our hypothesis that this is true if $k_1 + k_2 = 0$ or 1, so let us assume that it holds for $0 \leq k_1 + k_2 \leq k - 1$.

Suppose that $k_1 + k_2 = k \geq 2$ and $p \in V^{\otimes k}$. We define $\mathsf{E}_{ab}(p)$ as above, that is, we express p as $p = v \otimes \widetilde{p}$ and set $\mathsf{E}_{ab}(p) = [\mathsf{E}_{ac}(v), \mathsf{E}_{cb}(\widetilde{p})]$ for some $c \neq a, b$. By induction, we can assume that $\mathsf{E}_{cb}(\widetilde{p})$ is well defined. First, we prove that the definition of $\mathsf{E}_{ab}(p)$ does not depend on the choice of $c \neq a, b$. Write $\widetilde{p} = \widetilde{v} \otimes \overline{p}, \widetilde{v} \in V, \overline{p} \in V^{\otimes k-2}$ and choose d, e such that a, b, c, d, e are all distinct. Then

$$\begin{aligned}
\mathsf{E}_{ab}(p) &= \big[\mathsf{E}_{ac}(v), \mathsf{E}_{cb}(\widetilde{p})\big] = \big[\mathsf{E}_{ac}(v), [\mathsf{E}_{ce}(\widetilde{v}), \mathsf{E}_{eb}(\overline{p})]\big] \\
&= \big[[\mathsf{E}_{ac}(v), \mathsf{E}_{ce}(\widetilde{v})], \mathsf{E}_{eb}(\overline{p})\big] = \big[[\mathsf{E}_{ad}(v), \mathsf{E}_{de}(\widetilde{v})], \mathsf{E}_{eb}(\overline{p})\big] \\
&= \big[\mathsf{E}_{ad}(v), [\mathsf{E}_{de}(\widetilde{v}), \mathsf{E}_{eb}(\overline{p})]\big] = [\mathsf{E}_{ad}(v), \mathsf{E}_{db}(\widetilde{p})]
\end{aligned}$$

We have used $[\mathsf{E}_{ac}(v), \mathsf{E}_{eb}(\overline{p})] = 0 = [\mathsf{E}_{ad}(v), \mathsf{E}_{eb}(\overline{p})]$, a consequence of our inductive assumption. The definition of $\mathsf{E}_{ab}(p)$ also does not depend on the choice of v and $\widetilde{p}$.

Without loss of generality, we can assume that $k_1 \geq 2$ and write $p_1 = v_1 \otimes \widetilde{p}_1, v_1 \in V, \widetilde{p}_1 \in V^{\otimes k_1 - 1}$. We define $\mathsf{E}_{ac}(p_1 \otimes p_2)$ by $\mathsf{E}_{ac}(p_1 \otimes p_2) = [\mathsf{E}_{ad}(v_1), \mathsf{E}_{dc}(\widetilde{p}_1 \otimes p_2))]$, which does not depend on the choice of d. For $a \neq b \neq c \neq d \neq a$, choose $e \neq a, b, c, d$; then

$$[\mathsf{E}_{ab}(p_1), \mathsf{E}_{cd}(p_2)] = \big[[\mathsf{E}_{ae}(v_1), \mathsf{E}_{eb}(\widetilde{p}_1)], \mathsf{E}_{cd}(p_2)\big] = 0$$

since, by induction, $[\mathsf{E}_{ae}(v_1), \mathsf{E}_{cd}(p_2)] = 0 = [\mathsf{E}_{eb}(\widetilde{p}_1), \mathsf{E}_{cd}(p_2)]$.

Now, if $a \neq b \neq c \neq a$, choose $d \neq a, b, c$; then

$$\begin{aligned}
[\mathsf{E}_{ab}(p_1), \mathsf{E}_{bc}(p_2)] &= \big[[\mathsf{E}_{ad}(v_1), \mathsf{E}_{db}(\widetilde{p}_1)], \mathsf{E}_{bc}(p_2)\big] = \big[\mathsf{E}_{ad}(v_1), [\mathsf{E}_{db}(\widetilde{p}_1), \mathsf{E}_{bc}(p_2)]\big] \\
&= [\mathsf{E}_{ad}(v_1), \mathsf{E}_{dc}(\widetilde{p}_1 \otimes p_2)] = \mathsf{E}_{ac}(p_1 \otimes p_2)
\end{aligned}$$

This completes the induction step. We have proved our claim regarding $\mathfrak{f}$. We now observe that $\widehat{\mathfrak{sl}}_n(A)$ is the quotient of $\widehat{\mathfrak{sl}}_n(T_B V)$ by the Lie ideal generated by the elements $\mathsf{E}_{ab}(P)$; if we write such a P in the form $P = \pi_2(P) + \pi_1(P) + \pi_0(P)$ with $\pi_2(P) = \sum_k v_k \otimes \widetilde{v}_k \in V \otimes_B V$, then, for $a \neq b \neq c \neq a$,

$$\mathsf{E}_{ab}(P) = \sum_k [\mathsf{E}_{ac}(v_k), \mathsf{E}_{cb}(\widetilde{v}_k)] + \mathsf{E}_{ab}\big(\pi_1(P)\big) + \mathsf{E}_{ab}\big(\pi_0(P)\big),$$

which completes the proof of proposition 3.1. □

We will be interested in the following two situations:

1. $A = \mathbb{C}[u,v] \rtimes \Gamma$ where $\Gamma \subset SL_2(\mathbb{C})$ is a finite subgroup, so $B = \mathbb{C}[\Gamma]$ is the group algebra, $V = U \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]$, where $U = \text{span}\{u,v\} \cong \mathbb{C}^2$, and $P = u \otimes v - v \otimes u$ or, more generally, $P = u\otimes v - v \otimes u - z$ with $z \in \mathsf{Z}\Gamma$, $\mathsf{Z}\Gamma$ being the center of the group algebra $\mathbb{C}[\Gamma]$.

2. $A = \Pi(Q)$ is the preprojective algebra of a quiver Q which has no loop, so $B = \oplus_{i\in I} \mathbb{C} \cdot e_i$ is the semisimple algebra associated to the vertex set $I(Q)$ of Q with $e_i^2 = e_i, i \in I(Q)$, V is a vector space with basis given by the arrows of the double quiver $\overline{Q}$ and $(P_i)_{i\in I(Q)}$ is the ideal generated by the elements $P_i = \sum_{\{v\in Q | h(v)=i\}} \overline{v} \cdot v - \sum_{\{v \in Q | t(v)=i\}} v \cdot \overline{v}$ for each $i \in I(Q)$.

Proposition 3.2. *For these two algebras, proposition 3.1 is true also when $n = 4$.*

Proof. The case $\mathbb{C}[u,v] \rtimes \Gamma$ was treated in [Gu3], so we will explain how to adapt the proof of proposition 3.1 to the second case, following exactly the same steps as above, using induction and a similar notation. Actually, it will be enough to prove the statement for the path algebra $T_B E$. (Here $V = E$ and $E = E(Q)$ in the notation of section 6.) Note that, in $\widehat{\mathfrak{sl}}_n(T_B E)$, we can define elements $\mathsf{E}_{aa}(v)$ by $\mathsf{E}_{aa}(v) = [\mathsf{E}_{ab}(v), \mathsf{E}_{ba}(e_{t(v)})]$ for some $b \neq a$. (This does not depend on the choice of b.) Then it is true that $\mathsf{E}_{aa}(v) = [\mathsf{E}_{ac}(e_{h(v)}), \mathsf{E}_{ca}(v)]$ for any $c \neq a$.

Suppose that the path p equals $v \otimes \widetilde{p}, v \in E$ and $\widetilde{p} = \widetilde{v} \otimes \overline{p}, \widetilde{v} \in E$. Choose a,b,c,d all distinct. We can assume by induction that $\mathsf{E}_{db}(\widetilde{p}) = [\mathsf{E}_{dd}(\widetilde{v}), \mathsf{E}_{db}(\overline{p})]$. We want to show that the inductive definition of $\mathsf{E}_{ab}(p)$ does not depend on the choice of c. We have

$$\begin{aligned}
\mathsf{E}_{ab}(p) &= [\mathsf{E}_{ac}(v), \mathsf{E}_{cb}(\widetilde{p})] = \big[\mathsf{E}_{ac}(v), [\mathsf{E}_{cd}(e_{h(\widetilde{p})}), \mathsf{E}_{db}(\widetilde{p})]\big] = \big[\mathsf{E}_{ac}(v), [\mathsf{E}_{cc}(\widetilde{v}), \mathsf{E}_{cb}(\overline{p})]\big] \\
&= \Big[\mathsf{E}_{ac}(v), \big[\mathsf{E}_{cd}(e_{h(\widetilde{p})}), [\mathsf{E}_{dd}(\widetilde{v}), \mathsf{E}_{db}(\overline{p})]\big]\Big] = \big[[\mathsf{E}_{ad}(v), \mathsf{E}_{dc}(e_{t(v)})], [\mathsf{E}_{cc}(\widetilde{v}), \mathsf{E}_{cb}(\overline{p})]\big] \\
&= \big[\mathsf{E}_{ad}(v), [\mathsf{E}_{dd}(\widetilde{v}), \mathsf{E}_{db}(\overline{p})]\big] \text{ since } [\mathsf{E}_{ad}(v), \mathsf{E}_{cc}(\widetilde{v})] = 0 = [\mathsf{E}_{ad}(v), \mathsf{E}_{cb}(\overline{p})] \text{ by induction} \\
&= [\mathsf{E}_{ad}(v), \mathsf{E}_{db}(\widetilde{p})]
\end{aligned}$$

Hence, we have proved that the inductive definition of $\mathsf{E}_{ac}(p)$ does not depend on the choice of $c \neq a, b$.

Suppose now that $a \neq b \neq c \neq d \neq a, \deg(p_1) \geq 2, \deg(p_2) \geq 1$ and also $a \neq c$ (the case $\deg(p_1) \geq 2, \deg(p_2) = 0$ is easier); then, with $p_1 = v_1 \otimes \widetilde{p}_1$,

$$[\mathsf{E}_{ab}(p_1), \mathsf{E}_{cd}(p_2)] = \big[[\mathsf{E}_{aa}(v_1), \mathsf{E}_{ab}(\widetilde{p}_1)], \mathsf{E}_{cd}(p_2)\big] = 0$$

because $[\mathsf{E}_{aa}(v_1), \mathsf{E}_{cd}(p_2)] = 0 = [\mathsf{E}_{ab}(\widetilde{p}_1), \mathsf{E}_{cd}(p_2)]$ by induction. If $a = c$ and a,b,d are all distinct, then, writing $p_1 = \widehat{p}_1 \otimes \widehat{v}_1$, we get

$$[\mathsf{E}_{ab}(p_1), \mathsf{E}_{ad}(p_2)] = \big[[\mathsf{E}_{ab}(\widehat{p}_1), \mathsf{E}_{bb}(\widehat{v}_1)], \mathsf{E}_{ad}(p_2)\big] = 0$$

since $[\mathsf{E}_{ab}(\widehat{p}_1), \mathsf{E}_{ad}(p_2)] = 0 = [\mathsf{E}_{bb}(\widehat{v}_1), \mathsf{E}_{ad}(p_2)]$ by induction.

Now choose distinct a,b,c,d. Then

$$\begin{aligned}
[\mathsf{E}_{ab}(p_1), \mathsf{E}_{bc}(p_2)] &= \big[[\mathsf{E}_{ad}(v_1), \mathsf{E}_{db}(\widetilde{p}_1)], \mathsf{E}_{bc}(p_2)\big] \\
&= \big[\mathsf{E}_{ad}(v_1), [\mathsf{E}_{db}(\widetilde{p}_1), \mathsf{E}_{dc}(p_2)]\big] = [\mathsf{E}_{ad}(v_1), \mathsf{E}_{dc}(\widetilde{p}_1 \otimes p_2)] \\
&= \mathsf{E}_{ac}(p_1 \otimes p_2)
\end{aligned}$$

To complete the proof, we need to see that $[\mathsf{E}_{aa}(v), \mathsf{E}_{ac}(\widetilde{p})] = \mathsf{E}_{ac}(p)$. Choose $d \neq a, c$.

$$\begin{aligned}
[\mathsf{E}_{aa}(v), \mathsf{E}_{ac}(\widetilde{p})] &= \big[[\mathsf{E}_{ad}(v), \mathsf{E}_{da}(e_{t(v)})], \mathsf{E}_{ac}(\widetilde{p})\big] \\
&= [\mathsf{E}_{ad}(v), \mathsf{E}_{dc}(\widetilde{p})] = \mathsf{E}_{ac}(p) \text{ since } [\mathsf{E}_{ad}(\widetilde{v}), \mathsf{E}_{ac}(\widetilde{p})] = 0 \text{ as proved previously}
\end{aligned}$$

□

4 Universal central extension of $\mathfrak{g}(\mathbb{C}[u,v])$

The goal of this section is to give two presentations of the Lie algebra $\widehat{\mathfrak{g}}(\mathbb{C}[u,v])$, $\mathfrak{g}$ being a simple Lie algebra, in terms of generators and relations, which are similar to those obtained in [Le] and [MRY]. Let $C = (c_{ij})_{0\leq i,j\leq N}$ be the Cartan matrix of affine type associated to $\mathfrak{g}$. We will assume that the rank N of $\mathfrak{g}$ is ≥ 3 and denote by $\delta(\cdot)$ the usual δ-function: $\delta(\text{TRUE}) = 1, \delta(\text{FALSE}) = 0$.

Lemma 4.1. *The universal central extension $\widehat{\mathfrak{g}}(\mathbb{C}[u,v])$ of $\mathfrak{g}(\mathbb{C}[u,v])$ is isomorphic to the Lie algebra $\mathfrak{l}$ generated by the elements $X^{\pm}_{i,r}, H_{i,r}$ and $X^{+}_{0,r}$ for $1 \leq i \leq N, r \geq 0$ subjected to the following relations:*

$$[H_{i_1,r_1}, H_{i_2,r_2}] = 0, \quad [H_{i_1,0}, X^{\pm}_{i_3,r_3}] = \pm d_{i_1} c_{i_1 i_3} X^{\pm}_{i_3,r_3} \;\; \textit{for } 1 \leq i_1, i_2 \leq N,\; 0 \leq i_3 \leq N,\; r_1, r_2, r_3 \in \mathbb{Z}_{\geq 0} \tag{8}$$

$$[H_{i_1,r_1+1}, X^{\pm}_{i_2,r_2}] = [H_{i_1,r_1}, X^{\pm}_{i_2,r_2+1}], \quad [X^{\pm}_{i_1,r_1+1}, X^{\pm}_{i_2,r_2}] = [X^{\pm}_{i_1,r_1}, X^{\pm}_{i_2,r_2+1}],\; 0 \leq i_1, i_2 \leq N \tag{9}$$

$$[X^{+}_{i_1,r_1}, X^{-}_{i_2,r_2}] = \delta_{i_1 i_2} H_{i_1,r_1+r_2} \;\; \textit{for } 0 \leq i_1 \leq N,\; 1 \leq i_2 \leq N, r_1, r_2 \in \mathbb{Z}_{\geq 0}, \tag{10}$$

$$\sum_{\pi \in S_k} \left[X^{\pm}_{i_1,r_{\pi(1)}}, \left[\cdots, [X^{\pm}_{i_1,r_{\pi(k)}}, X^{\pm}_{i_2,s}] \cdots \right] \right] = 0 \;\; \textit{where } k = 1 - c_{i_1,i_2}, r_1, \ldots, r_k, s \in \mathbb{Z}_{\geq 0} \tag{11}$$

In (9), (11)*, when $i_1 = 0, i_2 = 0$ or $i_3 = 0$, there is a relation only in the "+"-case.*

Proof. This can be proved using the same ideas as in proposition 3.5 in [MRY], with one modification: since $\mathfrak{l}$ does not have the generator $X^{-}_{0,0}$, we cannot define the action of the affine Weyl group, but it is still possible to see that the root spaces corresponding to positive roots which are related by a simple reflection must have the same dimension by using the fact that the relations above for all $H_{i,r}, X^{+}_{i,r}$ define the non-negative part of a triangular decomposition of $\mathfrak{l}$. (This is not true for the relations in [MRY] involving the same elements, which explains why we have to consider more relations here.) □

Lemma 4.2. *The Lie algebra $\widehat{\mathfrak{g}}(\mathbb{C}[u,v])$ is isomorphic to the Lie algebra $\mathfrak{m}$ generated by the elements $X^{\pm}_{i,r}, H_{i,r}$ for $1 \leq i \leq N, r = 0, 1$ and $X^{+}_{0,0}, X^{+}_{0,1}$ subjected to the same relations as in lemma 4.1, but with the following restrictions: $r_1, r_2, r_3 = 0$ or 1 in* (8)*; $r_1, r_2 = 0$ in* (9)*; $r_1 + r_2 = 0, 1$ in* (10) *; $r_1, \ldots, r_k, s = 0$ in* (11).

Proof. This can be deduced from lemma 4.1 using computations similar to those in the proof of lemma 2.7 in [Gu2]. We need the observation that, if $c_{0i_1} \neq 0$, then it is possible to find i_2 such that $c_{i_1 i_2} \neq 0$ but $c_{i_2 0} = 0$. □

5 Deformed double current algebras

As an application of lemmas 4.1 and 4.2, we suggest a definition of deformed double current algebras associated to any simple Lie algebra $\mathfrak{g}$ of rank ≥ 3: the case $\mathfrak{g} = \mathfrak{sl}_n$ was treated in [Gu2] and we follow a similar approach, expressing them as limit forms of affine Yangians, which we have to define first.

Let us assume that $\mathfrak{g}$ is not of type A. (In type A, there are two deformation parameters in the definition of the affine Yangians, so that definition 5.1 would be less general in type A.) Under this assumption, we can fix k such that $c_{0k} \neq 0$ and $i = k$ is the only value of $i \in \{1, \ldots, N\}$ such that $c_{0i} \neq 0$. Let $d_0, \ldots, d_N$ be relatively prime integers such that DC is a symmetric matrix if D is the diagonal matrix with diagonal entries equal to $d_0, \ldots, d_N$. If A is any algebra and $a, b \in A$, we set $S(a, b) = ab + ba$.

Definition 5.1. *The affine Yangian* $\widehat{Y}(\mathfrak{g})$ *is the algebra generated by the elements* $X_{i,r}^{\pm}, H_{i,r}^{\pm}, 0 \leq i \leq N, r \geq 0$ *which satisfy the following relations for any* $0 \leq i_1, i_2 \leq N$,

$$[H_{i_1,r_1}, H_{i_2,r_2}] = 0, \quad [H_{i_1,0}, X_{i_2,s}^{\pm}] = \pm d_{i_1} c_{i_1,i_2} X_{i_2,s}^{\pm}, \;\; [X_{i_1,r_1}^{+}, X_{i_2,r_2}^{-}] = \delta_{i_1 i_2} H_{i_1,r+s} \tag{12}$$

$$[H_{i_1,r_1+1}, X_{i_2,r_2}^{\pm}] - [H_{i_1,r_1}, X_{i_2,r_2+1}^{\pm}] = \pm\frac{d_{i_1}}{2} c_{i_1,i_2} S(H_{i_1,r_1}, X_{i_2,r_2}^{\pm}) \tag{13}$$

$$[X_{i_1,r_1+1}^{\pm}, X_{i_2,r_2}^{\pm}] - [X_{i_1,r_1}^{\pm}, X_{i_2,r_2+1}^{\pm}] = \pm\frac{d_{i_1}}{2} c_{i_1,i_2} S(X_{i_1,r_1}^{\pm}, X_{i_2,r_2}^{\pm}) \tag{14}$$

$$\sum_{\pi \in S_j} [X_{i_1,r_{\pi(1)}}^{\pm}, [X_{i_1,r_{\pi(2)}}^{\pm}, \ldots, [X_{i_1,r_{\pi(j)}}^{\pm}, X_{i_2,s}^{\pm}] \ldots]] = 0 \tag{15}$$

where $j = 1 - c_{i_1 i_2}, r_1, \ldots, r_j, s \in \mathbb{Z}_{\geq 0}$.

Remark 5.1. *We could have introduced a deformation parameter* λ *in this definition by multipliying the right-hand side of relations* (13),(14) *by* λ. *However, for values of* $\lambda \neq 0$, *these algebras are all isomorphic, so we just set* $\lambda = 1$. *Setting* $\lambda = 0$ *yields the universal central extension* $\widehat{\mathfrak{g}}(\mathbb{C}[u^{\pm 1}, v])$ *of* $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[u^{\pm 1}, v]$.

When we exclude the generators for $i \neq 0$, we obtain a presentation of the Yangian of the corresponding finite type [Dr2]. These can also be defined in terms of generators $z, J(z)$ for $z \in \mathfrak{g}$ (see [Dr1],[ChPr1]) with the property that $[z_1, J(z_2)] = J([z_1, z_2])$. The relation between the two presentations is given by the following formula:

$$X_{i,1}^{\pm} = J(X_i^{\pm}) - \omega_i^{\pm} \text{ where } \omega_i^{\pm} = \pm\frac{1}{4} \sum_{\alpha \in \Delta^+} S\big([X_i^{\pm}, X_\alpha^{\pm}], X_\alpha^{\mp}\big) - \frac{1}{4} S(X_i^{\pm}, H_i) \tag{16}$$

where Δ^+ is the set of positive roots for $\mathfrak{g}$ and the root vectors $X_\alpha^{\pm}$ are those considered in [Dr2]. [1] This will be useful in the proof of theorem 5.1 below. There are also elements $\nu_i \in \mathfrak{U}(\mathfrak{g})$ such that $H_{i,1} = J(H_i) - \nu_i$: they are given by $\nu_i = [\omega_i^+, X_i^-]$.

Our goal in this section is to give some motivation for the next definition.

Definition 5.2. *The deformed double current algebra* $\mathsf{D}(\mathfrak{g})$ *is the algebra generated by* $\mathsf{X}_{i,r}^{\pm}, \mathsf{H}_{i,r}, \mathsf{X}_{0,r}^{+}$ *for* $1 \leq i \leq N, r = 0, 1$ *subjected to the same relations as in lemma 4.2, except that the following relations involving* $\mathsf{X}_{0,r}^{+}$ *must be modified:*

$$[\mathsf{X}_{k,1}^{+}, \mathsf{X}_{0,0}^{+}] - [\mathsf{X}_{k,0}^{+}, \mathsf{X}_{0,1}^{+}] = -\frac{d_0}{2} S(\mathsf{X}_{k,0}^{+}, \mathsf{X}_\theta^{-}) + [\omega_k^{+}, \mathsf{X}_\theta^{-}] + [\mathsf{X}_{k,0}^{+}, \omega_0^{+}] \tag{17}$$

$$[\mathsf{H}_{k,1}, \mathsf{X}_{0,0}^{+}] - [\mathsf{H}_{k,0}, \mathsf{X}_{0,1}^{+}] = -\frac{d_0}{2} S(\mathsf{H}_{k,0}^{+}, \mathsf{X}_\theta^{-}) + d_0 \omega_0^{+} + [\nu_k, \mathsf{X}_\theta^{-}] \tag{18}$$

$$[\mathsf{X}_{0,1}^{+}, \mathsf{X}_{k,0}^{-}] = [\mathsf{X}_{k,0}^{-}, \omega_0^{+}], \;\; [\mathsf{X}_{0,1}^{+}, \mathsf{X}_{0,0}^{+}] = 2d_0 \mathsf{X}_{0,0}^{+} \mathsf{X}_\theta^{-} \tag{19}$$

$$[\mathsf{X}_{0,0}^{+}, \mathsf{X}_{i,1}^{\pm}] = [\mathsf{X}_\theta^{-}, \omega_i^{\pm}], \;\; [\mathsf{X}_{0,1}^{+}, \mathsf{X}_{i,0}^{\pm}] = -[\omega_0^{+}, \mathsf{X}_{i,0}^{\pm}] \text{ for } i \neq k \tag{20}$$

The elements X_θ^{-} *and* ω_0^{+} *are defined at the beginning of the proof of theorem 5.1 below.*

Remark 5.2. *As with affine Yangians, we could have added a parameter* $\lambda \in \mathbb{C}$ *in this definition, but when* $\lambda \neq 0$, *these algebras would all be isomorphic to each other, while setting* $\lambda = 0$ *would give the universal central extension of* $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[u, v]$ *(see lemma 4.2).*

[1] In the published version, the root vectors were those from [ChPr1], but these were not the right choice. In formula (16) on p.378 in [ChPr1], two formulas should be $\varphi(X_i^{\pm}) = d_i^{-\frac{1}{2}} X_{i,0}^{\pm}$ and $\varphi(X_i^{\pm}) = d_i^{-\frac{1}{2}} X_{i,1}^{\pm} + \varphi(w_i^{\pm})$.

We have maps $\mathfrak{U}(\mathfrak{g}(\mathbb{C}[u])) \longrightarrow \mathsf{D}(\mathfrak{g})$, $\mathfrak{U}(\mathfrak{g}(\mathbb{C}[v])) \longrightarrow \mathsf{D}(\mathfrak{g})$ with images equal to the subalgebras generated by the elements $\mathsf{X}_{i,0}^{\pm}, \mathsf{H}_{i,0}, \mathsf{X}_{0,0}^{+}, 1 \le i \le N$ and by the elements $\mathsf{X}_{i,r}^{\pm}, \mathsf{H}_{i,r}$ with $i \neq 0, r = 0, 1$, respectively.

Let us start with $\widehat{Y}(\mathfrak{g})$ and its filtration given by $\deg(X_{i,r}^{\pm}) = r = \deg(H_{i,r})$. We need to introduce a new variable h. Let $\widetilde{S}$ be the subring of $\widehat{Y}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{C}[h]$ generated by $h^r X_{i,r}^{\pm}, h^r H_{i,r}, 0 \le i \le N, r \ge 0$. Set $\widehat{S} = \widetilde{S}/h\widetilde{S}$. Since $\widetilde{S}$ is the Rees ring of $\widehat{Y}(\mathfrak{g})$, $\widehat{S} \cong \mathrm{gr}(\widehat{Y}(\mathfrak{g}))$, the associated graded ring. There is a canonical map $\mathfrak{U}\big(\widehat{\mathfrak{g}}(\mathbb{C}[u^{\pm 1}, v])\big) \twoheadrightarrow \mathrm{gr}(\widehat{Y}(\mathfrak{g}))$: it was proven in [Gu3] that this map is an isomorphism when $\mathfrak{g} = \mathfrak{sl}_n$. In general, this is not known although believed to be true, so we will proceed in this section by assuming that we have an isomorphism $\mathfrak{U}\big(\widehat{\mathfrak{g}}(\mathbb{C}[u^{\pm 1}, v])\big) \xrightarrow{\sim} \mathrm{gr}(\widehat{Y}(\mathfrak{g}))$. We only need this assumption to prove theorem 5.1. Thus, under this assumption, we have a map $\widehat{S} \twoheadrightarrow \mathfrak{U}\big(\mathfrak{g}(\mathbb{C}[u^{\pm 1}, v])\big)$.

Consider the composite $\widetilde{S} \twoheadrightarrow \widehat{S} \twoheadrightarrow \mathfrak{U}(\mathfrak{sl}_n[u^{\pm 1}, v]) \twoheadrightarrow \mathfrak{U}(\mathfrak{sl}_n[v])$, where the last map is obtained by setting $u = 1$. Let K be its kernel. Let S be the $\mathbb{C}[h]$-subalgebra of $\widehat{Y}(\mathfrak{g}) \otimes_{\mathbb{C}[h]} \mathbb{C}[h, h^{-1}]$ generated by $\widetilde{S}$ and $h^{-1}\mathsf{K}$.

Theorem 5.1. *Assume that $\mathfrak{U}(\widehat{\mathfrak{g}}(\mathbb{C}[u^{\pm 1}, v])) \xrightarrow{\sim} \mathrm{gr}(\widehat{Y}(\mathfrak{g}))$. The algebra S/hS is a quotient of the deformed double current algebra $\mathsf{D}(\mathfrak{g})$.*

Proof. Let $\mathfrak{X}_{i,r}^{\pm} = h^r X_{i,r}^{\pm}$, $\mathfrak{H}_{i,r} = h^r H_{i,r}$ and denote by θ the highest positive root of $\mathfrak{g}$. We write X_{θ}^{-} as $X_{\theta}^{-} = [X_{k,0}^{-}, \widetilde{X}_{\theta-\alpha_k}^{-}]$, a root vector for the lowest root $-\theta$ of $\mathfrak{g}$, and $\mathsf{X}_{\theta}^{-} = [\mathsf{X}_{k,0}^{-}, \widetilde{\mathsf{X}}_{\theta-\alpha_k}^{-}] \in \mathsf{D}(\mathfrak{g})$. (Here, $\widetilde{\mathsf{X}}_{\theta-\alpha_k}^{-} = \mathsf{X}_{\theta-\alpha_k}^{-} \in \mathfrak{g} \subset \mathsf{D}(\mathfrak{g})$.) Let $\mathfrak{X}_{\theta,r}^{-} \in S$ be obtained by replacing $X_{k,0}^{-}$ in the previous expression for X_{θ}^{-} by $\mathfrak{X}_{k,r}^{-}$. We will also need the notation $\omega_0^{+} = -[\omega_k^{-}, \widetilde{X}_{\theta-\alpha_k}^{-}]$. The map $\varphi : \mathsf{D}(\mathfrak{g}) \longrightarrow S/hS$ is defined in terms of the generators of $\mathsf{D}(\mathfrak{g})$ in the following way: for $1 \le i \le N, r = 0, 1$, we set $\varphi(\mathsf{X}_{i,r}^{\pm}) = \mathfrak{X}_{i,r}^{\pm}, \varphi(\mathsf{H}_{i,r}) = \mathfrak{H}_{i,r}$, and set $\varphi(\mathsf{X}_{0,r}^{+}) = h^{-1}(\mathfrak{X}_{0,r}^{+} - \mathfrak{X}_{\theta,r}^{-}), r = 0, 1$.

We have to verify that the images of the generators satisfy the same relations as those given in definition 5.2. The relations with $i_1, i_2 \neq 0$ are easy to verify, so we focus on those involving $\varphi(\mathsf{X}_{0,r}^{+})$.

First, we compute

$$\begin{aligned}
[\varphi(\mathsf{X}_{k,1}^{+}), \varphi(\mathsf{X}_{0,0}^{+})] - [\varphi(\mathsf{X}_{k,0}^{+}), \varphi(\mathsf{X}_{0,1}^{+})] &= \left[hX_{k,1}^{+}, \frac{X_{0,0}^{+} - X_{\theta}^{-}}{h}\right] - \left[X_{k,0}^{+}, \frac{\mathfrak{X}_{0,1}^{+} - \mathfrak{X}_{\theta,1}^{-}}{h}\right] \\
&= \frac{d_k}{2} c_{k0} S(\mathfrak{X}_{k,0}^{+}, \mathfrak{X}_{0,0}^{+}) - \frac{[\mathfrak{X}_{k,1}^{+}, X_{\theta}^{-}] - [\mathfrak{X}_{k,0}^{+}, \mathfrak{X}_{\theta,1}^{-}]}{h} \\
&= \frac{d_0}{2} c_{0k} S\big(\varphi(\mathsf{X}_{k,0}^{+}), \varphi(\mathsf{X}_{\theta}^{-})\big) + [\varphi(\omega_k^{+}), \varphi(\mathsf{X}_{\theta}^{-})] + [\varphi(\mathsf{X}_{k,0}^{+}), \varphi(\omega_0^{+})]
\end{aligned}$$

since $\mathfrak{X}_{0,0}^{+} = X_{\theta}^{-}$ in S/hS. It is known that $c_{0k} = -1$. Note that the sum of the last two terms equals $\big[\varphi(\mathsf{X}_{k,0}^{-}), [\varphi(\omega_k^{+}), \varphi(\widetilde{\mathsf{X}}_{\theta-\alpha_k}^{-})]\big]$ when $-\theta + 2\alpha_k$ is not a root of $\mathfrak{g}$. Relation (18) can be deduced from (17) since $[\varphi(\mathsf{X}_{0,0}^{+}), \varphi(\mathsf{X}_{k,0}^{-})] = 0$ and

$$[\varphi(\mathsf{X}_{0,1}^{+}), \varphi(\mathsf{X}_{k,0}^{-})] = [X_{k,0}^{-}, X_{\theta,1}^{-}] = [X_{k,0}^{-}, \omega_0^{+}].$$

Now we look at

$$\begin{aligned}
[\varphi(\mathsf{X}_{0,1}^{+}), \varphi(\mathsf{X}_{0,0}^{+})] &= h^{-2}\Big([\mathfrak{X}_{0,1}^{+}, \mathfrak{X}_{0,0}^{+}] + [\mathfrak{X}_{\theta,1}^{-}, \mathfrak{X}_{\theta,0}^{-}] - [\mathfrak{X}_{0,1}^{+}, \mathfrak{X}_{\theta,0}^{-}] - [\mathfrak{X}_{\theta,1}^{-}, \mathfrak{X}_{0,0}^{+}]\Big) \\
&= h^{-1}\Big(d_0(\mathfrak{X}_{0,0}^{+})^2 - \big[[\omega_k^{-}, \mathfrak{X}_{\theta,0}^{-}], \widetilde{X}_{\theta-\alpha_k}^{-}\big]\Big) \\
&= h^{-1}\Big(d_0(\mathfrak{X}_{0,0}^{+})^2 - d_0(\mathfrak{X}_{\theta,0}^{-})^2\Big) = d_0 h^{-1}(\mathfrak{X}_{0,0}^{+} - \mathfrak{X}_{\theta,0}^{-})(\mathfrak{X}_{0,0}^{+} + \mathfrak{X}_{\theta,0}^{-}) \\
&= 2d_0 \varphi(\mathsf{X}_{0,0}^{+}) \varphi(\mathsf{X}_{\theta}^{-})
\end{aligned}$$

Here, we used

$$\begin{aligned}
\big[[\omega_k^-, X_\theta^-], \widetilde{X}_{\theta-\alpha_k}^-\big] &= -\frac{1}{4}\left[\sum_{\alpha\in\Delta^+} S\big([X_k^-, X_\alpha^-], [X_\alpha^+, X_\theta^-]\big) + S\big(X_k^-, [H_k, X_\theta^-]\big), \widetilde{X}_{\theta-\alpha_k}^-\right] \\
&= -\frac{1}{4}\left[S\big([X_k^-, X_{\theta-\alpha_k}^-], [X_{\theta-\alpha_k}^+, X_\theta^-]\big) + d_k c_{k0} S(X_k^-, X_\theta^-), \widetilde{X}_{\theta-\alpha_k}^-\right] \\
&= -\frac{1}{4}\left(S\big(X_\theta^-, [H_{\theta-\alpha_k}, X_\theta^-]\big) + 2d_0 c_{0k}(X_\theta^-)^2\right) \\
&= -\frac{1}{4}\left(d_k c_{k0} S(X_\theta^-, X_\theta^-) + 2d_0 c_{0k}(X_\theta^-)^2\right) \\
&= -d_0 c_{0k}(X_\theta^-)^2 = d_0(X_\theta^-)^2
\end{aligned}$$

Note that $d_k c_{k0} = d_0 c_{0k} = -d_0$.

For $i \neq k$, $c_{0i} = 0$ and $[X_\theta^-, X_{i,0}^\pm] = 0$ but we get some non-trivial relations:

$$[\varphi(\mathsf{X}_{0,0}^+), \varphi(\mathsf{X}_{i,1}^\pm)] = [\mathfrak{X}_{\theta,0}^-, \omega_i^\pm], \;\; [\varphi(\mathsf{X}_{0,1}^+), \varphi(\mathsf{X}_{i,0}^\pm)] = -[\omega_0^+, \mathfrak{X}_{i,0}^\pm] \text{ since } \big[[J(X_k^-), \widetilde{X}_{\theta-\alpha_k}^-], X_{i,0}^\pm\big] = 0.$$

Finally, we have to justify why φ is surjective. The kernel K is the two-sided ideal generated by the elements $\mathfrak{X}_{0,r}^+ - \mathfrak{X}_{\theta,r}^- \, \forall r \in \mathbb{Z}_{\geq 0}$, and we already know that the elements $h^{-1}(\mathfrak{X}_{0,r}^+ - \mathfrak{X}_{\theta,r}^-)$ for $r = 0, 1$ are in the image of φ as are $\mathfrak{X}_{i,r}^\pm \, \forall r \geq 0$, $\forall i \neq 0$. Since $\mathfrak{X}_{0,r}^+ = h\big(h^{-1}(\mathfrak{X}_{0,r}^+ - \mathfrak{X}_{\theta,r}^-)\big) + \mathfrak{X}_{\theta,r}^- = \mathfrak{X}_{\theta,r}^-$ in S/hS, $\mathfrak{X}_{0,r}^+$ is also in $\mathrm{Image}(\varphi)$ $\forall r \geq 0$.

Let us assume that we know that $h^{-1}(\mathfrak{X}_{0,\widetilde{r}}^+ - \mathfrak{X}_{\theta,\widetilde{r}}^-) \in \mathrm{Image}(\varphi)$. Then it is also the case that $\left[\mathfrak{H}_{k,1}, h^{-1}(\mathfrak{X}_{0,\widetilde{r}}^+ - \mathfrak{X}_{\theta,\widetilde{r}}^-)\right] \in \mathrm{Image}(\varphi)$. The subalgebra of S/hS generated by $\mathfrak{X}_{i,s}^\pm, \mathfrak{H}_{i,s}$ for $1 \leq i \leq N, s \geq 0$, is a quotient of $\mathfrak{U}\mathfrak{g}(\mathbb{C}[v])$ and one can see, from this observation, that $[\mathsf{H}_{k,1}, \mathsf{X}_{\theta,\widetilde{r}}^-] = d_k c_{k0} \mathsf{X}_{\theta,\widetilde{r}+1}^- + \kappa$ where $\kappa \in \widehat{Y}(\mathfrak{g})$ has filtration degree $\leq r$. Therefore,

$$\begin{aligned}
\left[\mathfrak{H}_{k,1}, h^{-1}(\mathfrak{X}_{0,\widetilde{r}}^+ - \mathfrak{X}_{\theta,\widetilde{r}}^-)\right] &= \frac{d_k c_{k0}}{h}\mathfrak{X}_{0,\widetilde{r}+1}^+ + \frac{d_k c_{k0}}{2} S(\mathfrak{H}_{k,0}, \mathfrak{X}_{0,\widetilde{r}}^+) - \frac{d_k c_{k0}}{h}\mathfrak{X}_{\theta,\widetilde{r}+1}^- + h^{\widetilde{r}}\kappa \\
&= d_k c_{k0} h^{-1}(\mathfrak{X}_{0,\widetilde{r}+1}^+ - \mathfrak{X}_{\theta,\widetilde{r}+1}^+) + \frac{d_k c_{k0}}{2} S(\mathfrak{H}_{k,0}, \mathfrak{X}_{0,\widetilde{r}}^+) + h^{\widetilde{r}}\kappa
\end{aligned}$$

We can conclude that $h^{-1}(\mathfrak{X}_{0,\widetilde{r}+1}^+ - \mathfrak{X}_{\theta,\widetilde{r}+1}^-) \in \mathrm{Image}(\varphi)$ and, by induction, this is true $\forall r \geq 0$. □

We conjecture that $\mathsf{D}(\mathfrak{g})$ and S/hS are isomorphic.

6 Gan-Ginzburg algebras

Let Q be an arbitrary finite quiver without loops, with arrow set $E(Q)$ and vertex set $I(Q)$ (sometimes abbreviated I). We will denote its double by $\overline{Q}$, the head of $v \in E(Q)$ by $h(v)$, its tail by $t(v)$ and the opposite arrow by $\overline{v} \in E(\overline{Q})$, so $h(\overline{v}) = t(v), t(\overline{v}) = h(v)$. For $i \in I, 1 \leq j \leq l$, let $B_i = \mathbb{C} \cdot e_i$ with e_i an idempotent, $B = \oplus_{i\in I} B_i$, $\mathsf{B} = B^{\otimes l}$, $\mathsf{E}_j = B^{\otimes(j-1)} \otimes_{\mathbb{C}} E(\overline{Q}) \otimes_{\mathbb{C}} B^{\otimes(l-j)}$ and $\mathsf{E} = \oplus_{j=1}^{l} \mathsf{E}_j$. The space E is a B-bimodule, so we can form the tensor algebra $T_{\mathsf{B}}\mathsf{E}$, which is a module for the symmetric group S_l, hence we have also the smash product $T_{\mathsf{B}}\mathsf{E} \rtimes S_l$. We set $1_B = \sum_{i\in I} e_i, e_i^{(j)} = 1_B^{\otimes(j-1)} \otimes e_i \otimes 1_B^{\otimes(l-j)}$, $v^{(j)} = 1_B^{\otimes(j-1)} \otimes v \otimes 1_B^{\otimes(l-j)} \in \mathsf{E}_j$ and

$$\rho_i^{(j)} = \sum_{\{v\in E(Q)|h(v)=i\}} \overline{v}^{(j)} \otimes v^{(j)} - \sum_{\{v\in E(Q)|t(v)=i\}} v^{(j)} \otimes \overline{v}^{(j)} \;\in\; \mathsf{E}_j^{\otimes 2}.$$

Definition 6.1 ([GaGi])**.** *Let $\lambda = (\lambda_i)_{i\in I} \in \mathbb{C}^{\oplus|I|}, \nu \in \mathbb{C}$. The deformed preprojective algebra $\Pi_l^{\lambda,\nu}(Q)$ (also called Gan-Ginzburg algebra in the literature) is defined as the quotient of $T_{\mathsf{B}}\mathsf{E} \rtimes S_l$ by the following relations:*

$$\text{For any } 1 \leq j \leq l,\ i \in I, \quad \rho_i^{(j)} - \lambda_i e_i^{(j)} = \nu \sum_{\substack{k=1\\k\neq j}}^{l} e_i^{(j)} e_i^{(k)} \sigma_{jk}; \tag{21}$$

For $1 \leq j \neq k \leq l, v_1, v_2 \in E(\overline{Q})$,

$$v_1^{(j)} \otimes v_2^{(k)} - v_2^{(k)} \otimes v_1^{(j)} = \nu \delta_{v_1\overline{v}_2}\big(1 - 2\delta(v_2 \in E(Q))\big) e^{(j)}_{t(v_1)} e^{(k)}_{h(v_1)} \sigma_{jk}. \tag{22}$$

It is possible to filter the algebra $\Pi_l^{\lambda,\nu}(Q)$ by assigning degree zero to the elements of B and degree one to those of E. One of the main results of [GaGi] is the next theorem.

Theorem 6.1 (Theorem 2.2.1 in [GaGi])**.** *Suppose that Q is a quiver whose underlying graph is an affine Dynkin diagram of type A,D or E. The canonical map $\Pi_l^{\lambda=\mathbf{0},\nu=0}(Q) \longrightarrow \mathrm{gr}(\Pi_l^{\lambda,\nu}(Q))$ is an isomorphism.*

7 Deformed enveloping quiver algebras

In this section, we introduce the algebras which will be our main objects of interest. For $i \in I(Q)$, set $\mathrm{nbh}(i) = \{j \in I | \exists\, v \in E(\overline{Q}), h(v) = i, t(v) = j\}$. Recall that, for any algebra A and elements $a, b \in A$, we set $S(a,b) = ab + ba$. We will assume that $n \geq 4$ for the rest of this paper.

Definition 7.1. *Let $\nu \in \mathbb{C}, \lambda = (\lambda_i)_{i\in I} \in \mathbb{C}^{\oplus|I|}$. The deformed enveloping quiver algebra $\mathsf{D}_n^{\lambda,\nu}(Q)$ is the algebra generated by elements $\mathsf{E}_{ab}(v), \mathsf{E}_{ab}(e)$ for any $1 \leq a, b \leq n, v \in E(\overline{Q}), e \in B$ which satisfy the following relations: The elements $\mathsf{E}_{ab}(e), 1 \leq a, b \leq n, e \in B$, generate a subalgebra isomorphic to $(\mathfrak{Ugl}_n)^{\otimes|I|} (= \mathfrak{Ugl}_n(B))$. $\forall\, 1 \leq a, b, c, d \leq n, \forall\, v \in E(\overline{Q}), \forall\, i \in I(Q)$,*

$$[\mathsf{E}_{ab}(e_i), \mathsf{E}_{cd}(v)] = \delta_{i,h(v)}\delta_{bc}\mathsf{E}_{ad}(v) - \delta_{i,t(v)}\delta_{ad}\mathsf{E}_{cb}(v) \tag{23}$$

For $a \neq b \neq c \neq a \neq d \neq c$, $v, \widehat{v} \in E(\overline{Q})$ and $\mathsf{H}_{bd}(e_{t(v)}) = \mathsf{E}_{bb}(e_{t(v)}) - \mathsf{E}_{dd}(e_{t(v)})$,

$$[\mathsf{E}_{ab}(v), \mathsf{E}_{bc}(\widehat{v})] - [\mathsf{E}_{ad}(v), \mathsf{E}_{dc}(\widehat{v})] = \frac{\nu}{2}\delta_{\widehat{v}\overline{v}}\big(1 - 2\delta(v \in E(Q))\big) S\big(\mathsf{H}_{bd}(e_{t(v)}), \mathsf{E}_{ac}(e_{h(v)})\big) \tag{24}$$

$$\sum_{i\in I(Q)} \sum_{a=1}^{n} \mathsf{E}_{aa}(e_i) \text{ is central } \forall\, i \in I(Q) \tag{25}$$

$$\begin{aligned}\sum_{\{v\in E|h(v)=i\}} [\mathsf{E}_{ab}(v), \mathsf{E}_{bc}(\overline{v})] \quad = \quad & \sum_{\{v\in E|t(v)=i\}} [\mathsf{E}_{ab}(\overline{v}), \mathsf{E}_{bc}(v)] + \left(\lambda_i - \frac{n\nu}{2}\right)\mathsf{E}_{ac}(e_i) \\ & + \frac{\nu}{2}\sum_{j,k=1}^{n} S\big([\mathsf{E}_{ab}(e_i), \mathsf{E}_{jk}(e_i)], [\mathsf{E}_{kj}(e_i), \mathsf{E}_{bc}(e_i)]\big) \\ & + \nu S\big(\mathsf{E}_{bb}(e_i), \mathsf{E}_{ac}(e_i)\big) - \frac{\nu}{2}\sum_{j\in\mathrm{nbh}(i)} S\big(\mathsf{E}_{ac}(e_i), \mathsf{E}_{bb}(e_j)\big)\end{aligned} \tag{26}$$

If $a \neq b \neq c \neq d \neq a$, then

$$[\mathsf{E}_{ab}(v), \mathsf{E}_{cd}(\widehat{v})] = \frac{\nu}{2}\delta_{\widehat{v}\overline{v}}\big(1 - 2\delta(v \in E(Q))\big) S\big(\mathsf{E}_{cb}(e_{t(v)}), \mathsf{E}_{ad}(e_{h(v)})\big) \quad \forall v, \widehat{v} \in E(\overline{Q}) \tag{27}$$

It follows from proposition 3.2 that $\mathsf{D}_n^{\lambda=\mathbf{0},\nu=0}(Q)$ is isomorphic to the enveloping algebra of a Lie algebra which properly contains $\widehat{\mathfrak{sl}}_n(\Pi(Q))$ since $\mathsf{E}_{aa}(e) \in \mathsf{D}_n^{\lambda,\nu}(Q)$ for any $1 \leq a \leq n, e \in B$. We denote this Lie algebra by $\check{\mathfrak{sl}}_n(\Pi(Q))$, so $\check{\mathfrak{sl}}_n(\Pi(Q)) = \widehat{\mathfrak{sl}}_n(\Pi(Q)) \oplus \big(\bigoplus_{i\in I} \mathbb{C}\cdot I(e_i)\big)$ where $I(e_i) \in \mathfrak{gl}_n(B_i)$ is the identity matrix and is central in $\check{\mathfrak{sl}}_n(\Pi(Q))$. More generally, $\mathsf{D}_n^{\lambda,\nu=0}(Q) = \mathfrak{U}\big(\check{\mathfrak{sl}}_n(\Pi^\lambda(Q))\big)$ with $\check{\mathfrak{sl}}_n(\Pi^\lambda(Q)) \supset \widehat{\mathfrak{sl}}_n(\Pi^\lambda(Q))$. When Q is an affine Dynkin quiver of type A,D or E, and $\lambda_i = \dim_{\mathbb{C}} N_i$ (N_i corresponds to i under the McKay correspondence, see theorem 9.1 below), we have $HC_1(\Pi^\lambda(Q)) = 0$, $HC_0(\Pi^\lambda(Q)) \cong \mathbb{C}^{\oplus(|I|-1)}$ (since $\Pi^\lambda(Q)$ is then Morita equivalent to the smash product of the first Weyl algebra with a certain finite group so that we can use the calculations in [AFLS]), hence $\widehat{\mathfrak{sl}}_n(\Pi^\lambda(Q)) \cong \mathfrak{sl}_n(\Pi^\lambda(Q))$ and $\check{\mathfrak{sl}}_n(\Pi^\lambda(Q)) = \mathfrak{gl}_n(\Pi^\lambda(Q)) \oplus \mathbb{C}\cdot I(1_B)$.

When Q satisfies the condition that $|\mathrm{nbh}(i)| = 2$ for any $i \in I(Q)$, relation (26) can be rewritten as:

$$\begin{aligned}\sum_{\{v\in E|h(v)=i\}} [\mathsf{E}_{ab}(v), \mathsf{E}_{bc}(\overline{v})] \quad &= \sum_{\{v\in E|t(v)=i\}} [\mathsf{E}_{ab}(\overline{v}), \mathsf{E}_{bc}(v)] - \frac{\nu}{2}\sum_{j\in\mathrm{nbh}(i)} S\big(\mathsf{E}_{ac}(e_i), \mathsf{E}_{bb}(e_j - e_i)\big) \\ &+ \Big(\lambda_i - \frac{n\nu}{2}\Big)\mathsf{E}_{ac}(e_i) + \frac{\nu}{2}\sum_{j,k=1}^{n} S\big([\mathsf{E}_{ab}(e_i), \mathsf{E}_{jk}(e_i)], [\mathsf{E}_{kj}(e_i), \mathsf{E}_{bc}(e_i)]\big)\end{aligned}$$

In this case, we can replace $\mathfrak{gl}_n(B)$ in definition 7.1 by its Lie subalgebra of codimension one generated by $\mathfrak{sl}_n(B)$ and $\mathsf{E}_{aa}(e_i - e_j)$ for $i \neq j, 1 \leq a \leq n$.

The algebra $\mathsf{D}_n^{\lambda,\nu}(Q)$ can be filtered by giving $\mathsf{E}_{ab}(e)$ degree zero and $\mathsf{E}_{ab}(v)$ degree one.

8 Schur-Weyl functor

In this section, we construct a functor which connects the category of modules over deformed preprojective algebras for wreath products to the category of modules over the deformed enveloping quiver algebra with the same parameters.

Let M be a right module over $\Pi_l^{\lambda,\nu}(Q)$ and $m\otimes \mathbf{u} \in M \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^n)^{\otimes l}, m \in M, \mathbf{u} \in (\mathbb{C}^n)^{\otimes l}$. We want to turn $M \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^n)^{\otimes l}$ into a left module over $\mathsf{D}_n^{\lambda,\nu}(Q)$ by letting $\mathsf{E}_{ab}(v)$ and $\mathsf{E}_{ab}(e_i)$ act on it according to the following formulas:

$$\mathsf{E}_{ab}(v)(m\otimes\mathbf{u}) = \sum_{j=1}^{l} mv^{(j)} \otimes E_{ab}^{(j)}(\mathbf{u}), \;\; \mathsf{E}_{ab}(e_i)(m\otimes\mathbf{u}) = \sum_{j=1}^{l} me_i^{(j)} \otimes E_{ab}^{(j)}(\mathbf{u}).$$

We have to verify that the relations in definition 7.1 are preserved by these operators.

For $v \in E(\overline{Q})$ and a,b,c,d all distinct, $\big([\mathsf{E}_{ab}(v), \mathsf{E}_{bc}(\widehat{v})] - [\mathsf{E}_{ad}(v), \mathsf{E}_{dc}(\widehat{v})]\big)(m\otimes\mathbf{u})$ equals

$$\begin{aligned} &= \sum_{1\leq j\neq k\leq l} m[\widehat{v}^{(j)}, v^{(k)}] \otimes \big(E_{ab}^{(k)}E_{bc}^{(j)} - E_{ad}^{(k)}E_{dc}^{(j)}\big)(\mathbf{u}) \\ &= \nu\delta_{\widehat{v}\overline{v}}\big(1 - 2\delta(v\in E(Q))\big) \sum_{1\leq j\neq k\leq l} me_{h(v)}^{(j)}e_{t(v)}^{(k)}\sigma_{jk} \otimes \big(E_{ab}^{(k)}E_{bc}^{(j)} - E_{ad}^{(k)}E_{dc}^{(j)}\big)(\mathbf{u}) \\ &= \nu\delta_{\widehat{v}\overline{v}}\big(1 - 2\delta(v\in E(Q))\big) \sum_{1\leq j\neq k\leq l} me_{h(v)}^{(j)}e_{t(v)}^{(k)} \otimes \big(E_{bb}^{(k)}E_{ac}^{(j)} - E_{dd}^{(k)}E_{ac}^{(j)}\big)(\mathbf{u}) \\ &= \frac{\nu}{2}\delta_{\widehat{v}\overline{v}}\big(1 - 2\delta(v\in E(Q))\big)S\big(\mathsf{H}_{bd}(e_{t(v)}), \mathsf{E}_{ac}(e_{h(v)})\big)(m\otimes\mathbf{u})\end{aligned}$$

and the relations (23) are checked in a similar manner.

The expression $\left(\sum_{\{v\in E|h(v)=i\}}[\mathsf{E}_{ab}(v),\mathsf{E}_{bc}(\overline{v})]-\sum_{\{v\in E|t(v)=i\}}[\mathsf{E}_{ab}(\overline{v}),\mathsf{E}_{bc}(v)]\right)(m\otimes\mathbf{u})$ equals

$$
\begin{aligned}
&= \sum_{k=1}^{l} m\left(\sum_{\{v\in E|h(v)=i\}}\overline{v}^{(k)}v^{(k)}-\sum_{\{v\in E|t(v)=i\}}v^{(k)}\overline{v}^{(k)}\right)\otimes E_{ac}^{(k)}(\mathbf{u})\\
&\quad+\sum_{\substack{j,k=1\\ j\neq k}}^{l} m\left(\sum_{\{v\in E|h(v)=i\}}[\overline{v}^{(j)},v^{(k)}]-\sum_{\{v\in E|t(v)=i\}}[v^{(j)},\overline{v}^{(k)}]\right)\otimes E_{ab}^{(k)}E_{bc}^{(j)}(\mathbf{u})\\
&= \lambda_i\sum_{k=1}^{l} me_i^{(k)}\otimes E_{ac}^{(k)}(\mathbf{u})+\nu\sum_{1\leq j\neq k\leq l} me_i^{(k)}e_i^{(j)}\sigma_{jk}\otimes E_{ac}^{(k)}(\mathbf{u})\\
&\quad-\nu\sum_{\{v\in E|h(v)=i\}}\sum_{1\leq j\neq k\leq l} me_{h(v)}^{(j)}e_{t(v)}^{(k)}\sigma_{jk}\otimes E_{ab}^{(k)}E_{bc}^{(j)}(\mathbf{u})\\
&\quad-\nu\sum_{\{v\in E|t(v)=i\}}\sum_{1\leq j\neq k\leq l} me_{h(v)}^{(k)}e_{t(v)}^{(j)}\sigma_{jk}\otimes E_{ab}^{(k)}E_{bc}^{(j)}(\mathbf{u})
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda_i\sum_{k=1}^{l} me_i^{(k)}\otimes E_{ac}^{(k)}(\mathbf{u})+\nu\sum_{d=1}^{n}\sum_{1\leq j\neq k\leq l} me_i^{(k)}e_i^{(j)}\otimes E_{ad}^{(j)}E_{dc}^{(k)}(\mathbf{u})\\
&\quad-\nu\sum_{\{v\in E|h(v)=i\}}\sum_{1\leq j\neq k\leq l} me_{h(v)}^{(j)}e_{t(v)}^{(k)}\otimes E_{bb}^{(k)}E_{ac}^{(j)}(\mathbf{u})\\
&\quad-\nu\sum_{\{v\in E|t(v)=i\}}\sum_{1\leq j\neq k\leq l} me_{h(v)}^{(k)}e_{t(v)}^{(j)}\otimes E_{bb}^{(k)}E_{ac}^{(j)}(\mathbf{u})\\
&= \left(\lambda_i-\frac{n\nu}{2}\right)\mathsf{E}_{ac}(e_i)(m\otimes\mathbf{u})+\frac{\nu}{2}\sum_{f,g=1}^{n} S\big([\mathsf{E}_{ab}(e_i),\mathsf{E}_{fg}(e_i)],[\mathsf{E}_{gf}(e_i),\mathsf{E}_{bc}(e_i)]\big)(m\otimes\mathbf{u})\\
&\quad+\nu S(\mathsf{E}_{bb}(e_i),\mathsf{E}_{ac}(e_i))(m\otimes\mathbf{u})-\frac{\nu}{2}\sum_{j\in\mathrm{nbh}(i)} S\big(\mathsf{E}_{ac}(e_i),\mathsf{E}_{bb}(e_j)\big)(m\otimes\mathbf{u})
\end{aligned}
$$

We have verified relation (26) and we now turn to (27). If $a\neq b\neq c\neq d\neq a$, then

$$
\begin{aligned}
[\mathsf{E}_{ab}(v),\mathsf{E}_{cd}(\widehat{v})](m\otimes\mathbf{u}) &= \sum_{1\leq j\neq k\leq l} m[\widehat{v}^{(j)},v^{(k)}]\otimes E_{ab}^{(k)}E_{cd}^{(j)}(\mathbf{u})\\
&= \nu\delta_{\widehat{v}\overline{v}}\big(1-2\delta(v\in E(Q))\big)\sum_{1\leq j\neq k\leq l} me_{h(v)}^{(j)}e_{t(v)}^{(k)}\sigma_{jk}\otimes E_{ab}^{(k)}E_{cd}^{(j)}(\mathbf{u})\\
&= \nu\delta_{\widehat{v}\overline{v}}\big(1-2\delta(v\in E(Q))\big)\sum_{1\leq j\neq k\leq l} me_{h(v)}^{(j)}e_{t(v)}^{(k)}\otimes E_{cb}^{(k)}E_{ad}^{(j)}(\mathbf{u})\\
&= \frac{\nu}{2}\delta_{\widehat{v}\overline{v}}\big(1-2\delta(v\in E(Q))\big)S\big(\mathsf{E}_{cb}(e_{t(v)}),\mathsf{E}_{ad}(e_{h(v)})\big)(m\otimes\mathbf{u})
\end{aligned}
$$

All these computations prove the first part of theorem 8.1 below, but first we need a couple of definitions.

Definition 8.1. *A module N over $\mathsf{D}_n^{\lambda,\nu}(Q)$ is said to be of level l if it is a (possibily infinite) direct sum of $\mathfrak{gl}_n(B_i)$-modules, each of which is a direct summand of $(\mathbb{C}^n)^{\otimes l}$, for each $i\in I(Q)$.*

Definition 8.2. *A module N over $\mathsf{D}_n^{\lambda,\nu}(Q)$ is said to be integrable if $\mathsf{E}_{ab}(e_i)$ and $\mathsf{E}_{ab}(v)$ act locally nilpotently on N for any $1\leq a\neq b\leq n$, $i\in I(Q)$, $v\in E(\overline{Q})$.*

We will denote by $\mathrm{mod}_L^{l,int} - \mathsf{D}_n^{\lambda,\nu}(Q)$ the category of integrable left modules of level l over $\mathsf{D}_n^{\lambda,\nu}(Q)$. Note that a map of right $\Pi_l^{\lambda,\nu}(Q)$-modules $M_1 \longrightarrow M_2$ induces a map $M_1 \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^n)^{\otimes l} \longrightarrow M_2 \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^n)^{\otimes l}$ of $\mathsf{D}_n^{\lambda,\nu}(Q)$-modules.

Theorem 8.1. *Let M be a right module over $\Pi_l^{\lambda,\nu}(Q)$ and set $\mathsf{SW}_l(M) = M \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^n)^{\otimes l}$. This formula defines a functor $\mathsf{SW}_l : \mathrm{mod}_R - \Pi_l^{\lambda,\nu}(Q) \longrightarrow \mathrm{mod}_L^{l,int} - \mathsf{D}_n^{\lambda,\nu}(Q)$. Furthermore, if $l+2 < n$, this functor is an equivalence of categories.*

Proof. It remains to prove the last statement, so let us assume that $l+2<n$. Let $N \in \mathrm{mod}_L^{l,int} - \mathsf{D}_n^{\lambda,\nu}(Q)$. From the classical Schur-Weyl duality, we know that $N = \mathsf{SW}_l(M)$ as a module over $\mathfrak{Ugl}_n(B)$ for some $B^{\otimes l} \rtimes S_l$-module M and $\mathsf{E}_{ab}(e_i)(m \otimes \mathbf{u}) = \sum_{j=1}^l \epsilon_i^j(m) \otimes E_{ab}^{(j)}(\mathbf{u})$ where $\epsilon_i^j \in \mathrm{End}_{\mathbb{C}}(M)$. Following the same approach as in [ChPr2, Gu3], one can show that, for any $v \in E(\overline{Q})$, there exists $\phi_j(v) \in \mathrm{End}_{\mathbb{C}}(M)$ such that $\mathsf{E}_{ab}(v)(m \otimes \mathbf{u}) = \sum_{j=1}^l \phi_j(v)(m) \otimes E_{ab}^{(j)}(\mathbf{u})$. Furthermore, since $[\mathsf{E}_{ab}(e_i), \mathsf{E}_{bc}(v)] = \delta_{i,h(v)} \mathsf{E}_{ac}(v)$, $[\mathsf{E}_{ab}(v), \mathsf{E}_{bc}(e_i)] = \delta_{i,t(v)} \mathsf{E}_{ac}(v)$ if $a \neq b \neq c \neq a$, we can show that $\epsilon_i^j \phi_k(v) = \phi_k(v) \epsilon_i^j$ if $j \neq k$, $\epsilon_i^k \phi_k(v) = \delta_{i,t(v)} \phi_k(v)$ and $\phi_k(v) \epsilon_i^k = \delta_{i,h(v)} \phi_k(v)$. This proves that M is a module over $T_{\mathsf{B}}\mathsf{E}$ and we want to show that it descends to $\Pi_l^{\lambda,\nu}(Q)$: this is similar to the proof of the Schur-Weyl duality in [Gu1, Gu2], using the other relations in definition 7.1 to deduce (21) and (22). ☐

9 Symplectic reflection algebras for wreath products

We need to recall the definition of symplectic reflection algebras for wreath products and why they are Morita equivalent to certain Gan-Ginzburg algebras via the McKay correspondence.

The definition of a symplectic reflection algebra depends on two parameters: $t \in \mathbb{C}$ and $\mathbf{c} = \kappa \cdot \mathrm{id} + \sum_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} c_\gamma \gamma \in \mathsf{Z}\Gamma$, which is an element in the center $\mathsf{Z}\Gamma$ of $\mathbb{C}[\Gamma]$. We have adapted the definition of the symplectic reflection algebra $\mathsf{H}_{t,\mathbf{c}}(\Gamma_l)$ from [GaGi]. For $\gamma \in \Gamma$, we write γ_k for $(\mathrm{id}, \ldots, \mathrm{id}, \gamma, \mathrm{id}, \ldots, \mathrm{id}) \in \Gamma_l = \Gamma^{\times l} \rtimes S_l$ where γ is in the k^{th} position. Let $U \cong U_k \cong \mathbb{C}^2, 1 \leq k \leq l$ be the two-dimensional symplectic plane with the standard symplectic form ω and set $U^l = \oplus_{k=1}^l U_k$. For each $1 \leq k \leq l$, choose a basis $x_k, y_k \in U_k$ such that $\omega(x_k, y_k) = 1$. Note that Γ_l acts on U^l.

Definition 9.1. *The symplectic reflection algebra $\mathsf{H}_{t,\mathbf{c}}(\Gamma_l)$ is defined as the algebra generated by the vectors in U^l and by $g \in \Gamma_l$ with the relations:*

$$g \cdot x_k \cdot g^{-1} = g(x_k), \;\; g \cdot y_k \cdot g^{-1} = g(y_k), \; k = 1, \ldots, l, \; \forall\, g \in \Gamma_l \tag{28}$$

$$[x_k, y_k] = t + \frac{\kappa}{2} \sum_{\substack{j=1 \\ j \neq k}}^{l} \sum_{\gamma \in \Gamma} \sigma_{jk} \gamma_k \gamma_j^{-1} + \sum_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} c_\gamma \gamma_k, \;\; k = 1, \ldots, l \tag{29}$$

For $1 \leq j \neq k \leq l$ and any $u_j \in U_j, v_k \in U_k$:

$$[u_j, v_k] = -\frac{\kappa}{2} \sum_{\gamma \in \Gamma} \omega(\gamma(u), v) \sigma_{jk} \gamma_j \gamma_k^{-1} \tag{30}$$

Remark 9.1. *When $\Gamma \cong \mathbb{Z}/d\mathbb{Z}$, $\mathsf{H}_{t,\mathbf{c}}(\Gamma_l)$ is a rational Cherednik algebra [GGOR].*

Fix $\Gamma \subset SL_2(\mathbb{C})$ and let $\mathrm{Irr}(\Gamma)$ be the set of its irreducible representations. One can define a graph $G(\Gamma)$ with vertices indexed by $\mathrm{Irr}(\Gamma)$ and with one edge between vertices N_1 and N_2 if $\mathrm{Hom}_\Gamma(N_1, N_2 \otimes \mathbb{C}^2) \neq 0$, and two edges if $\Gamma = \mathbb{Z}/2\mathbb{Z}$.

Theorem 9.1 (McKay correspondence)**.** *The map $\Gamma \longrightarrow G(\Gamma)$ establishes a bijection between isomorphism classes of finite subgroups of $SL_2(\mathbb{C})$ and affine Dynkin diagrams of type A, D, E.*

Let N_i be the irreducible representation of Γ corresponding to $i \in I(G(\Gamma))$. Let $f_N \in \mathbb{C}[\Gamma]$ be an idempotent such that $\mathbb{C}[\Gamma]f_N \cong N$ and set $f = \sum_{N\in \mathrm{Irr}(\Gamma)} f_N$. In the next section, we will obtain a result analogous to the following one, although weaker.

Theorem 9.2 (Theorem 3.5.2 in [GaGi], theorem 3.4 in [CBHo])**.** *Let Q be a quiver with underlying graph $G(\Gamma)$. Let λ_N be the trace of $t + \sum_{\gamma\neq \mathrm{id}} c_\gamma \gamma$ on N and set $\nu = \frac{\kappa|\Gamma|}{2}$. Then there is an isomorphism of algebras $\Pi_l^{\lambda,\nu}(Q) \stackrel{\sim}{\longrightarrow} f^{\otimes l}\mathsf{H}_{t,\mathbf{c}}(\Gamma_l)f^{\otimes l}$.*

This theorem implies that $\Pi_l^{\lambda,\nu}(Q)$ and $\mathsf{H}_{t,\mathbf{c}}(\Gamma_l)$ are Morita equivalent. The proof of this theorem is based on the next lemma. Let $C = U \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]$ and view it as a bimodule over $\mathbb{C}[\Gamma]$ where the left action is the diagonal one and the right action is simply right multiplication on the second factor. Set $f_i = f_{N_i}$ for $i \in I(G(\Gamma))$.

Lemma 9.1 (Lemma 3.3 in [CBHo])**.** *Suppose that the underlying graph of Q is $G(\Gamma)$. To each arrow $v \in E(Q)$, one can associate elements $\theta_v \in f_{t(v)}Cf_{h(v)}, \phi_v \in f_{h(v)}Cf_{t(v)}$ such that, for any $i \in I(Q)$,*

$$\sum_{\{v\in E(Q),h(v)=i\}} \phi_v\theta_v - \sum_{\{v\in E(Q),t(v)=i\}} \theta_v\phi_v = |N_i|f_i(xy-yx) \quad (\omega(x,y)=1, x,y\in U)$$

The isomorphism in theorem 9.2 is given by $e_i^{(k)} \mapsto f^{\otimes l} f_i^{(k)} f^{\otimes l}$, $\overline{v}^{(k)} \mapsto f^{\otimes l}\phi_v^{(k)} f^{\otimes l}$ and $v^{(k)} \mapsto f^{\otimes l}\theta_v^{(k)} f^{\otimes l}$. We write ϕ_v, θ_v in the form $\phi_v = f_{h(v)}\varphi_v f_{t(v)}$, $\theta_v = f_{t(v)}\vartheta_v f_{h(v)}$ with $\varphi_v, \vartheta_v \in C$.

10 A relation between deformed enveloping quiver algebras and Γ-DDCA

In this section, we will assume that Q is a quiver whose underlying graph is of affine Dynkin type A,D or E and related to the group Γ via the McKay correspondence (theorem 9.1). We will start by recalling the definition of Γ-DDCA from [Gu3]. In view of theorem 9.2, one could be led to conjecture that it is possible to realize $\mathsf{D}_n^{\lambda,\nu}(Q)$ as a subalgebra of an algebra slightly larger than $\mathsf{D}_n^{\beta,\mathbf{b}}(\Gamma)$, but this does not seem to be possible. Proposition 10.1 below gives us one relation between $\mathsf{D}_n^{\lambda,\nu}(Q)$ and $\mathsf{D}_n^{\beta,\mathbf{b}}(\Gamma)$.

We need an algebra slightly larger than the one studied in [Gu3] because, as defined here, the degree zero part of $\mathsf{D}_n^{\beta,\mathbf{b}}(\Gamma)$ (with respect to its natural filtration) is $\mathfrak{U}(\mathfrak{gl}_n(\mathbb{C}[\Gamma]))$ instead of $\mathfrak{U}(\overline{\mathfrak{gl}}_n(\mathbb{C}[\Gamma]))$. (The difference is simply $\mathfrak{gl}_n(\mathbb{C}[\Gamma]) \cong \overline{\mathfrak{gl}}_n(\mathbb{C}[\Gamma]) \oplus \mathbb{C}\cdot\mathrm{Id}$.)

Definition 10.1. *The Γ-deformed double current algebra $\mathsf{D}_n^{\beta,\mathbf{b}}(\Gamma)$ with parameters $\beta \in \mathbb{C}, \mathbf{b} \in Z\Gamma, \mathbf{b} = \widetilde{\lambda}\,\mathrm{id} + \sum_{\gamma\in\Gamma\setminus\{\mathrm{id}\}} b_\gamma\gamma$ is the algebra generated by the elements of $\mathfrak{gl}_n(\mathbb{C}[\Gamma]), \mathsf{E}_{ab}(t_1u+t_2v)$ for $1 \le a \neq b \le n, t_1, t_2 \in \mathbb{C}, u, v \in U$ which satisfy $\mathsf{E}_{ab}(t_1u+t_2v) = t_1\mathsf{E}_{ab}(u) + t_2\mathsf{E}_{ab}(v)$, $\mathrm{Id} \in \mathfrak{gl}_n(\mathbb{C}[\Gamma])$ is central in $\mathsf{D}_n^{\beta,\mathbf{b}}(\Gamma)$, and the following relations hold:*
If $a \neq b \neq c \neq a \neq d \neq c$,

$$[\mathsf{E}_{ab}(\gamma), \mathsf{E}_{bc}(u)] = [\mathsf{E}_{ad}(\gamma(u)), \mathsf{E}_{dc}(\gamma)], \quad [\mathsf{E}_{bb}(\gamma), \mathsf{E}_{bc}(u)] = [\mathsf{E}_{bc}(\gamma(u)), \mathsf{E}_{cc}(\gamma)] \tag{31}$$

$$
\begin{aligned}
[\mathsf{E}_{ab}(v), \mathsf{E}_{bc}(u)] &= [\mathsf{E}_{ad}(u), \mathsf{E}_{dc}(v)] + \omega(u,v)\mathsf{E}_{ac}\left(\mathbf{b}+\beta\right) + \frac{\widetilde{\lambda}}{8}\omega(u,v)\sum_{\gamma\in\Gamma}\sum_{j,k=1}^{n} \\
&\Big(S\big([\mathsf{E}_{ab}(\gamma^{-1}), \mathsf{E}_{kj}], [\mathsf{E}_{jk}, \mathsf{E}_{bc}(\gamma)]\big) + S\big([\mathsf{E}_{ad}(\gamma), \mathsf{E}_{kj}], [\mathsf{E}_{jk}, \mathsf{E}_{dc}(\gamma^{-1})]\big)\Big) \\
&-\frac{\widetilde{\lambda}}{2}\sum_{\gamma\in\Gamma}\big(\omega(\gamma(u),v) - \omega(u,v)\big)\big(\mathsf{E}_{bb}(\gamma^{-1})\mathsf{E}_{ac}(\gamma) + \mathsf{E}_{dd}(\gamma)\mathsf{E}_{ac}(\gamma^{-1})\big) \qquad (32)
\end{aligned}
$$

If $a \neq b \neq c \neq d \neq a$, $[\mathsf{E}_{bb}(\gamma), \mathsf{E}_{ac}(u)] = 0 = [\mathsf{E}_{ab}(\gamma), \mathsf{E}_{cd}(u)]$ and

$$
[\mathsf{E}_{ab}(u), \mathsf{E}_{cd}(v)] = -\frac{\widetilde{\lambda}}{4}\sum_{\gamma\in\Gamma}\omega(\gamma(u),v)S\big(\mathsf{E}_{ad}(\gamma^{-1}), \mathsf{E}_{cb}(\gamma)\big) \qquad (33)
$$

Lemma 9.1 entices us to consider the following algebra.

Definition 10.2. *We define $\mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma)$ to be the subalgebra of $\mathsf{D}_n^{\beta,\mathbf{b}}(\Gamma)$ generated by $\mathbf{E}_{ab}(f_i) = \mathsf{E}_{ab}(f_i)\,\forall\, 1 \leq a, b \leq n$ and by the elements $\mathbf{E}_{ab}(v) = \big[\mathsf{E}_{aa}(f), [\mathsf{E}_{ab}(\vartheta_v), \mathsf{E}_{bb}(f)]\big]$, $\mathbf{E}_{ab}(\overline{v}) = \big[\mathsf{E}_{aa}(f), [\mathsf{E}_{ab}(\varphi_v), \mathsf{E}_{bb}(f)]\big]$ for each $v \in E(Q), 1 \leq a \neq b \leq n$. (The idempotents f_i were defined in the previous section.)*

In [Gu3], we constructed a Schur-Weyl functor relating $\mathsf{H}_{t,c}(\Gamma_l)$ and $\mathsf{D}_n^{\beta,\mathbf{b}}(\Gamma)$, so we can put on the space $\mathsf{V}^l = \mathsf{H}_{t,c}(\Gamma_l)\otimes_{\mathbb{C}[S_l]}(\mathbb{C}^n)^{\otimes l}$ a structure of right module over $\mathsf{D}_n^{\beta,\mathbf{b}}(\Gamma)$ when $\beta = t - \frac{n\kappa|\Gamma|}{4} - \kappa, b_\gamma = c_{\gamma^{-1}}$ and $\widetilde{\lambda} = \kappa$.

We can view $f^{\otimes l}\mathsf{H}_{t,\mathbf{c}}(\Gamma_l)f^{\otimes l} \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^n)^{\otimes l}$ as a subspace of V^l and, as such, it is stabilized under the action of the subalgebra $\mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma)$. Indeed, the generators of $\mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma)$ act in the following way, for $m \in f^{\otimes l}\mathsf{H}_{t,\mathbf{c}}(\Gamma_l)f^{\otimes l}, \mathbf{u} \in (\mathbb{C}^n)^{\otimes l}, v \in E(Q)$:

$$
\mathbf{E}_{ab}(v)(m\otimes\mathbf{u}) = \sum_{k=1}^{l} m\theta_v^{(k)} \otimes E_{ab}^{(k)}(\mathbf{u}), \;\; \mathbf{E}_{ab}(\overline{v})(m\otimes\mathbf{u}) = \sum_{k=1}^{l} m\phi_v^{(k)} \otimes E_{ab}^{(k)}(\mathbf{u}).
$$

Let $\Psi_l : \mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma) \longrightarrow \mathrm{End}_{\mathbb{C}}\big(f^{\otimes l}\mathsf{H}_{t,\mathbf{c}}(\Gamma_l)f^{\otimes l} \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^n)^{\otimes l}\big)$ be the algebra map coming from this module structure. Let $\mathbf{J}_n^{\beta,\mathbf{b}} = \{x \in \mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma) | \Psi_l(x) = 0\,\forall\, l \in \mathbb{Z}_{\geq 1}\}$.

Proposition 10.1. *Suppose that $\beta = t - \frac{n\kappa|\Gamma|}{4} - \kappa$, $b_\gamma = c_{\gamma^{-1}}$, $\widetilde{\lambda} = \kappa$, $\nu = \frac{\kappa|\Gamma|}{2}$ and λ_j is the trace of $t + \sum_{\gamma\neq\mathrm{id}} c_\gamma\gamma$ on N_j. The algebra $\mathsf{D}_n^{\lambda,\nu}(Q)$ maps onto the quotient $\mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma)/\mathbf{J}_n^{\beta,\mathbf{b}}$.*

Proof. We want to define a map $\eta : \mathsf{D}_n^{\lambda,\nu}(Q) \longrightarrow \mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma)/\mathbf{J}_n^{\beta,\mathbf{b}}$ by $\mathsf{E}_{ab}(v) \mapsto \mathbf{E}_{ab}(v)$ for $v \in E(\overline{Q}), 1 \leq a \neq b \leq n$ and $\mathsf{E}_{ab}(e_i) \mapsto \mathbf{E}_{ab}(f_i)$ for any $1 \leq a, b \leq n$. We have to jutisfy why it respects the relations in definition 7.1. Let $\Phi_l : \mathsf{D}_n^{\lambda,\nu}(Q) \longrightarrow \mathrm{End}_{\mathbb{C}}\big(\Pi_l^{\lambda,\nu}(Q) \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^n)^{\otimes l}\big)$ be the algebra map coming from the module structure obtained in section 8 and note that $\Phi_l = \Psi_l \circ \eta\ \forall l \in \mathbb{Z}_{\geq 1}$ after the identification $\Pi_l^{\lambda,\nu}(Q) \xrightarrow{\sim} f^{\otimes l}\mathsf{H}_{t,\mathbf{c}}(\Gamma_l)^{\otimes l}f^{\otimes l}$ given by theorem 9.2 - see the formulas at the end of the previous section. From theorem 9.2 and the computations done in section 8, it follows that $\Psi_l(\mathbf{E}_{ab}(e_i))$ and $\Psi_l(\mathbf{E}(v))$ for all $i \in I(Q), v \in E(\overline{Q}), l \in \mathbb{Z}_{\geq 1}$ satisfy the relations in definition 7.1. Therefore, the same is true for $\eta(\mathsf{E}_{ab}(e_i)), \eta(\mathsf{E}_{ab}(v))$ in the quotient $\mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma)/\mathbf{J}_n^{\beta,\mathbf{b}}$ because of the way the ideal $\mathbf{J}_n^{\beta,\mathbf{b}}$ is defined. This proves that η is well defined. □

The case $\Gamma = \mathbb{Z}/d\mathbb{Z}$ and Q a cyclic quiver on d vertices is simpler to understand, for then $f = 1$, so that $\Pi_l^{\lambda,\nu}(Q) \cong \mathsf{H}_{t,\mathbf{c}}(\Gamma_l)$ and, moreover, $\mathsf{D}_n^{\beta,\mathbf{b}}(\Gamma) \cong \mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma) \cong \mathsf{D}_n^{\lambda,\nu}(Q)$. In [Gu3], in the case when $\Gamma = \mathbb{Z}/d\mathbb{Z}$, a second presentation of $\mathsf{D}_n^{\beta,\mathbf{b}}(\Gamma)$ was given which involves an infinite number of generators, but simpler relations. It thus provides another realization of $\mathsf{D}_n^{\lambda,\nu}(Q)$ when Q is the cyclic quiver with d vertices.

Let $F^1_\bullet$ be the filtration on $\mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma)$ inherited from the filtration on $\mathsf{D}_n^{\beta,\mathbf{b}}(\Gamma)$ considered in [Gu3] and let $F^2_\bullet$ be the one obtained by giving $\mathbf{E}_{ab}(f_i)$ degree zero and $\mathbf{E}_{ab}(v)$ degree one. Then $F^2_k\big(\mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma)\big) \subset F^1_k\big(\mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma)\big)$ for any $k \in \mathbb{Z}_{\geq 0}$, but it is not clear if this inclusion is an equality. We have a natural map $\mathbf{D}_n^{\beta=0,\mathbf{b}=\mathbf{0}}(\Gamma) \longrightarrow gr_{F^1}(\mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma))$ and an inclusion $gr_{F^1}\big(\mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma)\big) \subset gr_{F^1}\big(\mathsf{D}_n^{\beta,\mathbf{b}}(\Gamma)\big)$, so $gr_{F^1}(\mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma))$ is isomorphic to a certain subalgebra of $\mathsf{D}_n^{\beta=0,\mathbf{b}=\mathbf{0}}(\Gamma)$ [Gu3] which contains $\mathbf{D}_n^{\beta=0,\mathbf{b}=\mathbf{0}}(\Gamma)$.

We also have two filtrations on the quotient ring $\mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma)/\mathbf{J}_n^{\beta,\mathbf{b}}$; this family of algebras does not provide a flat deformation of $\mathbf{D}_n^{\beta=0,\mathbf{b}=\mathbf{0}}(\Gamma)/\mathbf{J}_n^{\beta=0,\mathbf{b}=\mathbf{0}}$. When $\beta=0, \mathbf{b}=\mathbf{0}$, $\mathbf{J}_n^{\beta,\mathbf{b}}$ is quite large, containing the center of $\widehat{\mathfrak{sl}}_n(\Pi(Q))$ (note that $\widehat{\mathfrak{sl}}_n(\Pi(Q)) = \widetilde{\mathfrak{sl}}_n(f(\mathbb{C}[u,v]\rtimes\Gamma)f) \subset \mathbf{D}_n^{\beta=0,\mathbf{b}=\mathbf{0}}(\Gamma) \subset \mathsf{D}_n^{\beta=0,\mathbf{b}=\mathbf{0}}(\Gamma) = \mathfrak{U}\check{\mathfrak{sl}}_n(\mathbb{C}[u,v]\rtimes\Gamma)$), but when $\kappa=0$ and $\lambda_i = \dim_{\mathbb{C}} N_i$ for any $i \in I(Q)$ (and corresponding $\beta, \mathbf{b}$ as in proposition 10.1), it is generated by $\sum_{v\in E(Q)}\sum_{k=1}^n [\mathbf{E}_{kk}(v), \mathbf{E}_{kk}(\overline{v})] - \sum_{i\in I}\lambda_i \mathbf{I}(f_i)$, where $\mathbf{E}_{kk}(v) = [\mathbf{E}_{k,k+1}(v), \mathbf{E}_{k+1,k}(e_{t(v)})]$. Actually, in the latter case, the map η above induces an isomorphism between $\mathbf{D}_n^{\beta,\mathbf{b}}(\Gamma)/\mathbf{J}_n^{\beta,\mathbf{b}}$ and the quotient of $\mathsf{D}_n^{\lambda,\nu}(Q)$ by the two-sided ideal generated by $\sum_{v\in E(Q)}\sum_{k=1}^n [\mathsf{E}_{kk}(v), \mathsf{E}_{kk}(\overline{v})] - \sum_{i\in I}\lambda_i \mathsf{I}(e_i)$. (When $\nu = 0, \lambda_N = \dim_{\mathbb{C}} N, \beta = 1, b_\gamma = 0$, these are isomorphic to a subalgebra of the enveloping algebra of $\mathfrak{gl}_n(A_1 \rtimes \Gamma)$.)

Finally, we note that, for any values of λ, ν, the following element is always in the kernel of the algebra map $\mathsf{D}_n^{\lambda,\nu}(Q) \longrightarrow \mathrm{End}_{\mathbb{C}}(\Pi_l^{\lambda,\nu}(Q) \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^n)^{\otimes l})$ from section 8:

$$\sum_{v\in E(Q)}\sum_{a=1}^n \Big([\mathsf{E}_{aa}(v), \mathsf{E}_{aa}(\overline{v})] + \frac{\nu}{2} S\big(\mathsf{E}_{aa}(e_{h(v)}), \mathsf{E}_{aa}(e_{t(v)})\big)\Big) - \sum_{i\in I(Q)} \Big(\overline{\lambda}_i \mathsf{I}(e_i) - \frac{\nu}{2}\sum_{a,b=1}^n S\big(\mathsf{E}_{ab}(e_i), \mathsf{E}_{ba}(e_i)\big)\Big).$$

This means that corollary 9.1 in [Gu3] does not hold for $\mathsf{D}_n^{\lambda,\nu}(Q)$.

11 Reflection functors

In [CBHo], the authors introduced reflection functors for deformed preprojective algebras of quivers: these provide equivalences between categories of modules over $\Pi^\lambda(Q)$ for values of λ related by a reflection of the Weyl group of the quiver. This was inspired by the classical work [BGP]. Their construction was generalized to $\Pi_l^{\lambda,\nu}(Q)$ in [Ga], and a second, more natural approach was given in [EGGO]. In this section, we construct reflection functors $R_{i_0,l}$ for the algebras $\mathsf{D}_n^{\lambda,\nu}(Q)$ where $i_0 \in I(Q)$ and $l \in \mathbb{Z}_{\geq 1}$.

The Ringel form of Q is the bilinear form on $\mathbb{Z}^{\oplus|I|}$ given by $\langle\alpha,\beta\rangle = \sum_{i\in I(Q)}\alpha_i\beta_i - \sum_{v\in E(Q)}\alpha_{t(v)}\beta_{h(v)}$ where $\alpha = (\alpha_i)_{i\in I(Q)}, \beta = (\beta_i)_{i\in I(Q)}$. Its symmetrization is given by $(\alpha,\beta) = \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle$. We write $\epsilon_i \in \mathbb{Z}^{\oplus|I|}$ for the coordinate vector corresponding to the vertex $i \in I(Q)$. If there is no loop at the vertex i, then we have a (simple) reflection $s_i : \mathbb{Z}^{\oplus|I|} \longrightarrow \mathbb{Z}^{\oplus|I|}$ defined by $s_i(\alpha) = \alpha - (\alpha,\epsilon_i)\epsilon_i$. The Weyl group W of the quiver Q is the group of automorphisms of $\mathbb{Z}^{\oplus|I|}$ generated by all these simple reflections. What will be more important for us is the dual reflection $r_i : B \longrightarrow B$ given by $r_i(\lambda) = \sum_{j\in I(Q)}(\lambda_j - (\epsilon_i,\epsilon_j)\lambda_i)e_j$ where $\lambda = \sum_{j\in I(Q)}\lambda_j e_j \in B$.

Fix a vertex $i_0 \in I(Q)$. We will assume that i_0 is a sink, so that no arrow in Q has a tail at i_0. This does not result in any loss of generality since $\mathsf{D}_n^{\lambda,\nu}(Q)$ does not depend on the orientation of Q: if Q_1 is obtained from Q by reversing an arrow v_1, then an isomorphism $\mathsf{D}_n^{\lambda,\nu}(Q) \xrightarrow{\sim} \mathsf{D}_n^{\lambda,\nu}(Q_1)$ is given by $\mathsf{E}_{ab}(v_1) \mapsto -\mathsf{E}_{ab}(v_1)$, $\mathsf{E}_{ab}(\overline{v}_1) \mapsto \mathsf{E}_{ab}(\overline{v}_1)$ and $\mathsf{E}_{ab}(v) \mapsto \mathsf{E}_{ab}(v)$ for $v \neq v_1, \overline{v}_1$. Let us set $\check{e}_{i_0} = \sum_{i\neq i_0} e_i$.

We need first to understand better $\check{e}_{i_0}\Pi_{l=1}^{\lambda}(Q)\check{e}_{i_0}$ and we start by looking at $\check{e}_{i_0}\mathbb{C}(\overline{Q})\check{e}_{i_0}$, $\mathbb{C}(\overline{Q})$ being the path algebra of $\overline{Q}$. The latter is the algebra of all paths in $\overline{Q}$ which starts and end at a vertex different from i_0. Let Q^0 be the quiver whose vertex set is $I(Q)\setminus\{i_0\}$ and whose arrows are $v\in E(\overline{Q})$ with $h(v)\neq i_0\neq t(v)$ and $v_{i_1,i_2}=v_{i_1}\otimes\overline{v}_{i_2}$ where v_{i_j} with $i_j\in \mathrm{nbh}(i_0)$ is an arrow in $E(Q)$ with $t(v_{i_j})=i_j, h(v_{i_j})=i_0$. In order to simplify the notation, we will assume that between two vertices in $I(Q)$ there is at most one arrow in $E(Q)$ and that $\mathrm{nbh}(i_0)=\{i_1,i_2,\ldots,i_M\}$ for some $M\in\mathbb{Z}_{\geq 1}$. The quiver Q^0 has a loop $\ell_i=v_{i,i}$ at every vertex $i\in\mathrm{nbh}(i_0)$ and $\check{e}_{i_0}\mathbb{C}(\overline{Q})\check{e}_{i_0}=\mathbb{C}(Q^0)$. Using this identification, the algebra $\check{e}_{i_0}\Pi^{\lambda}_{l=1}(Q)\check{e}_{i_0}$ can be viewed as the quotient of $\mathbb{C}(Q^0)$ by the ideal generated by the elements $\rho_i-\lambda_i e_i$ where ρ_i is as in definition 6.1 when $i\notin\mathrm{nbh}(i_0)$, whereas

$$\rho_i=\sum_{\{v\in E(Q)|h(v)=i\}}\overline{v}\otimes v-\sum_{\{v\in E(Q)|t(v)=i,h(v)\neq i_0\}}v\otimes\overline{v}-\ell_i\quad\text{if } i\in\mathrm{nbh}(i_0),$$

and generated also by the elements $\sum_{i\in\mathrm{nbh}(i_0)}v_{i_1,i}v_{i,i_2}-\lambda_{i_0}v_{i_1,i_2}$ for $i_1,i_2\in\mathrm{nbh}(i_0)$.

Let us now look at the higher rank case, that is, $l>1$. The algebra $\check{e}_{i_0}^{\otimes l}(T_{\mathsf{B}}\mathsf{E}\rtimes S_l)\check{e}_{i_0}^{\otimes l}$, which equals $(\check{e}_{i_0}^{\otimes l}T_{\mathsf{B}}\mathsf{E}\check{e}_{i_0}^{\otimes l})\rtimes S_l$, is generated by the idempotents $e_i^{(j)}$ with $i\neq i_0$, by $v^{(j)},\overline{v}^{(j)}$ if $h(v),t(v)\neq i_0$, by $v^{(j)}_{i_1,i_2}=v^{(j)}_{i_1}\otimes\overline{v}^{(j)}_{i_2}$ with $i_1,i_2\in\mathrm{nbh}(i_0)$ and by $g\in S_l$. (If $i_1=i_2=i\in\mathrm{nbh}(i_0)$, $v^{(j)}_{i_1,i_2}=\ell^{(j)}_i$.)

In view of proposition 11.2 below taken from [EGGO], the next proposition is most probably already known to some people, although it is not stated explicitly in *loc. cit.*. This will be useful later in this section and motivates our construction of $\mathsf{D}^{\lambda,\nu}_{n,i_0}(Q)$, following the ideas of section 3.

Proposition 11.1. *The algebra $\check{e}_{i_0}^{\otimes l}\Pi_l^{\lambda,\nu}(Q)\check{e}_{i_0}^{\otimes l}$ is the quotient of $\check{e}_{i_0}^{\otimes l}(T_{\mathsf{B}}\mathsf{E}\rtimes S_l)\check{e}_{i_0}^{\otimes l}$ by the ideal generated by:*

$$\sum_{\{v\in E(Q)|h(v)=i\}}\overline{v}^{(j)}_i\otimes v^{(j)}_i-\sum_{\{v\in E(Q)|t(v)=i,h(v)\neq i_0\}}v^{(j)}_i\otimes\overline{v}^{(j)}_i-\delta(i\in\mathrm{nbh}(i_0))\ell^{(j)}_i-\lambda_i e^{(j)}_i-\nu\sum_{\substack{k=1\\k\neq j}}^{l}e^{(j)}_i e^{(k)}_i\sigma_{jk}\tag{34}$$

$$\sum_{i\in\mathrm{nbh}(i_0)}v^{(j)}_{i_1,i}\otimes v^{(j)}_{i,i_2}-\lambda_{i_0}v^{(j)}_{i_1i_2}\ \textit{for } i_1,i_2\in\mathrm{nbh}(i_0).\ (i_1=i_2\ \textit{is allowed, } v^{(j)}_{i_1i_1}=\ell^{(j)}_{i_1}.)\tag{35}$$

For $1\leq j\neq k\leq l$, $h(v_1),h(v_2),t(v_1),t(v_2)\neq i_0$ and $i_1,i_2,i_3,i_4\in\mathrm{nbh}(i_0)$,

$$v^{(j)}_1\otimes v^{(k)}_2-v^{(k)}_2\otimes v^{(j)}_1-\nu\delta_{v_1,\overline{v_2}}\big(1-2\delta(v_2\in E(Q))\big)e^{(j)}_{t(v_1)}e^{(k)}_{h(v_1)}\sigma_{jk}\tag{36}$$

$$v^{(j)}_{i_1i_2}\otimes v^{(k)}_2-v^{(k)}_2\otimes v^{(j)}_{i_1i_2},\ \ell^{(j)}_{i_1}\otimes v^{(k)}_2-v^{(k)}_2\otimes\ell^{(j)}_{i_1}\ \textit{if } h(v_2),t(v_2)\neq i_0\tag{37}$$

$$v^{(j)}_{i_1i_3}\otimes v^{(k)}_{i_2i_4}-v^{(k)}_{i_2i_4}\otimes v^{(j)}_{i_1i_3}-\nu\big(\delta_{i_1i_4}v^{(k)}_{i_2i_3}e^{(j)}_{i_1}-\delta_{i_2i_3}v^{(j)}_{i_1i_4}e^{(k)}_{i_3}\big)\sigma_{jk}\tag{38}$$

In the last expression, $i_1=i_3$ and $i_2=i_4$ are allowed.

Proof. Let $J_l^{\lambda,\nu}$ be the defining ideal of $\Pi_l^{\lambda,\nu}(Q)$ as a quotient of $T_{\mathsf{B}}\mathsf{E}\rtimes S_l$, that is, $\Pi_l^{\lambda,\nu}(Q)=T_{\mathsf{B}}\mathsf{E}\rtimes S_l/J_l^{\lambda,\nu}$. It is described in definition 6.1. We must identify $J_l^{\lambda,\nu}\cap\check{e}_{i_0}^{\otimes l}T_{\mathsf{B}}\mathsf{E}\rtimes S_l\check{e}_{i_0}^{\otimes l}$.

Let us see from where relation (38) comes. Relation (22) with $v_1=\overline{v}_{i_3},v_2=v_{i_2}$ says that $\overline{v}^{(j)}_{i_3}v^{(k)}_{i_2}-v^{(k)}_{i_2}\overline{v}^{(j)}_{i_3}=-\nu\delta_{i_2i_3}e^{(j)}_{i_0}e^{(k)}_{i_3}\sigma_{jk}$. Multiplying on the left by $v^{(j)}_{i_1}$ and on the right by $\overline{v}^{(k)}_{i_4}$ yields

$$\begin{aligned}
v^{(j)}_{i_1i_3}v^{(k)}_{i_2i_4}-v^{(j)}_{i_1}v^{(k)}_{i_2}\overline{v}^{(j)}_{i_3}\overline{v}^{(k)}_{i_4}&=-\nu\delta_{i_2i_3}v^{(j)}_{i_1i_4}e^{(k)}_{i_3}\sigma_{jk}\\
\iff\quad v^{(j)}_{i_1i_3}v^{(k)}_{i_2i_4}-v^{(k)}_{i_2}v^{(j)}_{i_1}\overline{v}^{(k)}_{i_4}\overline{v}^{(j)}_{i_3}&=-\nu\delta_{i_2i_3}v^{(j)}_{i_1i_4}e^{(k)}_{i_3}\sigma_{jk}\\
\iff\quad v^{(j)}_{i_1i_3}v^{(k)}_{i_2i_4}-v^{(k)}_{i_2i_4}v^{(j)}_{i_1i_3}-\nu\delta_{i_1i_4}v^{(k)}_{i_2}e^{(j)}_{i_1}e^{(k)}_{i_0}\sigma_{jk}\overline{v}^{(j)}_{i_3}&=-\nu\delta_{i_2i_3}v^{(j)}_{i_1i_4}e^{(k)}_{i_3}\sigma_{jk}\\
\iff\quad \nu\delta_{i_1i_4}v^{(k)}_{i_2i_3}e^{(j)}_{i_1}\sigma_{jk}-\nu\delta_{i_2i_3}v^{(j)}_{i_1i_4}e^{(k)}_{i_3}\sigma_{jk}&=v^{(j)}_{i_1i_3}v^{(k)}_{i_2i_4}-v^{(k)}_{i_2i_4}v^{(j)}_{i_1i_3}
\end{aligned}$$

Relation (35) is the other one which requires some explanations. We start with relation (21) in the case $i=i_0$ and multiply it on the left by $v_{i_3}^{(k)}v_{i_1}^{(j)}$ and on the right by $\overline{v}_{i_2}^{(j)}\overline{v}_{i_4}^{(k)}$, which yields:

$$\sum_{i\in \mathrm{nbh}(i_0)} v_{i_3}^{(k)}v_{i_1,i}^{(j)}v_{i,i_2}^{(j)}\overline{v}_{i_4}^{(k)} - \lambda_{i_0}v_{i_3}^{(k)}v_{i_1i_2}^{(j)}\overline{v}_{i_4}^{(k)} - \nu v_{i_3}^{(k)}v_{i_1i_4}^{(j)}\overline{v}_{i_2}^{(k)}\sigma_{jk} - \nu\sum_{\substack{m=1\\m\neq j,k}}^{l} v_{i_3}^{(k)}v_{i_1}^{(j)}\overline{v}_{i_2}^{(m)}\overline{v}_{i_4}^{(k)}\sigma_{jm}$$

$$\begin{aligned}
= &\sum_{\substack{i\in \mathrm{nbh}(i_0)\\ i\neq i_3}} v_{i_1,i}^{(j)}v_{i}^{(j)}v_{i_3}^{(k)}\overline{v}_{i_2}^{(j)}\overline{v}_{i_4}^{(k)} + v_{i_1}^{(j)}v_{i_3}^{(k)}\overline{v}_{i_3}^{(j)}v_{i_3,i_2}^{(j)}\overline{v}_{i_4}^{(k)} - \lambda_{i_0}v_{i_3}^{(k)}v_{i_1i_2}^{(j)}\overline{v}_{i_4}^{(k)} - \nu v_{i_1}^{(j)}v_{i_3}^{(k)}\overline{v}_{i_4}^{(j)}\overline{v}_{i_2}^{(k)}\sigma_{jk}\\
&-\nu^2\delta_{i_3=i_2}\sum_{\substack{m=1\\m\neq j,k}}^{l} v_{i_1}^{(j)}e_{i_3}^{(k)}e_{i_0}^{(m)}\sigma_{km}\overline{v}_{i_4}^{(k)}\sigma_{jm} - \nu\sum_{\substack{m=1\\m\neq j,k}}^{l} v_{i_1}^{(j)}\overline{v}_{i_2}^{(m)}v_{i_3i_4}^{(k)}\sigma_{jm}\\
= &\ \nu\delta_{i_3=i_2}\sum_{\substack{i\in \mathrm{nbh}(i_0)\\ i\neq i_3}} v_{i_1,i}^{(j)}v_{i}^{(j)}e_{i_3}^{(k)}e_{i_0}^{(j)}\sigma_{jk}\overline{v}_{i_4}^{(k)} + \sum_{\substack{i\in \mathrm{nbh}(i_0)\\ i\neq i_3}} v_{i_1,i}^{(j)}v_{i,i_2}^{(j)}v_{i_3i_4}^{(k)} + \nu v_{i_1}^{(j)}e_{i_3}^{(k)}e_{i_0}^{(j)}\sigma_{jk}v_{i_3,i_2}^{(j)}\overline{v}_{i_4}^{(k)}\\
&+v_{i_1i_3}^{(j)}v_{i_3}^{(j)}v_{i_3}^{(k)}\overline{v}_{i_2}^{(j)}\overline{v}_{i_4}^{(k)} - \lambda_{i_0}v_{i_1}^{(j)}v_{i_3}^{(k)}\overline{v}_{i_2}^{(j)}\overline{v}_{i_4}^{(k)} - \nu^2\delta_{i_3i_4}v_{i_1}^{(j)}e_{i_3}^{(k)}e_{i_0}^{(j)}\sigma_{jk}\overline{v}_{i_2}^{(k)}\sigma_{jk} - \nu v_{i_1i_4}^{(j)}v_{i_3i_2}^{(k)}\sigma_{jk}\\
&-\nu^2\delta_{i_3,i_2}\sum_{\substack{m=1\\m\neq j,k}}^{l} v_{i_1}^{(j)}e_{i_3}^{(k)}\overline{v}_{i_4}^{(m)}\sigma_{km}\sigma_{jm} - \nu\sum_{\substack{m=1\\m\neq j,k}}^{l} v_{i_1}^{(j)}\overline{v}_{i_2}^{(m)}v_{i_3i_4}^{(k)}\sigma_{jm}\\
= &\ \nu\delta_{i_3i_2}\sum_{i\in \mathrm{nbh}(i_0),i\neq i_3} v_{i_1,i}^{(j)}v_{i,i_4}^{(j)}e_{i_3}^{(k)}\sigma_{jk} + \sum_{i\in \mathrm{nbh}(i_0),i\neq i_3} v_{i_1,i}^{(j)}v_{i,i_2}^{(j)}v_{i_3i_4}^{(k)}\\
&+\nu v_{i_1}^{(j)}v_{i_3}^{(k)}\overline{v}_{i_4}^{(j)}\overline{v}_{i_2}^{(k)}\sigma_{jk} + \nu\delta_{i_2i_3}v_{i_1i_3}^{(j)}v_{i_3}^{(j)}e_{i_3}^{(k)}e_{i_0}^{(j)}\sigma_{jk}\overline{v}_{i_4}^{(k)}\\
&+v_{i_1i_3}^{(j)}v_{i_3i_2}^{(j)}v_{i_3i_4}^{(k)} - \nu\delta_{i_2i_3}\lambda_{i_0}v_{i_1}^{(j)}e_{i_3}^{(k)}e_{i_0}^{(j)}\sigma_{jk}\overline{v}_{i_4}^{(k)} - \lambda_{i_0}v_{i_1i_2}^{(j)}v_{i_3i_4}^{(k)} - \nu^2\delta_{i_3i_4}v_{i_1i_2}^{(j)}e_{i_3}^{(k)}\\
&-\nu v_{i_1i_4}^{(j)}v_{i_3i_2}^{(k)}\sigma_{jk} - \nu^2\delta_{i_3i_2}\sum_{\substack{m=1\\m\neq j,k}}^{l} v_{i_1}^{(j)}e_{i_3}^{(k)}\overline{v}_{i_4}^{(m)}\sigma_{jm}\sigma_{jk} - \nu\sum_{\substack{m=1\\m\neq j,k}}^{l} v_{i_1}^{(j)}\overline{v}_{i_2}^{(m)}v_{i_3i_4}^{(k)}\sigma_{jm}
\end{aligned}$$

$$\begin{aligned}
= &\ \nu\delta_{i_3i_2}\sum_{i\in \mathrm{nbh}(i_0),i\neq i_3} v_{i_1,i}^{(j)}v_{i,i_4}^{(j)}e_{i_3}^{(k)}\sigma_{jk} + \sum_{i\in \mathrm{nbh}(i_0),i\neq i_3} v_{i_1,i}^{(j)}v_{i,i_2}^{(j)}v_{i_3i_4}^{(k)}\\
&+\nu^2\delta_{i_3i_4}v_{i_1}^{(j)}e_{i_3}^{(k)}e_{i_0}^{(j)}\sigma_{jk}\overline{v}_{i_2}^{(k)}\sigma_{jk} + \nu v_{i_1i_4}^{(j)}v_{i_3i_2}^{(k)}\sigma_{jk} + \nu\delta_{i_2i_3}v_{i_1i_3}^{(j)}v_{i_3i_4}^{(j)}e_{i_3}^{(k)}\sigma_{jk}\\
&+v_{i_1i_3}^{(j)}v_{i_3i_2}^{(j)}v_{i_3i_4}^{(k)} - \nu\delta_{i_2i_3}\lambda_{i_0}v_{i_1i_4}^{(j)}e_{i_3}^{(k)}\sigma_{jk} - \lambda_{i_0}v_{i_1i_2}^{(j)}v_{i_3i_4}^{(k)} - \nu^2\delta_{i_3i_4}v_{i_1i_2}^{(j)}e_{i_3}^{(k)} - \nu v_{i_1i_4}^{(j)}v_{i_3i_2}^{(k)}\sigma_{jk}\\
&-\nu^2\delta_{i_3i_2}\sum_{\substack{m=1\\m\neq j,k}}^{l} v_{i_1}^{(j)}\overline{v}_{i_4}^{(m)}e_{i_3}^{(k)}\sigma_{jm}\sigma_{jk} - \nu\sum_{\substack{m=1\\m\neq j,k}}^{l} v_{i_1}^{(j)}\overline{v}_{i_2}^{(m)}\sigma_{jm}v_{i_3i_4}^{(k)}\\
= &\ \nu\delta_{i_3i_2}\left(\sum_{i\in \mathrm{nbh}(i_0)} v_{i_1,i}^{(j)}v_{i,i_4}^{(j)} - \lambda_{i_0}v_{i_1i_4}^{(j)} - \nu\sum_{\substack{m=1\\m\neq j,k}}^{l} v_{i_1}^{(j)}\overline{v}_{i_4}^{(m)}\sigma_{jm}\right)e_{i_3}^{(k)}\sigma_{jk}\\
&+\left(\sum_{i\in \mathrm{nbh}(i_0)} v_{i_1,i}^{(j)}v_{i,i_2}^{(j)} - \lambda_{i_0}v_{i_1i_2}^{(j)} - \nu\sum_{\substack{m=1\\m\neq j,k}}^{l} v_{i_1}^{(j)}\overline{v}_{i_2}^{(m)}\sigma_{jm}\right)v_{i_3i_4}^{(k)}
\end{aligned}$$

Now, multiplying on the left by $v_{i_5}^{(p)}$ and on the right by $\overline{v}_{i_6}^{(p)}$ with $p\neq j,k$, we see that we get a linear combination of elements of the form $\left(\sum_{i\in \mathrm{nbh}(i_0)} v_{i_1,i}^{(j)}v_{i,i_7}^{(j)} - \lambda_{i_0}v_{i_1i_7}^{(j)} - \nu\sum_{m\neq j,k,p} v_{i_1}^{(j)}\overline{v}_{i_7}^{(m)}\sigma_{jm}\right)q$

with $q \in \Pi_l^{\lambda,\nu}(Q)\check{e}_{i_0}^{\otimes l}$ and $i_7 \in \mathrm{nbh}(i_0)$ after performing similar computations. Continuing this way, we see that any element in $J_l^{\lambda,\nu} \cap \check{e}_{i_0}^{\otimes l}\Pi_l^{\lambda,\nu}(Q)\check{e}_{i_0}^{\otimes l}$ of the form $q_1q_2q_3$ with $q_1 \in \check{e}_{i_0}^{\otimes l}\Pi_l^{\lambda,\nu}(Q), q_3 \in \Pi_l^{\lambda,\nu}(Q)\check{e}_{i_0}^{\otimes l}$ and $q_2 = r_{i_0}^{(j)} - \lambda_{i_0}e_{i_0}^{(j)} - \nu\sum_{\substack{k=1\\k\neq j}}^{l} e_{i_0}^{(j)}e_{i_0}^{(k)}\sigma_{jk}$ as in (21) is a sum of elements of the form $p_1p_2p_3$ with $p_1, p_3 \in \check{e}_{i_0}^{\otimes l}\Pi_l^{\lambda,\nu}(Q)\check{e}_{i_0}^{\otimes l}$ and $p_2 = \left(\sum_{i\in\mathrm{nbh}(i_0)} v_{i_1,i}^{(j)}v_{i,i_2}^{(j)}\right) - \lambda_{i_0}v_{i_1i_2}^{(j)}$ for some $i_1, i_2 \in \mathrm{nbh}(i_0)$. □

The following proposition was first proved in [EGGO] as a consequence of the results of [Ga]. It is fundamental for the construction of the reflection functors.

Proposition 11.2. *The algebras $\check{e}_{i_0}^{\otimes l}\Pi_l^{\lambda,\nu}(Q)\check{e}_{i_0}^{\otimes l}$ and $\check{e}_{i_0}^{\otimes l}\Pi_l^{r_{i_0}(\lambda),\nu}(Q)\check{e}_{i_0}^{\otimes l}$ are isomorphic.*

Proof. An isomorphism is given explicitly in terms of the generators in proposition 11.1 by:

$$v^{(j)} \mapsto v^{(j)} \text{ if } h(v), t(v) \neq i_0, \ \ v_{i_1i_2}^{(j)} \mapsto v_{i_1i_2}^{(j)} \text{ if } i_1 \neq i_2, \ \ \ell_i^{(j)} \mapsto \ell_i^{(j)} + \lambda_{i_0}e_i^{(j)} \text{ for } i \in \mathrm{nbh}(i_0)$$

□

To simplify the notation, we will set $\Pi_{l,i_0}^{\lambda,\nu}(Q) = \check{e}_{i_0}^{\otimes l}\Pi_l^{\lambda,\nu}(Q)\check{e}_{i_0}^{\otimes l}$. Given a right module M over $\Pi_{l,i_0}^{\lambda,\nu}(Q)$, we can form the tensor product $M \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^n)^{\otimes l}$. In particular, we can simply choose $M = \Pi_{l,i_0}^{\lambda,\nu}(Q)$. We want to identify a subalgebra of $\mathsf{D}_n^{\lambda,\nu}(Q)$ which stabilizes this subspace of $\Pi_l^{\lambda,\nu}(Q) \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^n)^{\otimes l}$. This leads us to the following definition.

Definition 11.1. *We let $\mathsf{D}_{n,i_0}^{\lambda,\nu}(Q)$ be the subalgebra of $\mathsf{D}_n^{\lambda,\nu}(Q)$ which is generated by the elements $\mathsf{E}_{ab}(e_i)$ for $i \neq i_0$ and $1 \leq a, b \leq n$, by $\mathsf{E}_{ab}(v), \mathsf{E}_{ab}(\overline{v})$ if $h(v), t(v) \neq i_0$, by $\mathsf{E}_{ab}(\ell_i) = [\mathsf{E}_{ac}(\overline{v}_i), \mathsf{E}_{cb}(v_i)] - \frac{\nu}{2}S(\mathsf{E}_{cc}(e_{i_0}), \mathsf{E}_{ab}(e_i))$ for $1 \leq a \neq b \leq n$ and some $c \neq a, b$ if $i \in \mathrm{nbh}(i_0)$ and by $\mathsf{E}_{ab}(v_{i_1,i_2}) = [\mathsf{E}_{ac}(\overline{v}_{i_2}), \mathsf{E}_{cb}(v_{i_1})]$ for $i_1, i_2 \in \mathrm{nbh}(i_0), i_1 \neq i_2$ and any $1 \leq a \neq b \leq n$. We also need to define $\mathsf{E}_{aa}(\ell_{i_0})$ by*

$$\begin{aligned} \mathsf{E}_{aa}(\ell_{i_0}) \ &= \sum_{\{v\in E(Q)|h(v)=i_0\}} [\mathsf{E}_{ac}(\overline{v}), \mathsf{E}_{ca}(v)] + \left(\lambda_{i_0} - \frac{n\nu}{2}\right)\mathsf{E}_{cc}(e_{i_0}) - \frac{\nu}{2}\mathsf{I}(e_{i_0}) \\ &+ \frac{\nu}{2}\sum_{f=1}^{n} S\big(\mathsf{E}_{cf}(e_{i_0}), \mathsf{E}_{fc}(e_{i_0})\big) - \frac{\nu}{2}\sum_{i\in\mathrm{nbh}(i_0)} S\big(\mathsf{E}_{aa}(e_{i_0}), \mathsf{E}_{cc}(e_{i_0})\big) \end{aligned}$$

(These elements do not depend on the choice of $c \neq a, b$.)

The algebra $\mathsf{D}_{n,i_0}^{\lambda,\nu}(Q)$ inherits a filtration from $\mathsf{D}_n^{\lambda,\nu}(Q)$. We can identify a quotient of $\mathsf{D}_{n,i_0}^{\lambda,\nu}(Q)$ with the following algebra defined in terms of generators and relations - see theorem 11.1 below.

Definition 11.2. *Denote by $\mathcal{D}_{n,i_0}^{\lambda,\nu}(Q)$ the algebra generated by the elements $\mathcal{E}_{ab}(e_i)$ for $i \neq i_0$, by $\mathcal{E}_{ab}(v), \mathcal{E}_{ab}(\overline{v})$ if $h(v), t(v) \neq i_0$, by $\mathcal{E}_{ab}(v_{i_1,i_2})$ for $i_1, i_2 \in \mathrm{nbh}(i_0), i_1 \neq i_2, 1 \leq a, b \leq n$, by $\mathcal{E}_{ab}(\ell_i)$ for $i \in \mathrm{nbh}(i_0)$ and $1 \leq a \neq b \leq n$, and by $\mathcal{E}_{aa}(\ell_{i_0})$ for $1 \leq a \leq n$, which satisfy the following list of relations (many of these are quite similar to some others.).*
The elements $\mathcal{E}_{ab}(e_i), 1 \leq a, b \leq n, i \in I(Q) \setminus \{i_0\}$, generate a subalgebra isomorphic to $\mathfrak{U}\mathfrak{gl}_n(\oplus_{i\neq i_0}B_i)$.
For any $1 \leq a, b, c, d \leq n$,

$$[\mathcal{E}_{ab}(e_j), \mathcal{E}_{cd}(v)] = \delta_{bc}\delta_{j,h(v)}\mathcal{E}_{ad}(v) - \delta_{ad}\delta_{j,t(v)}\mathcal{E}_{cb}(v) \tag{39}$$

$$[\mathcal{E}_{ab}(e_j), \mathcal{E}_{cd}(v_{i_1i_2})] = \delta_{bc}\delta_{j,i_2}\mathcal{E}_{ad}(v_{i_1i_2}) - \delta_{ad}\delta_{j,i_1}\mathcal{E}_{bc}(v_{i_1i_2}) \tag{40}$$

$$[\mathcal{E}_{ab}(e_j), \mathcal{E}_{cc}(\ell_{i_0})] = (\delta_{bc} - \delta_{ac})\delta_{j,i_k}\mathcal{E}_{ac}(\ell_{i_k}) \tag{41}$$

We set $\mathtt{H}_{bd}(v_{i_2i_3}) = [\mathcal{E}_{bd}(v_{i_2i_3}), \mathcal{E}_{db}(e_{i_2}) + (1-\delta_{i_2i_3})\mathcal{E}_{db}(e_{i_3})]$. *For* $a \neq b \neq c \neq a \neq d \neq c$ *and* $h(v), t(v), h(\widehat{v}), t(\widehat{v}) \neq i_0$,

$$[\mathcal{E}_{ab}(v), \mathcal{E}_{bc}(\widehat{v})] - [\mathcal{E}_{ad}(v), \mathcal{E}_{dc}(\widehat{v})] = \frac{\nu}{2}\delta_{\widehat{v}\overline{v}}\big(1 - 2\delta(v \in E(Q))\big)S\big(\mathtt{H}_{bd}(e_{t(v)}), \mathcal{E}_{ac}(e_{h(v)})\big) \tag{42}$$

$$[\mathcal{E}_{ab}(v), \mathcal{E}_{bc}(v_{i_1i_2})] = [\mathcal{E}_{ad}(v), \mathcal{E}_{dc}(v_{i_1i_2})], \quad [\mathcal{E}_{aa}(\ell_{i_0}), \mathcal{E}_{ac}(v)] = [\mathcal{E}_{ab}(\ell_{i_0}), \mathcal{E}_{bc}(v)] \tag{43}$$

$$\begin{aligned}[\mathcal{E}_{ab}(v_{i_2i_4}), \mathcal{E}_{bc}(v_{i_1i_3})] - [\mathcal{E}_{ad}(v_{i_2i_4}), \mathcal{E}_{dc}(v_{i_1i_3})] \quad = \quad & \frac{\nu}{2}\delta_{i_1i_4}S\big(\mathtt{H}_{bd}(v_{i_2i_3}), \mathcal{E}_{ac}(e_{i_1})\big) \\ & -\frac{\nu}{2}\delta_{i_2i_3}S\big(\mathtt{H}_{bd}(e_{i_3}), \mathcal{E}_{ac}(v_{i_1i_4})\big)\end{aligned} \tag{44}$$

$$[\mathcal{E}_{aa}(\ell_{i_0}), \mathcal{E}_{ac}(v_{i_1i_2})] - [\mathcal{E}_{ab}(\ell_{i_0}), \mathcal{E}_{bc}(v_{i_1i_2})] = \frac{\nu}{2}S\big(\mathtt{H}_{ab}(v_{i_1i_2}), \mathcal{E}_{ac}(e_{i_1})\big) - \frac{\nu}{2}S\big(\mathtt{H}_{ab}(e_{i_2}), \mathcal{E}_{ac}(v_{i_1i_2})\big) \tag{45}$$

$$[\mathcal{E}_{ac}(v_{i_1i_2}), \mathcal{E}_{cc}(\ell_{i_0})] - [\mathcal{E}_{ab}(v_{i_1i_2}), \mathcal{E}_{bc}(\ell_{i_0})] = \frac{\nu}{2}S\big(\mathcal{E}_{ac}(v_{i_1i_2}), \mathtt{H}_{bc}(e_{i_1})\big) - \frac{\nu}{2}S\big(\mathcal{E}_{ac}(e_{i_2}), \mathtt{H}_{bc}(v_{i_1i_2})\big) \tag{46}$$

$$\begin{aligned}\sum_{\{v\in E|h(v)=i\}} [\mathcal{E}_{ab}(v), \mathcal{E}_{bc}(\overline{v})] \quad = \quad & \sum_{\{v\in E|t(v)=i, h(v)\neq i_0\}} [\mathcal{E}_{ab}(\overline{v}), \mathcal{E}_{bc}(v)] + \delta(i \in \mathrm{nbh}(i_0))\mathcal{E}_{ac}(\ell_i) \\ (\textit{For } i \neq i_0) \qquad & +\frac{\nu}{2}\sum_{j,k=1}^{n} S\big([\mathcal{E}_{ab}(e_i), \mathcal{E}_{jk}(e_i)], [\mathcal{E}_{kj}(e_i), \mathcal{E}_{bc}(e_i)]\big) + \nu S\big(\mathcal{E}_{bb}(e_i), \mathcal{E}_{ac}(e_i)\big) \\ & -\frac{\nu}{2}\sum_{j\in\mathrm{nbh}(i), j\neq i_0} S\big(\mathcal{E}_{ac}(e_i), \mathcal{E}_{bb}(e_j)\big) + \Big(\lambda_i - \frac{n\nu}{2}\Big)\mathcal{E}_{ac}(e_i)\end{aligned} \tag{47}$$

$$\begin{aligned}\sum_{i\in\mathrm{nbh}(i_0)} [\mathcal{E}_{ab}(v_{i,i_2}), \mathcal{E}_{bc}(v_{i_1,i})] \quad = \quad & \lambda_{i_0}\mathcal{E}_{ac}(v_{i_1i_2}) - \frac{\nu}{2}\sum_{i\in\mathrm{nbh}(i_0)} S\big(\mathcal{E}_{ac}(v_{i_1i_2}), \mathcal{E}_{bb}(e_i)\big) \\ (i_1 = i_2 \textit{ is allowed}) \qquad & +\frac{\nu}{2}\delta_{i_1i_2}S\big(\mathcal{E}_{ac}(e_{i_1}), \mathcal{E}_{bb}(\ell_{i_0})\big)\end{aligned} \tag{48}$$

If $a \neq b \neq c \neq d \neq a$ *and* $h(v), t(v), h(\widehat{v}), t(\widehat{v}) \neq i_0$, *then*

$$[\mathcal{E}_{ab}(v), \mathcal{E}_{cd}(\widehat{v})] = \frac{\nu}{2}\delta_{\widehat{v}\overline{v}}\big(1 - 2\delta(v \in E(Q))\big)S\big(\mathcal{E}_{cb}(e_{t(v)}), \mathcal{E}_{ad}(e_{h(v)})\big) \tag{49}$$

$$[\mathcal{E}_{ab}(v_{i_1i_2}), \mathcal{E}_{cd}(v)] = 0 \quad [\mathcal{E}_{aa}(\ell_{i_0}), \mathcal{E}_{cd}(v)] = 0 \tag{50}$$

$$[\mathcal{E}_{ab}(v_{i_2i_4}), \mathcal{E}_{cd}(v_{i_1i_3})] \quad = \quad \frac{\nu}{2}\delta_{i_1i_4}S\big(\mathcal{E}_{ad}(e_{i_1}), \mathcal{E}_{cb}(v_{i_2i_3})\big) - \frac{\nu}{2}\delta_{i_2i_3}S\big(\mathcal{E}_{ad}(v_{i_1i_4}), \mathcal{E}_{cb}(e_{i_3})\big) \tag{51}$$

$$[\mathcal{E}_{aa}(\ell_{i_0}), \mathcal{E}_{bb}(\ell_{i_0})] = \frac{\nu}{2}\sum_{i\in\mathrm{nbh}(i_0)} \big(S(\mathcal{E}_{ab}(e_i), \mathcal{E}_{ba}(\ell_i)) - S(\mathcal{E}_{ab}(\ell_i), \mathcal{E}_{ba}(e_i))\big) \tag{52}$$

$$\textit{If } a \neq c \neq d \neq a,\ [\mathcal{E}_{aa}(\ell_{i_0}), \mathcal{E}_{cd}(v_{i_1i_2})] = \frac{\nu}{2}\big(S(\mathcal{E}_{ad}(e_{i_1}), \mathcal{E}_{ca}(v_{i_1i_2})) - S(\mathcal{E}_{ad}(v_{i_1i_2}), \mathcal{E}_{ca}(e_{i_2}))\big) \tag{53}$$

Remark 11.1. *It is possible to filter* $\mathcal{D}^{\lambda,\nu}_{n,i_0}(Q)$ *by giving* $\mathcal{E}_{ab}(e_i)$ *degree zero and* $\mathcal{E}_{ab}(v), \mathcal{E}_{ab}(\ell_{i_j}), \mathcal{E}_{ab}(v_{i_1i_2})$ *and* $\mathcal{E}_{aa}(\ell_{i_0})$ *degree one. It could also be filtered by giving each of* $\mathcal{E}_{ab}(\ell_{i_j}), \mathcal{E}_{ab}(v_{i_1i_2}), \mathcal{E}_{aa}(\ell_{i_0})$ *degree two instead.*

Theorem 11.1. *The algebra* $\mathcal{D}^{\lambda,\nu}_{n,i_0}(Q)$ *maps onto the quotient of* $\mathsf{D}^{\lambda,\nu}_{n,i_0}(Q)$ *by the ideal* $\mathsf{J}^{\lambda,\nu}_{n,i_0} \cap \mathsf{D}^{\lambda,\nu}_{n,i_0}(Q)$ *where* $\mathsf{J}^{\lambda,\nu}_{n,i_0}$ *is the two-sided ideal of* $\mathsf{D}^{\lambda,\nu}_{n}(Q)$ *generated by* $\mathsf{E}_{ab}(v_{i_1})\mathsf{E}_{cd}(\overline{v}_{i_2})$ *and* $\mathsf{E}_{ab}(e_{i_0})$ *for any* $1 \leq a, b, c, d \leq n, i, i_1, i_2 \in \mathrm{nbh}(i_0)$.

Proof. The epimorphism $\varphi : \mathcal{D}^{\lambda,\nu}_{n,i_0}(Q) \twoheadrightarrow \mathsf{D}^{\lambda,\nu}_{n,i_0}(Q)/\mathsf{J}^{\lambda,\nu}_{n,i_0} \cap \mathsf{D}^{\lambda,\nu}_{n,i_0}(Q)$ is simply defined by $\mathcal{E}_{ab}(e_i) \mapsto \mathsf{E}_{ab}(e_i), \mathcal{E}_{ab}(v) \mapsto \mathsf{E}_{ab}(v), \mathcal{E}_{ab}(\ell_i) \mapsto \mathsf{E}_{ab}(\ell_i)$, $\mathcal{E}_{ab}(v_{i_1i_2}) \mapsto \mathsf{E}_{ab}(v_{i_1i_2})$ and $\mathcal{E}_{aa}(\ell_{i_0}) \mapsto \mathsf{E}_{aa}(\ell_{i_0})$. (Note that this map respects the filtrations on the two algebras if we give $\mathcal{E}_{ab}(v_{i_1i_2}), \mathcal{E}_{ab}(\ell_i)$ and $\mathcal{E}_{aa}(\ell_{i_0})$ filtration degree two.) We have to verify that φ is well defined, that is, that it respects the relations in definition 11.2. We will verify only (44),(48),(51): the other relations are easier to check or can be verified in a similar way.

First, we find that $[\varphi(\mathcal{E}_{ab}(v_{i_2i_4})), \varphi(\mathcal{E}_{bc}(v_{i_1i_3}))] - [\varphi(\mathcal{E}_{ad}(v_{i_2i_4})), \varphi(\mathcal{E}_{dc}(v_{i_1i_3}))]$ equals

$$
\begin{aligned}
& \big[[\mathsf{E}_{ac}(\overline{v}_{i_4}), \mathsf{E}_{cb}(v_{i_2})] - \frac{\nu}{2}\delta_{i_2i_4} S(\mathsf{E}_{cc}(e_{i_0}), \mathsf{E}_{ab}(e_{i_2})), [\mathsf{E}_{ba}(\overline{v}_{i_3}), \mathsf{E}_{ac}(v_{i_1})] \\
& -\frac{\nu}{2}\delta_{i_1i_3} S(\mathsf{E}_{aa}(e_{i_0}), \mathsf{E}_{bc}(e_{i_1}))\big] - \big[[\mathsf{E}_{ac}(\overline{v}_{i_4}), \mathsf{E}_{cd}(v_{i_2})] \\
& -\frac{\nu}{2}\delta_{i_2i_4} S(\mathsf{E}_{cc}(e_{i_0}), \mathsf{E}_{ad}(e_{i_2})), [\mathsf{E}_{da}(\overline{v}_{i_3}), \mathsf{E}_{ac}(v_{i_1})] - \frac{\nu}{2}\delta_{i_1i_3} S(\mathsf{E}_{aa}(e_{i_0}), \mathsf{E}_{dc}(e_{i_1}))\big] \\
= & \Big[[\mathsf{E}_{ac}(\overline{v}_{i_4}), [\mathsf{E}_{cb}(v_{i_2}), \mathsf{E}_{ba}(\overline{v}_{i_3})]], \mathsf{E}_{ac}(v_{i_1})\Big] + \Big[\mathsf{E}_{ba}(\overline{v}_{i_3}), [[\mathsf{E}_{ac}(\overline{v}_{i_4}), \mathsf{E}_{ac}(v_{i_1})], \mathsf{E}_{cb}(v_{i_2})]\Big] \\
& -\frac{\nu}{2}\delta(i_1 = i_3 = i_2)\Big(\big[\mathsf{E}_{ac}(\overline{v}_{i_4}), S(\mathsf{E}_{aa}(e_{i_0}), \mathsf{E}_{cc}(v_{i_2}))\big] - \big[\mathsf{E}_{ac}(\overline{v}_{i_4}), S(\mathsf{E}_{aa}(e_{i_0}), \mathsf{E}_{cc}(v_{i_2}))\big]\Big) \\
& -\frac{\nu}{2}\delta(i_2 = i_4 = i_3)\Big(S\big(\mathsf{E}_{cc}(e_{i_0}), [\mathsf{E}_{aa}(\overline{v}_{i_3}), \mathsf{E}_{ac}(v_{i_1})]\big) - S\big(\mathsf{E}_{cc}(e_{i_0}), [\mathsf{E}_{aa}(\overline{v}_{i_3}), \mathsf{E}_{ac}(v_{i_1})]\big)\Big) \\
& +\frac{\nu^2}{4}\delta(i_1 = i_2 = i_3 = i_4)\Big(S\big(\mathsf{E}_{cc}(e_{i_0}), S(\mathsf{E}_{aa}(e_{i_0}), \mathsf{E}_{ac}(e_{i_1}))\big) - S\big(\mathsf{E}_{cc}(e_{i_0}), S(\mathsf{E}_{aa}(e_{i_0}), \mathsf{E}_{ac}(e_{i_1}))\big)\Big) \\
& -\big[[\mathsf{E}_{ac}(\overline{v}_{i_4}), \mathsf{E}_{cd}(v_{i_2})], [\mathsf{E}_{da}(\overline{v}_{i_3}), \mathsf{E}_{ac}(v_{i_1})]\big] \\
= & \Big[[\mathsf{E}_{ac}(\overline{v}_{i_4}), [\mathsf{E}_{cb}(v_{i_2}), \mathsf{E}_{ba}(\overline{v}_{i_3})]], \mathsf{E}_{ac}(v_{i_1})\Big] + \Big[\mathsf{E}_{ba}(\overline{v}_{i_3}), [[\mathsf{E}_{ac}(\overline{v}_{i_4}), \mathsf{E}_{ac}(v_{i_1})], \mathsf{E}_{cb}(v_{i_2})]\Big] \\
& -\big[[\mathsf{E}_{ac}(\overline{v}_{i_4}), \mathsf{E}_{cd}(v_{i_2})], [\mathsf{E}_{da}(\overline{v}_{i_3}), \mathsf{E}_{ac}(v_{i_1})]\big] \\
= & \Big[[\mathsf{E}_{ac}(\overline{v}_{i_4}), [\mathsf{E}_{cd}(v_{i_2}), \mathsf{E}_{da}(\overline{v}_{i_3})]], \mathsf{E}_{ac}(v_{i_1})\Big] - \frac{\nu}{2}\delta_{i_2i_3}\big[[\mathsf{E}_{ac}(\overline{v}_{i_4}), S(\mathsf{H}_{bd}(e_{i_2}), \mathsf{E}_{ca}(e_{i_0}))], \mathsf{E}_{ac}(v_{i_1})\big] \\
& +\frac{\nu}{2}\delta_{i_1i_4}\big[\mathsf{E}_{ba}(\overline{v}_{i_3}), [S(\mathsf{E}_{ac}(e_{i_0}), \mathsf{E}_{ac}(e_{i_1})), \mathsf{E}_{cb}(v_{i_2})]\big] - \big[[\mathsf{E}_{ac}(\overline{v}_{i_4}), \mathsf{E}_{cd}(v_{i_2})], [\mathsf{E}_{da}(\overline{v}_{i_3}), \mathsf{E}_{ac}(v_{i_1})]\big] \\
= & \Big[[[\mathsf{E}_{ac}(\overline{v}_{i_4}), \mathsf{E}_{cd}(v_{i_2})], \mathsf{E}_{ac}(v_{i_1})], \mathsf{E}_{da}(\overline{v}_{i_3})\Big] - \frac{\nu}{2}\delta_{i_2i_3}\big[S(\mathsf{H}_{bd}(e_{i_2}), \mathsf{E}_{aa}(\overline{v}_{i_4})), \mathsf{E}_{ac}(v_{i_1})\big] \\
& +\frac{\nu}{2}\delta_{i_1i_4}\big[\mathsf{E}_{ba}(\overline{v}_{i_3}), S(\mathsf{E}_{ab}(v_{i_2}), \mathsf{E}_{ac}(e_{i_1}))\big] \\
= & \frac{\nu}{2}\delta_{i_1i_4}\big[[S(\mathsf{E}_{ac}(e_{i_0}), \mathsf{E}_{ac}(e_{i_1})), \mathsf{E}_{cd}(v_{i_2})], \mathsf{E}_{da}(\overline{v}_{i_3})\big] - \frac{\nu}{2}\delta_{i_2i_3} S(\mathsf{H}_{bd}(e_{i_2}), \mathsf{E}_{ac}(v_{i_1i_4})) \\
& -\delta_{i_1i_4}\left(\frac{\nu^2}{4}\delta_{i_2i_3} S\big(\mathsf{H}_{bd}(e_{i_2}), S(\mathsf{E}_{aa}(e_{i_0}), \mathsf{E}_{ac}(e_{i_1}))\big) - \frac{\nu}{2} S\big([\mathsf{E}_{ba}(\overline{v}_{i_3}), \mathsf{E}_{ab}(v_{i_2})], \mathsf{E}_{ac}(e_{i_1})\big)\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{\nu}{2}\delta_{i_1i_4} S\big([\mathsf{E}_{ad}(v_{i_2}), \mathsf{E}_{da}(\overline{v}_{i_3})] - [\mathsf{E}_{ab}(v_{i_2}), \mathsf{E}_{ba}(\overline{v}_{i_3})], \mathsf{E}_{ac}(e_{i_1})\big) \\
& -\frac{\nu}{2}\delta_{i_2i_3} S(\mathsf{H}_{bd}(e_{i_2}), \mathsf{E}_{ac}(v_{i_1i_4})) - \frac{\nu^2}{4}\delta_{i_2i_3}\delta_{i_1i_4} S\big(\mathsf{H}_{bd}(e_{i_2}), S(\mathsf{E}_{aa}(e_{i_0}), \mathsf{E}_{ac}(e_{i_1}))\big) \\
= & \frac{\nu}{2}\delta_{i_1i_4} S\big(\varphi(\mathtt{H}_{bd}(v_{i_2i_3})), \varphi(\mathcal{E}_{ac}(e_{i_1}))\big) - \frac{\nu}{2}\delta_{i_2i_3} S\big(\varphi(\mathcal{E}_{bb}(e_{i_3})) - \varphi(\mathcal{E}_{dd}(e_{i_3})), \varphi(\mathcal{E}_{ac}(v_{i_1i_4}))\big)
\end{aligned}
$$

This verifies (44). Next, we compute that $\sum_{i\in \mathrm{nbh}(i_0)}[\varphi(\mathcal{E}_{ab}(v_{i,i_2})),\varphi(\mathcal{E}_{bc}(v_{i_1,i}))]$ is equal to

$$
\begin{aligned}
=\;& \sum_{i\in\mathrm{nbh}(i_0)} [\mathsf{E}_{ab}(v_{i,i_2}),\mathsf{E}_{bc}(v_{i_1,i})] \\
=\;& \sum_{i\in\mathrm{nbh}(i_0)} \big[[\mathsf{E}_{ac}(\overline{v}_{i_2}),\mathsf{E}_{cb}(v_i)],[\mathsf{E}_{bd}(\overline{v}_i),\mathsf{E}_{dc}(v_{i_1})]\big] - \frac{\nu}{2}\big[S(\mathsf{E}_{cc}(e_{i_0}),\mathsf{E}_{ab}(e_{i_2})),[\mathsf{E}_{bd}(\overline{v}_{i_2}),\mathsf{E}_{dc}(v_{i_1})]\big] \\
& -\frac{\nu}{2}\big[[\mathsf{E}_{ac}(\overline{v}_{i_2}),\mathsf{E}_{cb}(v_{i_1})],S(\mathsf{E}_{dd}(e_{i_0}),\mathsf{E}_{bc}(e_{i_1}))\big] + \frac{\nu^2}{4}\delta_{i_1i_2}S\big(\mathsf{E}_{cc}(e_{i_0}),S(\mathsf{E}_{dd}(e_{i_0}),\mathsf{E}_{ac}(e_{i_1}))\big) \\
=\;& \sum_{i\in\mathrm{nbh}(i_0)} \Big[\big[\mathsf{E}_{ac}(\overline{v}_{i_2}),[\mathsf{E}_{cb}(v_i),\mathsf{E}_{bd}(\overline{v}_i)]\big],\mathsf{E}_{dc}(v_{i_1})\Big] - \frac{\nu}{2}S\big(\mathsf{E}_{cc}(e_{i_0}),[\mathsf{E}_{ad}(\overline{v}_{i_2}),\mathsf{E}_{dc}(v_{i_1})]\big) \\
& +\frac{\nu}{2}\delta_{i_1i_2}\sum_{i\in\mathrm{nbh}(i_0)} \big[\mathsf{E}_{bd}(\overline{v}_i),[S(\mathsf{E}_{dc}(e_{i_0}),\mathsf{E}_{ac}(e_{i_2})),\mathsf{E}_{cb}(v_i)]\big] - \frac{\nu}{2}S(\mathsf{E}_{dd}(e_{i_0}),[\mathsf{E}_{ac}(\overline{v}_{i_2}),\mathsf{E}_{cc}(v_{i_1})]) \\
& +\frac{\nu^2}{4}\delta_{i_1i_2}S\big(\mathsf{E}_{cc}(e_{i_0}),S(\mathsf{E}_{dd}(e_{i_0}),\mathsf{E}_{ac}(e_{i_1}))\big) \\
=\;& \Big[\Big[\mathsf{E}_{ac}(\overline{v}_{i_2}),\Big(\lambda_{i_0}-\frac{n\nu}{2}\Big)\mathsf{E}_{cd}(e_{i_0}) + \frac{\nu}{2}\sum_{f=1}^{n} S\big(\mathsf{E}_{cf}(e_{i_0}),\mathsf{E}_{fd}(e_{i_0})\big) \\
& -\frac{\nu}{2}\sum_{i\in\mathrm{nbh}(i_0)} S(\mathsf{E}_{bb}(e_i),\mathsf{E}_{cd}(e_{i_0}))\Big],\mathsf{E}_{dc}(v_{i_1})\Big] - \frac{\nu}{2}S\big(\mathsf{E}_{cc}(e_{i_0}),[\mathsf{E}_{ad}(\overline{v}_{i_2}),\mathsf{E}_{dc}(v_{i_1})]\big) \\
& +\frac{\nu}{2}\delta_{i_1i_2}\sum_{i\in\mathrm{nbh}(i_0)} \big[\mathsf{E}_{bd}(\overline{v}_i),S(\mathsf{E}_{db}(v_i),\mathsf{E}_{ac}(e_{i_2}))\big] - \frac{\nu}{2}S\big(\mathsf{E}_{dd}(e_{i_0}),[\mathsf{E}_{ac}(\overline{v}_{i_2}),\mathsf{E}_{cc}(v_{i_1})]\big) \\
& +\frac{\nu^2}{4}\delta_{i_1i_2}S\big(\mathsf{E}_{cc}(e_{i_0}),S(\mathsf{E}_{dd}(e_{i_0}),\mathsf{E}_{ac}(e_{i_1}))\big) \\
=\;& \Big[\Big(\lambda_{i_0}-\frac{n\nu}{2}\Big)\mathsf{E}_{ad}(\overline{v}_{i_2}) + \frac{\nu}{2}\sum_{f=1}^{n} S(\mathsf{E}_{af}(\overline{v}_{i_2}),\mathsf{E}_{fd}(e_{i_0})) + \frac{\nu}{2}S(\mathsf{E}_{cc}(e_{i_0}),\mathsf{E}_{ad}(\overline{v}_{i_2})) \\
& -\frac{\nu}{2}\sum_{i\in\mathrm{nbh}(i_0)} S(\mathsf{E}_{bb}(e_i),\mathsf{E}_{ad}(\overline{v}_{i_2})),\mathsf{E}_{dc}(v_{i_1})\Big] - \frac{\nu}{2}S\big(\mathsf{E}_{cc}(e_{i_0}),[\mathsf{E}_{ad}(\overline{v}_{i_2}),\mathsf{E}_{dc}(v_{i_1})]\big) \\
& -\frac{\nu}{2}S\big(\mathsf{E}_{dd}(e_{i_0}),[\mathsf{E}_{ac}(\overline{v}_{i_2}),\mathsf{E}_{cc}(v_{i_1})]\big) - \frac{\nu}{2}\delta_{i_1i_2}\sum_{i\in\mathrm{nbh}(i_0)} S\big([\mathsf{E}_{db}(v_i),\mathsf{E}_{bd}(\overline{v}_i)],\mathsf{E}_{ac}(e_{i_2})\big) \\
& +\frac{\nu^2}{4}\delta_{i_1i_2}S\big(\mathsf{E}_{cc}(e_{i_0}),S(\mathsf{E}_{dd}(e_{i_0}),\mathsf{E}_{ac}(e_{i_1}))\big) \\
=\;& \Big(\lambda_{i_0}-\frac{n\nu}{2}\Big)\mathsf{E}_{ac}(v_{i_1i_2}) + \frac{\nu}{2}\Big(\lambda_{i_0}-\frac{n\nu}{2}\Big)\delta_{i_1i_2}S(\mathsf{E}_{dd}(e_{i_0}),\mathsf{E}_{ac}(e_{i_1})) + \frac{\nu}{2}\sum_{f=1}^{n} S(\mathsf{E}_{af}(\overline{v}_{i_2}),\mathsf{E}_{fc}(v_{i_1})) \\
& +\frac{\nu}{2}\sum_{\substack{f=1\\ f\neq d}}^{n} S\big([\mathsf{E}_{af}(\overline{v}_{i_2}),\mathsf{E}_{dc}(v_{i_1})],\mathsf{E}_{fd}(e_{i_0})\big) + \frac{\nu}{2}S\big([\mathsf{E}_{ad}(\overline{v}_{i_2}),\mathsf{E}_{dc}(v_{i_1})]-[\mathsf{E}_{ac}(\overline{v}_{i_2}),\mathsf{E}_{cc}(v_{i_1})],\mathsf{E}_{dd}(e_{i_0})\big) \\
& -\frac{\nu}{2}\sum_{i\in\mathrm{nbh}(i_0)} S\big(\mathsf{E}_{bb}(e_i),[\mathsf{E}_{ad}(\overline{v}_{i_2}),\mathsf{E}_{dc}(v_{i_1})]\big) - \frac{\nu}{2}\delta_{i_1i_2}S\Big(\Big(\lambda_{i_0}-\frac{n\nu}{2}\Big)\mathsf{E}_{dd}(e_{i_0}) \\
& +\frac{\nu}{2}\sum_{f=1}^{n} S(\mathsf{E}_{df}(e_{i_0}),\mathsf{E}_{fd}(e_{i_0})) - \frac{\nu}{2}\sum_{i\in\mathrm{nbh}(i_0)} S(\mathsf{E}_{bb}(e_i),\mathsf{E}_{dd}(e_{i_0})) - \mathsf{E}_{bb}(\ell_{i_0}) - \frac{\nu}{2}\mathsf{I}(e_{i_0}),\mathsf{E}_{ac}(e_{i_1})\Big) \\
& +\frac{\nu^2}{4}\delta_{i_1i_2}S\big(\mathsf{E}_{cc}(e_{i_0}),S(\mathsf{E}_{dd}(e_{i_0}),\mathsf{E}_{ac}(e_{i_1}))\big)
\end{aligned}
$$

$$\begin{aligned}
&= \Big(\lambda_{i_0} - \frac{n\nu}{2}\Big)\,\mathsf{E}_{ac}(v_{i_1 i_2}) + \frac{\nu^2}{4}\delta_{i_1 i_2}\Big(S\big(S(\mathsf{H}_{dc}(e_{i_0}), \mathsf{E}_{ac}(e_{i_1})), \mathsf{E}_{dd}(e_{i_0})\big) + S(\mathsf{I}(e_{i_0}), \mathsf{E}_{ac}(e_{i_1}))\Big) \\
&\quad -\frac{\nu^2}{4}\delta_{i_1 i_2} S\big(S(\mathsf{E}_{dd}(e_{i_0}), \mathsf{E}_{dd}(e_{i_0})), \mathsf{E}_{ac}(e_{i_1})\big) + \frac{\nu}{2}\delta_{i_1 i_2} S(\mathsf{E}_{bb}(\ell_{i_0}), \mathsf{E}_{ac}(e_{i_1})) \\
&\quad +\frac{\nu^2}{4}\delta_{i_1 i_2} \sum_{i \in \mathrm{nbh}(i_0)} S\big(S(\mathsf{E}_{bb}(e_i), \mathsf{E}_{dd}(e_{i_0})), \mathsf{E}_{ac}(e_{i_1})\big) + \frac{\nu}{2}\sum_{f=1}^{n} S(\mathsf{E}_{af}(\overline{v}_{i_2}), \mathsf{E}_{fc}(v_{i_1})) \\
&\quad +\frac{\nu^2}{4}\delta_{i_1 i_2} S\big(\mathsf{E}_{cc}(e_{i_0}), S(\mathsf{E}_{dd}(e_{i_0}), \mathsf{E}_{ac}(e_{i_1}))\big) - \frac{\nu}{2}\sum_{i \in \mathrm{nbh}(i_0)} S\big([\mathsf{E}_{ad}(\overline{v}_{i_2}), \mathsf{E}_{dc}(v_{i_1})], \mathsf{E}_{bb}(e_i)\big) \\
&= \lambda_{i_0}\mathsf{E}_{ac}(v_{i_1 i_2}) - \frac{\nu}{2}\sum_{i \in \mathrm{nbh}(i_0)} S\big(\mathsf{E}_{ac}(v_{i_1 i_2}), \mathsf{E}_{bb}(e_i)\big) + \frac{\nu}{2}\delta_{i_1 i_2} S\big(\mathsf{E}_{bb}(\ell_{i_0}), \mathsf{E}_{ac}(e_{i_1})\big) \\
&\quad +\nu\sum_{f=1}^{n} \mathsf{E}_{fc}(v_{i_1})\mathsf{E}_{af}(\overline{v}_{i_2}) + \frac{3\nu^2}{4}\delta_{i_1 i_2} S(\mathsf{I}(e_{i_0}), \mathsf{E}_{ac}(e_{i_1})) \qquad (54) \\
&= \lambda_{i_0}\varphi(\mathcal{E}_{ac}(v_{i_1 i_2})) - \frac{\nu}{2}\sum_{i \in \mathrm{nbh}(i_0)} S\big(\varphi(\mathcal{E}_{ac}(v_{i_1 i_2})), \varphi(\mathcal{E}_{bb}(e_i))\big) \\
&\quad +\frac{\nu}{2}\delta_{i_1 i_2}\sum_{i \in \mathrm{nbh}(i_0)} S\big(\varphi(\mathcal{E}_{bb}(\ell_i)), \varphi(\mathcal{E}_{ac}(e_{i_1}))\big)
\end{aligned}$$

To obtain the last equality, note that the elements on line (54) are in the ideal $\mathsf{J}_{n,i_0}^{\lambda,\nu} \cap \mathsf{D}_{n,i_0}^{\lambda,\nu}$.

Finally, we check that, if $a \neq b \neq c \neq d \neq a$, $[\varphi(\mathcal{E}_{ab}(v_{i_2 i_4})), \varphi(\mathcal{E}_{cd}(v_{i_1 i_3}))]$ equals

$$\begin{aligned}
&= [\mathsf{E}_{ab}(v_{i_2 i_4}), \mathsf{E}_{cd}(v_{i_1 i_3})] \\
&= \big[[\mathsf{E}_{ac}(\overline{v}_{i_4}), \mathsf{E}_{cb}(v_{i_2})], [\mathsf{E}_{ca}(\overline{v}_{i_3}), \mathsf{E}_{ad}(v_{i_1})]\big] - \frac{\nu}{2}\delta_{i_2 i_4}\big[S(\mathsf{E}_{cc}(e_{i_0}), \mathsf{E}_{ab}(e_{i_2})), [\mathsf{E}_{ca}(\overline{v}_{i_3}), \mathsf{E}_{ad}(v_{i_1})]\big] \\
&\quad -\frac{\nu}{2}\delta_{i_1 i_3}\big[[\mathsf{E}_{ac}(\overline{v}_{i_4}), \mathsf{E}_{cb}(v_{i_2})], S(\mathsf{E}_{aa}(e_{i_0}), \mathsf{E}_{cd}(e_{i_1}))\big] \\
&\quad +\frac{\nu^2}{4}\delta_{i_2 i_4}\delta_{i_1 i_3}\big[S(\mathsf{E}_{cc}(e_{i_0}), \mathsf{E}_{ab}(e_{i_2})), S(\mathsf{E}_{aa}(e_{i_0}), \mathsf{E}_{cd}(e_{i_1}))\big] \\
&= \Big[\big[\mathsf{E}_{ac}(\overline{v}_{i_4}), [\mathsf{E}_{cb}(v_{i_2}), \mathsf{E}_{ca}(\overline{v}_{i_3})]\big], \mathsf{E}_{ad}(v_{i_1})\Big] + \Big[\mathsf{E}_{ca}(\overline{v}_{i_3}), \big[[\mathsf{E}_{ac}(\overline{v}_{i_4}), \mathsf{E}_{ad}(v_{i_1})], \mathsf{E}_{cb}(v_{i_2})\big]\Big] \\
&= -\frac{\nu}{2}\delta_{i_2 i_3}\big[[\mathsf{E}_{ac}(\overline{v}_{i_4}), S(\mathsf{E}_{cb}(e_{i_2}), \mathsf{E}_{ca}(e_{i_0}))], \mathsf{E}_{ad}(v_{i_1})\big] \\
&\quad +\frac{\nu}{2}\delta_{i_1 i_4}\big[\mathsf{E}_{ca}(\overline{v}_{i_3}), [S(\mathsf{E}_{ac}(e_{i_0}), \mathsf{E}_{ad}(e_{i_1})), \mathsf{E}_{cb}(v_{i_2})]\big] \\
&= -\frac{\nu}{2}\delta_{i_2 i_3} S\big(\varphi(\mathcal{E}_{cb}(e_{i_2})), \varphi(\mathcal{E}_{ad}(v_{i_1 i_4}))\big) + \frac{\nu}{2}\delta_{i_1 i_4} S\big(\varphi(\mathcal{E}_{cb}(v_{i_2 i_3})), \varphi(\mathcal{E}_{ad}(e_{i_1}))\big)
\end{aligned}$$

□

It may be tempting to think that $\mathsf{D}_{n,i_0}^{\lambda,\nu}(Q)$ gives a flat deformation of $\mathsf{D}_{n,i_0}^{\lambda=0,\nu=0}(Q) \subset \mathfrak{Us}\check{\mathfrak{l}}_n(\Pi(Q))$. (Here, we consider the filtration on $\mathsf{D}_{n,i_0}^{\lambda,\nu}(Q)$ inherited from the one on $\mathsf{D}_{n}^{\lambda,\nu}(Q)$.) This is not true. As the computations in the previous proof show, when $\nu \neq 0$, there exist elements p such that $p \in F_2(\mathsf{D}_{n}^{\lambda,\nu}(Q))$ and $0 \neq \bar{p} \in F_2(\mathsf{D}_{n,i_0}^{\lambda,\nu}(Q))/F_1(\mathsf{D}_{n,i_0}^{\lambda,\nu}(Q))$, but $\bar{p}$ is not in the image of $\mathsf{D}_{n,i_0}^{\lambda=\mathbf{0},\nu=0}(Q) \longrightarrow \mathrm{gr}_{F_\bullet}(\mathsf{D}_{n,i_0}^{\lambda,\nu}(Q))$. The induced filtration on $\mathsf{D}_{n,i_0}^{\lambda,\nu}(Q)$ is different from the one obtained by giving generators $\mathsf{E}_{ab}(e_i)$ degree zero, $\mathsf{E}_{ab}(v)$ degree one, and $\mathsf{E}_{ab}(\ell_i), \mathsf{E}_{ab}(v_{i_1 i_2}), \mathsf{E}_{aa}(\ell_{i_0})$ degree two, which we denote $\widetilde{F}$. Again, from the computations above, we see that we can find elements p such that $p \in F_2(\mathsf{D}_{n}^{\lambda,\nu}(Q))$, but $p \notin \widetilde{F}_2(\mathsf{D}_{n}^{\lambda,\nu}(Q))$ although $p \in \widetilde{F}_4(\mathsf{D}_{n}^{\lambda,\nu}(Q))$.

We have the following analog of proposition 11.2.

Proposition 11.3. *The algebras* $\mathcal{D}_{n,i_0}^{\lambda,\nu}(Q)$ *and* $\mathcal{D}_{n,i_0}^{r_{i_0}(\lambda),\nu}(Q)$ *are isomorphic.*

Proof. An explicit isomorphism is given by: $\mathcal{E}_{ab}(e_i) \mapsto \mathcal{E}_{ab}(e_i)$ for $i \neq i_0$,

$$\mathcal{E}_{ab}(v) \mapsto \mathcal{E}_{ab}(v) \text{ if } h(v), t(v) \neq i_0, \;\; \mathcal{E}_{aa}(\ell_{i_0}) \mapsto \lambda_{i_0} \sum_{i \in \mathrm{nbh}(i_0)} \mathcal{E}_{aa}(e_i) + \mathcal{E}_{aa}(\ell_{i_0})$$

$$\mathcal{E}_{ab}(v_{i_1 i_2}) \mapsto \mathcal{E}_{ab}(v_{i_1 i_2}) \text{ if } i_1 \neq i_2, \;\; \mathcal{E}_{ab}(\ell_i) \mapsto \lambda_{i_0} \mathcal{E}_{ab}(e_i) + \mathcal{E}_{ab}(\ell_i) \text{ for } i \in \mathrm{nbh}(i_0).$$

□

The algebras $\Pi_{l,i_0}^{\lambda,\nu}(Q)$ and $\mathcal{D}_{n,i_0}^{\lambda,\nu}(Q)$ are also related by a functor of Schur-Weyl type.

Theorem 11.2. *Given a left* $\Pi_{l,i_0}^{\lambda,\nu}(Q)$*-module* M*, the space* $(\mathbb{C}^n)^{\otimes l} \otimes_{\mathbb{C}[S_l]} M$ *can be given the structure of a right module over* $\mathcal{D}_{n,i_0}^{\lambda,\nu}(Q)$*. (Here, we view* $\mathbb{C}^n$ *as the space of row vectors of length* n *on which* $M_n(\mathbb{C})$ *acts on the right by matrix multiplication.) Thus, we have a functor, which we denote also* SW_l*, from* $\mathrm{mod}_L - \Pi_{l,i_0}^{\lambda,\nu}(Q)$ *to* $\mathrm{mod}_R - \mathcal{D}_{n,i_0}^{\lambda,\nu}(Q)$

Proof. We let the generators of $\mathcal{D}_{n,i_0}^{\lambda,\nu}(Q)$ act on this space according to the following formulas:

$$(\mathbf{u} \otimes m)\mathcal{E}_{ab}(e_i) = \sum_{j=1}^{l} (\mathbf{u}) E_{ab}^{(j)} \otimes e_i^{(j)} m, \; i \neq i_0, \;\; (\mathbf{u} \otimes m)\mathcal{E}_{ab}(v) = \sum_{j=1}^{l} (\mathbf{u}) E_{ab}^{(j)} \otimes v^{(j)} m \text{ if } h(v), t(v) \neq i_0$$

$$(\mathbf{u} \otimes m)\mathcal{E}_{ab}(v_{i_1 i_2}) = \sum_{j=1}^{l} (\mathbf{u}) E_{ab}^{(j)} \otimes v_{i_1 i_2}^{(j)} m, \;\; (\mathbf{u} \otimes m)\mathcal{E}_{aa}(\ell_{i_0}) = \sum_{j=1}^{l} \sum_{i \in \mathrm{nbh}(i_0)} (\mathbf{u}) E_{aa}^{(j)} \otimes \ell_i^{(j)} m.$$

One can check, as in section 8, that these operators satisfy the relations in definition 11.2. We do it here only for equations (42) and (48).

$$\begin{aligned}
&(\mathbf{u} \otimes m)\big([\mathcal{E}_{ab}(v), \mathcal{E}_{bc}(\widehat{v})] - [\mathcal{E}_{ad}(v), \mathcal{E}_{dc}(\widehat{v})]\big) \\
&\quad = \sum_{1 \leq j \neq k \leq l} (\mathbf{u})(E_{ab}^{(j)} E_{bc}^{(k)} - E_{ad}^{(j)} E_{dc}^{(k)}) \otimes [\widehat{v}^{(k)}, v^{(j)}] m \\
&\quad = \nu \delta_{\widehat{v}\overline{v}} \big(1 - 2\delta(v \in E(Q))\big) \sum_{1 \leq j \neq k \leq l} (\mathbf{u})(E_{ab}^{(j)} E_{bc}^{(k)} - E_{ad}^{(j)} E_{dc}^{(k)}) \otimes \sigma_{jk} e_{h(v)}^{(j)} e_{t(v)}^{(k)} m \\
&\quad = \nu \delta_{\widehat{v}\overline{v}} \big(1 - 2\delta(v \in E(Q))\big) \sum_{1 \leq j \neq k \leq l} (\mathbf{u})(E_{ac}^{(j)} E_{bb}^{(k)} - E_{ac}^{(j)} E_{dd}^{(k)}) \otimes e_{h(v)}^{(j)} e_{t(v)}^{(k)} m \\
&\quad = \frac{\nu}{2} \delta_{\widehat{v}\overline{v}} \big(1 - 2\delta(v \in E(Q))\big) (\mathbf{u} \otimes m) S\big(\mathcal{E}_{ac}(e_{h(v)}), \mathcal{E}_{bb}(e_{t(v)}) - \mathcal{E}_{dd}(e_{t(v)})\big)
\end{aligned}$$

$$(\mathbf{u}\otimes m)\left(\sum_{i\in \mathrm{nbh}(i_0)}[\mathcal{E}_{ab}(v_{i,i_2}),\mathcal{E}_{bc}(v_{i_1,i})]\right)$$

$$= \sum_{j=1}^{l}\sum_{i\in\mathrm{nbh}(i_0)}(\mathbf{u})E_{ac}^{(j)}\otimes v_{i_1,i}^{(j)}v_{i,i_2}^{(j)}m+\sum_{\substack{j,k=1\\ j\neq k}}^{l}\sum_{i\in\mathrm{nbh}(i_0)}(\mathbf{u})E_{ab}^{(j)}E_{bc}^{(k)}\otimes[v_{i_1,i}^{(k)},v_{i,i_2}^{(j)}]m$$

$$= \lambda_{i_0}\sum_{j=1}^{l}(\mathbf{u})E_{ac}^{(j)}\otimes v_{i_1i_2}^{(j)}m-\nu\sum_{i\in\mathrm{nbh}(i_0)}\sum_{\substack{j,k=1\\ j\neq k}}^{l}(\mathbf{u})E_{ab}^{(j)}E_{bc}^{(k)}\otimes\sigma_{jk}v_{i_1,i_2}^{(j)}e_i^{(k)}m$$

$$+\nu\delta_{i_1i_2}\sum_{i\in\mathrm{nbh}(i_0)}\sum_{j\neq k}(\mathbf{u})E_{ab}^{(j)}E_{bc}^{(k)}\otimes\sigma_{jk}\ell_i^{(k)}e_{i_1}^{(j)}m$$

$$= \lambda_{i_0}(\mathbf{u}\otimes m)\mathcal{E}_{ab}(v_{i_1i_2})-\frac{\nu}{2}\sum_{i\in\mathrm{nbh}(i_0)}(\mathbf{u}\otimes m)S(\mathcal{E}_{ac}(v_{i_1i_2}),\mathcal{E}_{bb}(e_i))$$

$$+\frac{\nu}{2}\delta_{i_1i_2}\sum_{i\in\mathrm{nbh}(i_0)}(\mathbf{u}\otimes m)S(\mathcal{E}_{ac}(e_{i_1}),\mathcal{E}_{bb}(\ell_i))$$

□

Remark 11.2. *We can view $\Pi_{l,i_0}^{\lambda,\nu}(Q)\otimes_{\mathbb{C}[S_l]}(\mathbb{C}^n)^{\otimes l}$ and $\Pi_l^{\lambda,\nu}(Q)\check{e}_{i_0}^{\otimes l}\otimes_{\mathbb{C}[S_l]}(\mathbb{C}^n)^{\otimes l}$ as subspaces of the $\mathsf{D}_n^{\lambda,\nu}(Q)$-module $\Pi_l^{\lambda,\nu}(Q)\otimes_{\mathbb{C}[S_l]}(\mathbb{C}^n)^{\otimes l}$ and, as such, they are stable under the action of the subalgebra $\mathsf{D}_{n,i_0}^{\lambda,\nu}(Q)$.*

The following diagram is commutative, the vertical arrows being the equivalences of categories coming from the isomorphisms given in the proofs of propositions 11.2 and 11.3:

$$\begin{array}{ccc}
\mathrm{mod}_L-\Pi_{l,i_0}^{\lambda,\nu}(Q) & \xrightarrow{\ \mathsf{SW}_l\ } & \mathrm{mod}_R-\mathcal{D}_{n,i_0}^{\lambda,\nu}(Q)\\
\downarrow\sim & & \downarrow\sim\\
\mathrm{mod}_L-\Pi_{l,i_0}^{r_{i_0}(\lambda),\nu}(Q) & \xrightarrow{\ \mathsf{SW}_l\ } & \mathrm{mod}_R-\mathcal{D}_{n,i_0}^{r_{i_0}(\lambda),\nu}(Q)
\end{array}$$

Now we need to introduce the space $\mathcal{V}_{l,i_0}=(\mathbb{C}^{\otimes n})^{\otimes l}\otimes_{\mathbb{C}[S_l]}\check{e}_{i_0}^{\otimes l}\Pi_l^{r_{i_0}(\lambda),\nu}(Q)\otimes_{\mathbb{C}[S_l]}(\mathbb{C}^n)^{\otimes l}$. It is a left module over $\mathsf{D}_n^{r_{i_0}(\lambda),\nu}(Q)$ by theorem 8.1, a right module over $\mathcal{D}_{n,i_0}^{r_{i_0}(\lambda),\nu}(Q)$ by theorem 11.2 and these two module structures commute, so that $\mathcal{V}_{l,i_0}$ is a bimodule. Proposition 11.3 implies that it is also a right module over $\mathcal{D}_{n,i_0}^{\lambda,\nu}(Q)$.

We are now in a position to construct the reflection functor $R_{i_0,l}$. Let N be a left $\mathsf{D}_n^{\lambda,\nu}(Q)$-module. We want to obtain a new module $R_{i_0,l}(N)$ over $\mathsf{D}_n^{r_{i_0}(\lambda),\nu}(Q)$. Let $\widetilde{N}^{i_0}=\{x\in N|I(e_{i_0})x=0\}$, note that $\widetilde{N}^{i_0}$ is a module for $\mathsf{D}_{n,i_0}^{\lambda,\nu}(Q)$, set $\mathsf{J}^{i_0}=\mathsf{J}_{n,i_0}^{\lambda,\nu}\cap\mathsf{D}_{n,i_0}^{\lambda,\nu}(Q)$ (see theorem 11.1) and $N^{i_0}=\widetilde{N}^{i_0}/\mathsf{J}^{i_0}\widetilde{N}^{i_0}$. We can view N^{i_0} as a module over $\mathsf{D}_{n,i_0}^{\lambda,\nu}(Q)/\mathsf{J}^{i_0}$, hence over $\mathcal{D}_{n,i_0}^{\lambda,\nu}(Q)$ according to theorem 11.1.

Definition 11.3. *The reflection functor $R_{i_0,l}:\mathrm{mod}_L-\mathsf{D}_n^{\lambda,\nu}(Q)\longrightarrow\mathrm{mod}_L-\mathsf{D}_n^{r_{i_0}(\lambda),\nu}(Q)$ is defined by $R_{i_0,l}(N)=\mathcal{V}_{l,i_0}\otimes_{\mathcal{D}_{n,i_0}^{\lambda,\nu}(Q)}N^{i_0}$.*

Before proceeding further, we need to recall the construction of the reflection functors $F_{i_0,l}$ for $\Pi_l^{\lambda,\nu}(Q)$ given in [EGGO]. (When $\lambda_{i_0}+k\nu\neq 0$ for $k=0,\dots,l-1$, they are the same as the reflection functors studied in [Ga], see corollary 6.6.3 in [EGGO].) Given a left module M over $\Pi_l^{\lambda,\nu}(Q)$, $\check{e}_{i_0}^{\otimes l}M$ is a module over $\Pi_{l,i_0}^{\lambda,\nu}(Q)$. Proposition 6.6.1 in [EGGO] states that $\Pi_{l,i_0}^{\lambda,\nu}(Q)$ is isomorphic to $\Pi_{l,i_0}^{r_{i_0}(\lambda),\nu}(Q)$.

Therefore, it is possible to view $\Pi_l^{r_{i_0}(\lambda),\nu}(Q)\check{e}_{i_0}^{\otimes l}$ as a right module over $\Pi_{l,i_0}^{\lambda,\nu}(Q)$. The tensor product $\Pi_l^{r_{i_0}(\lambda),\nu}(Q)\check{e}_{i_0}^{\otimes l}\otimes_{\Pi_{l,i_0}^{\lambda,\nu}(Q)}\check{e}_{i_0}^{\otimes l}M$ is thus a left module over $\Pi_l^{r_{i_0}(\lambda),\nu}(Q)$. Since we need to work with right modules, we will consider instead the functor $F_{i_0,l}: M\mapsto M\check{e}_{i_0}^{\otimes l}\otimes_{\Pi_{l,i_0}^{\lambda,\nu}(Q)}\check{e}_{i_0}^{\otimes l}\Pi_l^{r_{i_0}(\lambda),\nu}(Q)$.

The functors $R_{i_0,l}$ and $F_{i_0,l}$ are related as stated in the next proposition.

Proposition 11.4. *Suppose that $l+1\leq n$. Then the following diagram is commutative:*

$$\begin{array}{ccc}
\mathrm{mod}_R-\Pi_l^{\lambda,\nu}(Q) & \xrightarrow{\mathsf{SW}_l} & \mathrm{mod}_L-\mathsf{D}_n^{\lambda,\nu}(Q)\\
{\scriptstyle F_{i_0,l}}\downarrow & & \downarrow{\scriptstyle R_{i_0,l}}\\
\mathrm{mod}_R-\Pi_l^{r_{i_0}(\lambda),\nu}(Q) & \xrightarrow[\mathsf{SW}_l]{} & \mathrm{mod}_L-\mathsf{D}_n^{r_{i_0}(\lambda),\nu}(Q)
\end{array}$$

Proof. Under the assumption $n\geq l$, the Schur-Weyl functor from $\mathrm{mod}_R-\mathbb{C}[S_l]$ to $\mathrm{mod}_L^l-\mathfrak{sl}_n$ is an equivalence of categories and its inverse is given by $N\mapsto(\mathbb{C}^n)^{\otimes l}\otimes_{\mathfrak{U}(\mathfrak{sl}_n)}N$: this follows from the decomposition of $(\mathbb{C}^n)^{\otimes l}$ into irreducible $S_l\times SL_n(\mathbb{C})$-modules, which is the classical Schur-Weyl reciprocity.

Let $M\in\mathrm{mod}_R-\Pi_l^{\lambda,\nu}(Q), l+1\leq n$ and observe that $\widetilde{\mathsf{SW}(M)}^{i_0}=M\check{e}_{i_0}^{\otimes l}\otimes_{\mathbb{C}[S_l]}(\mathbb{C}^n)^{\otimes l}=\mathsf{SW}(M)^{i_0}$. This can be seen by decomposing M as the direct sum of its subspaces $Me_{i_1}\otimes\cdots\otimes e_{i_l}$ for $i_1,\ldots,i_l\in I(Q)$ as in [Ga] and observing that $\mathsf{J}^{i_0}\widetilde{\mathsf{SW}(M)}^{i_0}=0$. Any element $\mathbf{u}_1\otimes p\otimes m\otimes\mathbf{u}_2$ in $\big((\mathbb{C}^n)^{\otimes l}\otimes_{\mathbb{C}[S_l]}\Pi_{l,i_0}^{r_{i_0}(\lambda),\nu}(Q)\big)\otimes_{\mathcal{D}_{n,i_0}^{\lambda,\nu}}\big(M\check{e}_{i_0}^{\otimes l}\otimes_{\mathbb{C}[S_l]}(\mathbb{C}^n)^{\otimes l}\big)$ can be written as a linear combination of elements of the form $\tilde{\mathbf{u}}_1\otimes\check{e}_{i_0}^{\otimes l}\otimes\widetilde{m}\otimes\tilde{\mathbf{u}}_2$ where $\tilde{\mathbf{u}}_1=u_{j_1}\otimes\cdots\otimes u_{j_l}$ with $j_1,\ldots,j_l$ all distinct, $u_1,\ldots,u_n$ being the standard basis of $\mathbb{C}^n$. Using this observation and the one in the previous paragraph, we can conclude that $\big((\mathbb{C}^n)^{\otimes l}\otimes_{\mathbb{C}[S_l]}\Pi_{l,i_0}^{r_{i_0}(\lambda),\nu}(Q)\big)\otimes_{\mathcal{D}_{n,i_0}^{\lambda,\nu}}\big(M\check{e}_{i_0}^{\otimes l}\otimes_{\mathbb{C}[S_l]}(\mathbb{C}^n)^{\otimes l}\big)$ is isomorphic to $M\check{e}^{\otimes l}$ as a right module over $\Pi_{l,i_0}^{r_{i_0}(\lambda),\nu}(Q)$. Therefore,

$$\begin{aligned}
R_{i_0,l}\circ\mathsf{SW}_l(M) &= R_{i_0,l}\big(M\otimes_{\mathbb{C}[S_l]}V^{\otimes l}\big)\\
&= \mathcal{V}_{l,i_0}\otimes_{\mathcal{D}_{n,i_0}^{\lambda,\nu}}\big(M\check{e}_{i_0}^{\otimes l}\otimes_{\mathbb{C}[S_l]}(\mathbb{C}^n)^{\otimes l}\big)\\
&= \Big(\big((\mathbb{C}^n)^{\otimes l}\otimes_{\mathbb{C}[S_l]}\check{e}_{i_0}^{\otimes l}\Pi_l^{r_{i_0}(\lambda),\nu}(Q)\big)\otimes_{\mathcal{D}_{n,i_0}^{\lambda,\nu}}\big(M\check{e}_{i_0}^{\otimes l}\otimes_{\mathbb{C}[S_l]}(\mathbb{C}^n)^{\otimes l}\big)\Big)\otimes_{\mathbb{C}[S_l]}(\mathbb{C}^n)^{\otimes l}\\
&= \big(M\check{e}_{i_0}^{\otimes l}\otimes_{\Pi_{l,i_0}^{\lambda,\nu}(Q)}\check{e}_{i_0}^{\otimes l}\Pi_l^{r_{i_0}(\lambda),\nu}(Q)\big)\otimes_{\mathbb{C}[S_l]}(\mathbb{C}^n)^{\otimes l}\\
&= \mathsf{SW}_l\circ F_{i_0,l}(M)
\end{aligned}$$

□

We end with a few open questions. As remarked above, the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{mod}_R-\Pi_l^{\lambda,\nu}(Q) & \xrightarrow{\mathsf{SW}_l} & \mathrm{mod}_L-\mathsf{D}_n^{\lambda,\nu}(Q)\\
{\scriptstyle \cdot e_{i_0}^{\otimes l}}\downarrow & & \downarrow{\scriptstyle \bullet^{i_0}}\\
\mathrm{mod}_R-e_{i_0}^{\otimes l}\Pi_l^{\lambda,\nu}(Q)e_{i_0}^{\otimes l} & \xrightarrow[\mathsf{SW}_l]{} & \mathrm{mod}_L-\mathsf{D}_{n,i_0}^{\lambda,\nu}(Q)
\end{array}$$

Here, the first vertical arrow is the functor $M\mapsto Me_{i_0}^{\otimes l}$, which is a Morita equivalence when $\lambda_{i_0}\neq 0$ according to lemma 6.6.2 in [EGGO]. The second one is the functor $N\mapsto N^{i_0}$ introduced above. In view of this diagram and lemma 6.6.2 in *loc.cit.*, one may ask if the functor $\bullet^{i_0}$ can sometimes be a Morita equivalence, at least when restricted to certain subcategories of $\mathrm{mod}_L-\mathsf{D}_n^{\lambda,\nu}(Q)$ and $\mathrm{mod}_L-\mathsf{D}_{n,i_0}^{\lambda,\nu}(Q)$.

An answer to this question would require a better understanding of the ideal $\mathsf{J}^{\lambda,\nu}_{n,i_0}$. As a particular case, one can ask if there are equivalences between certain categories of modules over the Lie algebra $\widehat{\mathfrak{sl}}_n(\Pi(Q))$ and over $\widehat{\mathfrak{sl}}_n(e_{i_0}\Pi(Q)e_{i_0})$. One can also ask the same question with $\widehat{\mathfrak{sl}}_n$ replaced by $\mathfrak{gl}_n$ and $\Pi(Q)$ replaced by $\Pi^{\lambda_i=1\forall i}(Q)$. In the same line of thought, one can wonder about the relationship between the categories of modules over $\widehat{\mathfrak{sl}}_n(\mathbb{C}[u,v]\rtimes\Gamma)$ and $\widehat{\mathfrak{sl}}_n(\mathbb{C}[u,v]^\Gamma)$, or over $\mathfrak{gl}_n(A_1\rtimes\Gamma)$ and $\mathfrak{gl}_n(A_1^\Gamma)$ where A_1 is the first Weyl algebra.

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