Embeddings of Symmetric Varieties

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Abstract

We generalize to the case of a symmetric variety the construction of the enveloping semigroup of a semisimple algebraic group due to E.B. Vinberg, and we establish a connection with the wonderful completion of the associated adjoint symmetric variety due to C. De Concini and C. Procesi.

Introduction

In [Vin], Vinberg classifies linear algebraic semigroups in characteristic zero which are equivariant, dominant,normal, affine embeddings of reductive algebraic groups, and studies some of their properties. Furthermore, to a semisimple algebraic group G_0, he associates an affine algebraic monoid with certain nice properties, the enveloping semigroup $\mathrm{Env}(G_0)$ and shows how the wonderful completion of the adjoint group of G_0 can be obtained from $\mathrm{Env}(G_0)$.

We generalize this construction to the case of a symmetric variety of a semisimple algebraic group. We adopt the following definition: a homogeneous space G/H of the reductive group G is called symmetric if there exists an involution τ of G such that $G^\tau \subseteq H \subseteq N_G(G^\tau)$, G^τ being the subgroup of fixed points. Every symmetric variety is isomorphic to one arising from a simply connected group ([Vust2]). A classification of equivariant normal embeddings of symmetric spaces can be found in [Vust2], and those which are affine can be identified using the affinity criterion for spherical varieties given in [Knop]. Let G_0 be a semisimple simply connected algebraic group of rank n over an algebraically closed field k of characteristic zero. Fix a non-trivial involution σ of G_0 with fixed-point subgroup K_0, whose normalizer in G_0 is written H_0. If Y is an affine G-variety, G a reductive algebraic group, $\Lambda(Y)$ will denote the group formed by the B-weights of the elements of the set $k(Y)^{(B)}$ of semi-invariants for the action of B, B a Borel subgroup of G. Let $G_1 = G_0 \times S_0$, with S_0 a maximal anisotropic torus of G_0, and $H_1 = \Delta^{1,-1}(N_0)(K_0 \times {S_0}^\sigma), \Delta^{1,-1}(N_0) = \{(s, s^{-1}) | s \in N_0\}, N_0 = N_{S_0}(K_0)$. We define $\mathrm{Env}(G_0/K_0)$ to be the affine variety over k which is the spectrum of the ring $\oplus_{\nu \in \mathcal{L}} k[G_1/H_1]_\nu$, where $k[G_1/H_1]_\nu$ is the isotypic component of $k[G_1/H_1]$ corresponding to the integral dominant weight ν, and $\mathcal{L}$ is the $\mathbb{Q}^+$-cone in $(\Lambda(G_0/K_0) \oplus X^*(S_{0K_0}))^{\Delta^{1,-1}(N_0)} \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by $\{(\overline{\eta}_i, w_0\overline{\eta}_i), (0, -\overline{\alpha_i})\}_{i=1}^l$, S_{0K_0} being the group $S_0/S_0 \cap K_0$. Here, the $\overline{\alpha_i}$ are the simple roots of a root system in $X^*(S_0) \otimes_{\mathbb{Z}} \mathbb{Q}$, the $\overline{\eta}_i$ are the corresponding fundamental weights, and w_0 is the longuest element of its Weyl group. (See section 1.2 for more information on the restricted root system.) Note that since G_1/H_1 is G_1-spherical, $k[G_1/H_1]$ is multiplicity free, so $k[G_1/H_1]_\nu$ is actually irreducible.

After some preliminary notions concerning symmetric varieties, the first section is devoted to the theory of spherical varieties developed by Brion, Luna and Vust ([BLV],[LuVu]); a concise exposition can be found in [Knop], but the main reference for us is [Vust2]. The language of colored cones developed there will be used throughout. Afterwards, we recall some properties of the wonderful compactification $\overline{G_{ad}/K_{ad}}$, constructed by De Concini and Procesi in [DP1], of the symmetric variety G_{ad}/K_{ad} of the adjoint group G_{ad}, and in section 3 we elaborate on the definition of $\mathrm{Env}(G_0/K_0)$.

Section 4 is devoted to establishing a connection between $\mathrm{Env}(G_0/K_0)$ and $\overline{G_{ad}/K_{ad}}$ (cf. propositions 1,2,3): $\mathrm{Env}(G_0/K_0)$ is a fiber product, over an affine toric variety, of affine space with the normalization of a multicone over $\overline{G_{ad}/K_{ad}}$. The next one concerns properties of $\mathrm{Env}(G_0/K_0)$: we study its orbit decomposition (propositions 4,6), certain toric sub-varieties, and prove in section 5.3 that it enjoys a universal property (theorem 3) like Vinberg's enveloping semigroup. In the last section, we show how to construct $\overline{G_{ad}/K_{ad}}$ as a geometric quotient of an open subvariety Σ of $\mathrm{Env}(G_0/K_0)$; our approach is similar to Vinberg's, with one noticeable difference: we take the B_1-stable cell $\mathcal{B}_\Sigma$ in Σ (B_1 a Borel subgroup of G_1) to be the canonical affine B_1-stable subset introduced in [Knop].

Remark 1. *All embeddings of homogeneous varieties will be assumed normal or will be shown to be so, unless otherwise specified. All varieties will be defined over the algebraically closed field k of characteristic zero.*

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1 Preliminaries

1.1 Notation

Let's introduce the rest of the notation that we will need. If $L, M < G$, G any group, $L \cap M \lhd L$, then $L_M = L/L \cap M$. Z_0 is the center of G_0, the adjoint group of which is $G_{ad} = G_0/Z_0$; note that σ descends to G_{ad}, so we can define similarly K_{ad} ($= (G_{ad})^\sigma$) and the symmetric variety G_{ad}/K_{ad} ($\cong G_0/H_0$). We fix a maximal σ-stable torus T_0 of G_0 containing S_0. R_0 is the root system of G_0 with respect to T_0, and $\alpha_1, \ldots, \alpha_n$ are a choice of simple roots, the α_i with $i > m$ being exactly the simple roots which vanish on $\mathfrak{s}_0$ (=Lie(S_0)), i.e. those which are fixed by σ. (See the next subsection for more concerning our choice of basis of R_0.) $T_0/T_0^\sigma \cong S_{0K_0}$, and the multiplication morphism $T_0^\sigma \times S_0 \longrightarrow T_0$ is an isogeny. $N_0 = N_{S_0}(K_0)$, and by lemma 1.7 in [DP1], N_0 is the subset of elements $s \in S_0$ such that $s^2 \in Z_0$, so N_0 is a finite group.

We will need to extend these notions to G_1. H_1 was defined in the introduction and it is equal to $\{(ks, s^{-1}) \in G_1 | k \in K_0, s \in N_0\}$. σ gives rise to an involution of G_1 with $G_1^\sigma = K_1 = K_0 \times S_0^\sigma$. Let $T_1 = T_0 \times S_0$, $S_1 = S_0 \times S_0$. Furthermore, if $G_0 = \widetilde{G_0} \times \widetilde{G_0}$ ($\widetilde{G_0}$ being any reductive algebraic group over k) and σ is the transposition $(x, y) \longrightarrow (y, x)$, then $K_0 = \Delta\widetilde{G_0}$, $G_0/K_0 \cong \widetilde{G_0}$, $S_0 = \Delta^{1,-1}(\widetilde{T_0}) = \{(t, t^{-1}) | t \in \widetilde{T_0}\}$, $T_0 = \widetilde{T_0} \times \widetilde{T_0}$, $N_{G_0}(K_0) = (\widetilde{Z_0} \times \widetilde{Z_0})\Delta\widetilde{G_0}$,

$N_0 = \{(s, s^{-1}) \in S_0 | s^2 \in \widetilde{Z_0}\}$, $S_0 \cap K_0 = \{(s, s^{-1}) \in S_0 | s^2 = 1\}$. We claim that, in this case, $G_1/H_1 \cong \widetilde{G_0} \times \widetilde{T_0}/\Delta^{1-1}(\widetilde{Z_0})$. Consider the morphism $\varphi : \widetilde{G_0} \times \widetilde{G_0} \times \Delta^{1,-1}(\widetilde{T_0})/M_0 \longrightarrow \widetilde{G_0} \times \widetilde{T_0}/\Delta^{1,-1}(\widetilde{Z_0})$, where $M_0 = \Delta^{1,-1}(N_0)(\Delta(\widetilde{G_0}) \times (\Delta^{1,-1}(\widetilde{T_0}) \cap \Delta(\widetilde{G_0})))$, defined by $\varphi((g_1, g_2, t, t^{-1})M_0) = (g_1 g_2^{-1}, t^2)\Delta^{1,-1}(\widetilde{Z_0})$. φ is a bijective quotient morphism, hence an isomorphism.

1.2 Restricted roots and weights

For an arbitrary algebraic group G, let $X_*(G)$ be the set of its one parameter subgroups and $X^*(G)$ be its set of characters. According to [Vust1], $\exists \tilde{\lambda} \in X_*(S_0)$ such that $P_0(\tilde{\lambda})$ is a parabolic subgroup of G_0 with an open dense orbit in G_0/K_0. Here $P_0(\tilde{\lambda})$ is defined as the parabolic subgroup of G_0 containing T_0 and corresponding to the roots $\{\alpha \in R_0 | \langle \tilde{\lambda}, \alpha \rangle \geq 0\}$. Set $P_0 = P_0(\tilde{\lambda})$; then $Z_{G_0}(S_0) = Z_{G_0}(\tilde{\lambda}) = P_0 \cap \sigma(P_0)$. Moreover, $\exists \tilde{\mu} \in X_*(T_0)$ such that $B_0 = P_0(\tilde{\mu})$ is a Borel subgroup of G_0 contained in P_0 and $B_0 K_0 = P_0 K_0$ is open in G_0 (similarly for K_0 replaced by H_0). We can assume that B_0 corresponds to our previous choice of simple roots $\alpha_1, \ldots, \alpha_n$. We will need the following lemma in order to be able to use our choice of B_0 (actually, B_0^-) in section 2.

Lemma 1. *Our choice of root system satisfies the condition in lemma 1.2 of [DP1], that is, if α is a positive root which is not identically zero on $\mathfrak{s}_0$, then $\sigma(\alpha)$ is a negative root.*

Proof. If α is a positive root in $\{\alpha \in R_0 | \langle \tilde{\lambda}, \alpha \rangle > 0\}$, then $\sigma(\alpha)$ is negative because $\sigma(\tilde{\lambda}) = -\tilde{\lambda}$ ([Vust1], prop. 4) and $B_0 \subseteq P_0$. Therefore,it is enough to notice that $\{\alpha \in R_0 | \langle \tilde{\lambda}, \alpha \rangle > 0\}$ is $\{\alpha \in R_0^+ | \alpha \not\equiv 0 \text{ on } \mathfrak{s}_0\}$. Indeed, if $\langle \tilde{\lambda}, \alpha \rangle = 0, \alpha \in R_0^+$, then $U_\alpha \subseteq Z_{G_0}(\tilde{\lambda}) = Z_{G_0}(S_0)$, hence $\alpha \equiv 0$ on $\mathfrak{s}_0$. □ □

Set $\overline{R_0} = \{\overline{\alpha} = \alpha - \sigma(\alpha) | \alpha \in R_0\}$. Lemma 2.3 in [Vust2] says that $\overline{R_0}$ is a root system in $X^*(S_{0H_0}) \otimes_{\mathbb{Z}} \mathbb{Q}$, which is a $\mathbb{Q}$-vector space of dimension l, l being the rank of S_0, which is also the rank of the symmetric variety G_0/K_0. We can order the simple roots of R_0 in such a way that $\alpha_{m+1}, \ldots, \alpha_n$ are exactly those fixed by σ, and $\{\overline{\alpha_1}, \ldots, \overline{\alpha_l}\}$ is a set of simple roots for $\overline{R_0}$ ($l < m \leq n$); furthermore, if $i > l$ and $\sigma(\alpha_i) \neq \alpha_i$, there is an $s \leq l$ such that $\overline{\alpha_i} = \overline{\alpha_s}$. $\overline{\alpha_1}^\vee, \ldots, \overline{\alpha_l}^\vee$ are the simple dual coroots. The character group of S_{0K_0} is $\{\overline{\chi} = \chi - \sigma(\chi) | \chi \in X^*(T_0)\}$. We denote by $\overline{\eta}_i, i = 1, \ldots, l$, the fundamental weights of the root system $\overline{R_0}$; by lemma 3.1 in [Vust2], the weight lattice of $\overline{R_0}$ is $X^*(S_{0K_0})$ and the root lattice is $X^*(S_{0H_0})$.

We will need to know later how the weights $\overline{\eta}_i$ are related to the fundamental weights of R_0. We can partition these into two sets

$$\{\omega_1, \ldots, \omega_m\}, \{\zeta_1, \ldots, \zeta_k\}, m + k = n$$

as in [DP1] §1.3, that is, such that $\langle \omega_i, \alpha_j^\vee \rangle = 0$ if $j > m, \langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ if $j \leq m, \langle \zeta_i, \alpha_j^\vee \rangle = \delta_{(i+m)j}$ if $j > m, \langle \zeta_i, \alpha_j^\vee \rangle = 0$ if $j \leq m$. It is proved in [DP1] that if $i \leq m$ then $\sigma(\alpha_i) = -\alpha_j - \sum_{r>m} n_{ri} \alpha_r$ where the n_{ri} are non-negative integers, $j \leq m$, and $\sigma(\omega_i) = -\omega_j$. In the same article, the authors argue that, for $i \leq m$, $\overline{\eta}_i = c\omega_i$ or $\overline{\eta}_i = c(\omega_i + \omega_j)$, where $c = 1$ or 2. We can be a little more precise. According to [Vust2], there are three possible cases for a simple root, although only the first two are of interest to us: $(i \leq l \leq m)$

1. $\sigma(\alpha_i) = -\alpha_i$, so $\sigma(\omega_i) = -\omega_i$, $\langle(\overline{\alpha_i})^\vee, \overline{\omega_i}\rangle = \frac{1}{2}\langle\alpha_i^\vee, \overline{\omega_i}\rangle = \langle\alpha_i^\vee, \omega_i\rangle = 1$ and $\langle(\overline{\alpha_j})^\vee, \overline{\omega_i}\rangle = 0$ if $j \neq i$. Hence $\overline{\eta}_i = 2\omega_i$.

2. $\langle\alpha_i^\vee, \sigma(\alpha_i)\rangle = 0$; then $\langle(\overline{\alpha_i})^\vee, \omega_i + \omega_j\rangle = \langle\alpha_i^\vee, \omega_i + \omega_j\rangle = 1$ or 2, $\langle(\overline{\alpha_t})^\vee, \omega_i + \omega_j\rangle = \langle\alpha_t^\vee, \omega_i + \omega_j\rangle = 0$ if $t \neq i, j$. It follows that $\overline{\eta}_i$ is either $\omega_i + \omega_j$ if $j \neq i$, or ω_i.

3. $\langle\alpha_i^\vee, \sigma(\alpha_i)\rangle = 1$. If this happens, then $\overline{R_0}$ is not reduced and any simple root $\overline{\alpha_i}$ of $\overline{R_0}$ comes from a root of the first two types.

$\mathcal{W}_0$ ($\overline{\mathcal{W}_0}$) denotes the Weyl group of R_0 ($\overline{R_0}$), and w_0 is the longuest element of $\mathcal{W}_0$. $\overline{\mathcal{W}_0}$ is isomorphic to $N_{K_0}(S_0)/Z_{K_0}(S_0)$ [Rich], and this is isomorphic to $N_{H_0}(S_0)/Z_{H_0}(S_0)$ because ([Vust2]) $H_0 = (S_0 \cap H_0)K_0 \implies N_{H_0}(S_0) = (S_0 \cap H_0)N_{K_0}(S_0), Z_{H_0}(S_0) = (S_0 \cap H_0)Z_{K_0}(S_0)$.

1.3 Regular functions on G_1/H_1

$k[G_0] = \oplus_\lambda V_\lambda \otimes_k V_\lambda^*$, where λ runs over all the dominant integral weights of B_0 and V_λ is the irreducible representation of G_0 of highest weight λ with respect to B_0. To obtain $k[G_0/K_0] = k[G_0]^{K_0}$, we have to take the sum over those λ such that V_λ^* contains a K_0-fixed non-zero vector, which is unique up to a scalar because B_0K_0 is dense in G_0. If this is the case, then $V_\lambda \cong V_\lambda^{*,\sigma}$ ([DP1] lemma 1.6), so V_λ contains also a K_0-fixed non-zero vector, and vice-versa. (By $V_\lambda^{*,\sigma}$, we mean the G_0-module V_λ^* with the action twisted by σ.) Therefore,

$$k[G_0/K_0] \cong \bigoplus_{\dim\, V_\lambda^{K_0}=1} V_\lambda.$$

Suppose that $\dim V_\lambda^{K_0} = 1$ and let $v \in V_\lambda^{K_0}$. We claim that N_0 acts on the line spanned by v by the character λ. Indeed, it follows from the analysis done in section 1.7 of [DP1] that $v \otimes v = v_\lambda \otimes v_\lambda + \sum_{i=1}^m u_i \otimes v_i$, where v_λ is a highest weight vector of V_λ and $u_i \otimes v_i$ is a weight vector of smaller weight. This implies that $v = v_\lambda + \sum \tilde{v}_i$, $\tilde{v}_i$ being a weight vector of weight $\lambda - \sum_j a_i^j \alpha_j$, say. Let $s \in N_0$; then sv is a multiple of v, and $sv = \chi^\lambda(s)v + \sum_{i=1}^m \chi^\lambda(s)\prod_j \chi^{-a_i^j\alpha_j}(s)\tilde{v}_i = \chi^\lambda(s)(v_\lambda + \sum_{i=1}^m \prod_j \chi^{-a_i^j\alpha_j}(s)\tilde{v}_i)$; therefore $\prod_j \chi^{-a_i^j\alpha_j}(s) = 1\ \forall i$. ($\chi^\lambda$ is the multiplicative character corresponding to λ.)

The isotypic component $k[G_0/K_0]_\lambda$ of $k[G_0/K_0]$ under left multiplication by G_0 is spanned by the functions $f \otimes_k f_\lambda^*$, where $f_\lambda^* \in V_\lambda^{*,K_0}$. The argument above shows that N_0 acts by right multiplication by the character $-w_0(\lambda)$ on $k[G_0/K_0]_\lambda$. If $k[G_0/K_0]_\lambda \otimes_k \chi^\mu$ is an irreducible component of $k[G_0/K_0 \times S_{0K_0}]$, then $\Delta^{1,-1}(N_0)$ acts on it (by multiplication on the right) by the character $-w_0(\lambda)-\mu$. Therefore,

$$k[G_1/H_1] = \bigoplus_{-w_0(\lambda)-\mu\in\mathbb{Z}\{\overline{\alpha_1},\ldots,\overline{\alpha_l}\}} k[G_0/K_0]_\lambda \otimes_k \chi^\mu$$

$$k[G_1/H_1]_{(\lambda,-\mu)} = k[G_0/K_0]_\lambda \otimes_k \chi^\mu.$$

(Note that S_0-acts on the function χ^μ by the character $-\mu$ under the action given by $(s_1\chi^\mu)(s_2) = \chi^\mu(s_1^{-1}s_2)$; this explains the minus sign.) By $(\Lambda(G_0/K_0)\oplus X^*(S_{0K_0}))^{\Delta^{1,-1}(N_0)}$, we mean the B_1-weights of the rational functions on $G_0/K_0\times S_{0K_0}$ which are also rational functions on G_1/H_1, so they are all the weights (λ,μ) of $k[G_0/K_0]\otimes_k k[S_{0K_0}]$ such that $\mu - w_0(\lambda)\in\mathbb{Z}\{\overline{\alpha_1},\ldots,\overline{\alpha_l}\}$. Note that $\Lambda(G_0/K_0)$ is a subgroup of $X^*(T_0)$ stable under $-w_0$, and $\chi^{-\lambda}|_{N_0} = \chi^{-w_0(\lambda)}|_{N_0}$, i.e. $\lambda - w_0(\lambda)\in\mathbb{Z}\overline{R_0}$.

Finally, if for any affine G-variety Y - G a reductive group - we denote by $\Lambda_+(Y)$ the set of highest weights of the G-module $k[Y]$, then restriction of weights from T_0 to S_0 establishes an isomorphism between $\Lambda_+(G_0/K_0)$ and $X^+(S_{0K_0})$ ([Vust2]).

1.4 Classification of embeddings of symmetric varieties

We will be interested in normal embeddings of the varieties G_0/K_0, G_1/H_1 and G_{ad}/K_{ad}, but only in the last two cases will we consider dominant ones, that is, embeddings containing the given symmetric variety as a dense subset. We present in this section the combinatorial data associated to these varieties. Since G_0 is semisimple and simply connected, we can apply directly the results of [Vust2]. However, this is not the case for G_1, so we have to make some slight modifications.

Spherical varieties (i.e. normal, irreducible G-varieties which contain an open orbit under the action of a Borel subgroup of a reductive group G, e.g. symmetric varieties) can be classified in terms of certain combinatorial data (see e.g. [Knop]). Let $\mathcal{D}(G_0/H_0)$ denote the set of B_0-stable irreducible divisors of G_0/H_0; these are the colors of G_0/H_0. For a simple (i.e. having only one closed orbit) embedding E_0 of the homogeneous space G_0/H_0, $\mathcal{D}(E_0)$ is just the set of B_0-stable prime divisors of E_0. The set of colors $\mathcal{F}(E_0)$ of E_0 consists of the B_0-stable prime divisors D of G_0/H_0 whose closure $\overline{D}$ in E_0 contains the (unique) closed orbit of E_0. For $D\in\mathcal{D}(E_0)$, v_D denotes the normalized discrete valuation of $k(G_0/H_0)$ associated to D.

Let $\mathcal{V}(G_0/H_0)$ be the set of normalized G_0-invariant discrete valuations of $k(G_0/H_0)$. Each G_0-stable prime divisor in E_0 determines an element of $\mathcal{V}(G_0/H_0)$; the set of all valuations arising in this way is written $\mathcal{V}(E_0)$.

Theorem 1 (cf. [LuVu]). *A simple normal embedding E_0 of G_0/H_0 is uniquely determined by the data $(\mathcal{F}(E_0),\mathcal{V}(E_0))$.*

Denote by $\mathcal{P}_0^{H_0}$ the subgroup of $k(G_0/H_0)^\times$ consisting of the normalized eigenvectors for the action of P_0 (i.e. those taking the value 1 at $1\cdot H_0$);

$\mathcal{P}_0^{H_0} \cong \Lambda(G_0/H_0)$. Each valuation v gives us an element $\rho(v)$ in $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{P}_0^{H_0}, \mathbb{Z})$. The map $\mathcal{V}(G_0/H_0) \xrightarrow{\rho} \mathrm{Hom}_{\mathbb{Z}}(\mathcal{P}_0^{H_0}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective, but not the one $\mathcal{D}(G_0/H_0) \xrightarrow{\rho} \mathrm{Hom}_{\mathbb{Z}}(\mathcal{P}_0^{H_0}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ in general. (The latter is one-to-one when, for instance, the symmetric variety is an algebraic group.)

If E_0 is a simple embedding of G_0/H_0, we let $\mathcal{C}(E_0)$ be the $\mathbb{Q}^+$-cone inside $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{P}_0^{H_0}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the finite sets $\rho(\mathcal{V}(E_0))$ and $\rho(\mathcal{F}(E_0))$.The pair $(\mathcal{C}(E_0), \mathcal{F}(E_0))$ is called the colored cone of E_0. More generally, we can state the following definition.

Definition 1. *A colored cone is a pair $(\mathcal{C}, \mathcal{F})$ with $\mathcal{C} \subseteq \mathrm{Hom}_{\mathbb{Z}}(\mathcal{P}_0^{H_0}, \mathbb{Q})$ and $\mathcal{F} \subseteq \mathcal{D}(G_0/H_0)$, such that $\mathcal{C}$ is a cone generated by $\rho(\mathcal{F})$ and a finite subset of $\mathbb{Q}^+\mathcal{V}(G_0/H_0)$, and $\mathcal{C}^\circ \cap \mathbb{Q}^+\mathcal{V}(G_0/H_0) \neq \phi$.*

Note that $\mathcal{C}(E_0)$ is a fortiori generated also by $\rho(\mathcal{F}(E_0))$ and by $\mathcal{C}(E_0) \cap \mathbb{Z}^+\mathcal{V}(G_0/H_0)$.

Theorem 2 (cf. [Knop] §4.1). *There is a bijection between the set of simple normal embeddings of G_0/H_0 and the strongly convex rational polyhedral colored cones in the vector space $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{P}_0^{H_0}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$. This correspondence is the one described above.*

As proved in [Vust2], $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{P}_0^{H_0}, \mathbb{Z})$ is isomorphic to $X_*(S_{0H_0})$. An isomorphism is induced by the isomorphism $\phi : \mathcal{P}_0^{H_0} \longrightarrow X^*(S_{0H_0})$ given by $f \to -\omega$ if f is an eigenvector for P_0 of weight ω. Therefore, we can view the colored cone of a simple embedding of G_0/H_0 as a cone in $X_*(S_{0H_0}) \otimes_{\mathbb{Z}} \mathbb{Q} = X_*(S_0) \otimes_{\mathbb{Z}} \mathbb{Q}$. ($X_*(S_{0H_0})$ has finite index in $X_*(S_0)$.) From the previous section, we know that $X^*(S_{0H_0})$ is stable under w_0. w_0 thus induces an automorphism of $X_*(S_{0H_0})$ also.

Everything said so far (in particular theorem 1 and 2) is valid also for embeddings of the symmetric variety G_1/H_1 with P_0 replaced by $P_1 = P_0 \times S_0$, B_0 by $B_1 = B_0 \times S_0$, and the maximal anisotropic torus being $S_1 = S_0 \times S_0$. $\mathcal{P}_1^{H_1}$ is the subgroup of $k(G_1/K_1)^\times$ consisting of the normalized eigenvectors for the action of P_1. We can define similarly $\mathcal{V}(G_1/H_1)$, $\mathcal{D}(G_1/H_1)$, and maps from these two sets to the lattice $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{P}_1^{H_1}, \mathbb{Z})$. The colored cone $\mathcal{C}(E)$ and the colors $\mathcal{F}(E)$ of a simple embedding E of G_1/H_1 are defined as before. We obtain also an isomorphism between $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{P}_1^{H_1}, \mathbb{Z})$ and $X_*(S_{1H_1})$ by sending a B_1-weight vector of weight (w_1, w_2) to the character $(-w_1|_{S_0}, -w_2|_{S_0})$; this follows from results in [Vust1] §2.1, 2.2 which are valid for any reductive group.

Proposition 1 §2.4 in [Vust2] says that the $\rho(u_D), D \in \mathcal{D}(G_0/H_0)$, get identified, under the isomorphism $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{P}_0^{H_0}, \mathbb{Z}) \longrightarrow X_*(S_{0H_0})$, to the negative of the simple coroots of $\overline{R_0}$.

The B_1-stable (or P_1-stable) divisors of G_1/H_1 are exactly the images of $D \times S_0/S_0^\sigma$ under the quotient morphism $G_0/K_0 \times S_0/S_0^\sigma \longrightarrow G_1/H_1$, where D is a B_0-stable (or P_0-stable) divisor of G_0/K_0. Therefore, the $\rho(u_D)$, with $D \in \mathcal{D}(G_1/H_1)$, can be identified with the subset $\{(-(\overline{\alpha_i})^\vee, 0) | 1 \leq i \leq l\}$. We can reach this conclusion also by mimicking the proof in [Vust2] since $k[G_0 \times S_0]$ is also a UFD.

According to proposition 2, §2.4 in [Vust2], the set $\mathcal{V}(G_0/H_0)$ corresponds to the set of indecomposable elements in $\overline{C_0} \cap X_*(S_{0H_0})$, where $\overline{C_0}$ is the chamber determined by the choice of positive roots in R_0 (i.e. the one containing $\tilde{\lambda}$). As for $\mathcal{V}(G_1/H_1)$, it corresponds to the indecomposable elements in $\overline{C} \cap X_*(S_{1H_1})$, $\overline{C}$ being again defined by our choice of positive roots $\{(\overline{\alpha_i}, 0)\}_{i=1}^{l}$; the proof in [Vust2] applies to this case too.

1.5 Valuations and one-parameter subgroups

The set $\mathcal{V}(E)$, E an embedding of G_1/H_1, can be described in terms of certain one-parameter subgroups. Let λ be a one-parameter subgroup of G_1. λ induces a valuation $v_\lambda \in \mathbb{Z}^+\mathcal{V}(G_1/H_1)$ in the following way. Let $f \in k[G_1]$; then $f = \sum_{n\in\mathbb{Z}} f_n$ where $\lambda(t)f_n = t^n f_n$ for all $t \in k^*$; we set $v_\lambda(f) = \inf\{n \in \mathbb{Z} | f_n \neq 0\}$, extend v_λ to $k(G_1)$, and restrict it to $k(G_1)^{H_1}$.

An elementary embedding of G_1/H_1 is a normal (a fortiori smooth) embedding consisting of two orbits: G_1/H_1 and a closed orbit of codimension 1. It follows from the general theory of elementary embeddings in [LuVu] (§4.10, §7.5) that there exists a bijection (denoted $E' \leftrightarrow v_{E'}$, where $v_{E'}$ is the valuation associated to the unique closed orbit of E') between elementary embeddings and G_1-invariant, discrete, normalized valuations of $k(G_1/H_1)$. If E' is such an embedding, $x \in E'$ a point with isotropy group equal to H_1, then there exists a one-parameter subgroup $\lambda_{E'}$ of S_{1H_1} such that $\lim_{t\to 0} \lambda_{E'}(t)x$ belongs to the open P_1-orbit in the unique closed G_1-orbit of E' ([BLV] §4). Furthermore, $v_{\lambda_{E'}}$ is equivalent to $v_{E'}$, and we can choose $\lambda_{E'}$ in $X_+(S_{1H_1})$.

Now let E be an embedding of G_1/H_1 and $\mathcal{O}$ a G_1-orbit of codimension 1 in E. Then $G_1/H_1 \cup \mathcal{O}$ is an elementary embedding of G_1/H_1. It follows from the previous paragraph that there exists a one-parameter subgroup $\lambda \in X_*(S_{1H_1})$ such that v_λ is equivalent to the G_1-invariant, discrete, normalized valuation of $k(G_1/H_1)$ corresponding to $\mathcal{O}$. In conclusion, one way to find $\mathcal{V}(E)$ is to identify the one-parameter subgroups of S_{1H_1} for which $\lambda(t)x$ converges in E when $t \to 0$ to a point in the open P_1-orbit of a G_1-stable prime divisor.

Using the bijection $E' \leftrightarrow v_{E'}$, it is possible to give more information on the set $\mathcal{C}(E) \cap \mathbb{Z}^+\mathcal{V}(G_1/H_1)$. If $v_{E'} \in \mathcal{C}(E) \cap \mathbb{Z}^+\mathcal{V}(G_1/H_1)$, then we can find a morphism $E' \xrightarrow{\varphi} E$ ($\varphi|_{G_1/H_1} = id$), and $\lim_{t\to 0} \lambda_{E'}(t)1 \cdot H_1$ exists in E', hence $\lim_{t\to 0} \lambda_{E'}(t)1 \cdot H_1$ exists also in E via φ. Conversely, if $\lambda \in X^*(S_{1H_1})$ and, without lost of generality, λ lies in the positive Weyl chamber, and if $\lambda(t)1 \cdot H_1$ converges in E as $t \to 0$, then we can extend the identity map on G_1/H_1 to a morphism $E' \longrightarrow E$ (by [LuVu]§4.9), which implies that $v_{E'} \in \mathcal{C}(E)$, E' being the elementary embedding such that $v_{E'}$ is the normalized invariant valuation equivalent to v_λ.

2 The wonderful completion of G_{ad}/K_{ad}

In this section, we recall some of the properties of the wonderful compactification $\overline{G_{ad}/K_{ad}}$ of G_{ad}/K_{ad} ([DP1]). $\overline{G_{ad}/K_{ad}}$ is a smooth complete variety over

k containing G_{ad}/K_{ad} as a dense G_0-orbit, and the complement of G_{ad}/K_{ad} consists of l smooth, normal crossing divisors X_i. Moreover, the G_0-orbits of $\overline{G_{ad}/K_{ad}}$ are in a bijective correspondence with the subsets of $\{1,\dots,l\}$, and the orbit closures are exactly the intersections $X_{\{i_1,\dots,i_k\}} = X_{i_1}\cap\ldots\cap X_{i_k}$.

$\overline{G_{ad}/K_{ad}}$ can be constructed as the closure of the G_0-orbit in $\mathbb{P}(V_{2\lambda})$ of the class of the unique - up to a scalar multiple - vector h' in $V_{2\lambda}$ fixed by K_0, where λ is a regular special weight, dominant with respect to B_0^-. (We choose B_0^- instead of B_0 for convenience.) The geometric analysis of $\overline{G_{ad}/K_{ad}}$ can be carried out by studying a certain affine cell (i.e. locally closed subvariety isomorphic to affine space), denoted $\mathcal{B}$, which enjoys the following properties: $\mathcal{B}$ is B_0-stable and isomorphic to $U_{S_0}\times\mathbb{A}^l$ where U_{S_0} is the unipotent group generated by the root subgroups corresponding to the positive roots in R_0 whose restrictions to $\mathfrak{s}_0$ are non-zero, the torus T_0 acts on it by multiplication by $\chi^{\overline{\alpha_i}}(t)$ on the i^{th}-coordinate of $\mathbb{A}^l$, and the intersection of the G_0-orbit of $[h']$ with $\mathcal{B}$ is the open set where the last l coordinates are non-zero. Furthermore, the unique closed G_0-orbit Y in $\overline{G_{ad}/K_{ad}}$ is the closure of $U_{S_0}\times\{0\}$, and the intersection of $\mathcal{B}$ with X_i is the variety of codimension one given by the vanishing of the i^{th} coordinate of $\mathbb{A}^l$.

Let's determine the combinatorial data of $\overline{G_{ad}/K_{ad}}$ as a G_0-spherical variety with respect to the choice of B_0 as Borel subgroup. It follows from the description given in the previous paragraph that Y is not contained in the closure of any of the B_0-stable divisors of G_{ad}/K_{ad} because these are in the complement of $\mathcal{B}$. This means that $\overline{G_{ad}/K_{ad}}$ has no colors, so $\overline{G_{ad}/K_{ad}}$ is an example of a toroidal spherical variety.

Let x_j be a local equation for $X_j\cap\mathcal{B}$ as in [DP1]. x_j is a rational function on G_{ad}/K_{ad} which is a B_0-eigenvector and its weight is $w_0(\overline{\alpha_j}) = \overline{w_0(\alpha_{i_j})} = -\overline{\alpha}_{w_0(i_j)}$ (up to reordering the x_j). Here is what we mean by this. B_0^- is the Borel subgroup corresponding to the choice $\{w_0(\alpha_i)\}_{i=1}^n$ of simple roots. This basis satisfies also the condition of lemma 1.7 in [DP1]. w_0 induces a permutation, also denoted w_0, of the set $\{1,\dots,n\}$ by $w_0(\alpha_i) = -\alpha_{w_0(i)}$. If $\overline{w_0(\alpha_{j_1})},\dots,\overline{w_0(\alpha_{j_l})}$ are all independent (distinct and non-zero), then we can assume that $\{w_0(j_1),\dots,w_0(j_l)\} = \{1,\dots,l\}$, $\{\overline{w_0(\alpha_{j_1})},\dots,\overline{w_0(\alpha_{j_l})}\} = -\{\overline{\alpha_1},\dots,\overline{\alpha_l}\}$. In particular, $w_0(\overline{\alpha_j}) = -\overline{\alpha_k}$ (for some k) $=\overline{w_0(\alpha_{j_i})}$ for some $j_i, 1\le i\le l$.

Let v_k be the G_0-invariant valuation corresponding to X_k. Then

$$\rho(v_k)(-(\overline{w_0(\alpha_{i_j})})) = v_k(x_j) = \delta_{jk} \Longrightarrow \rho(v_k) = \widetilde{\eta}_{w_0(i_k)} \in X_*(S_{0H_0}).$$

Therefore, $\mathcal{C}(\overline{G_{ad}/K_{ad}}) = \mathbb{Q}^+\{\rho(v_1),\dots,\rho(v_l)\} = \overline{C_0} = \mathbb{Q}^+\mathcal{V}(G_0/H_0)$.

We can also characterize $\overline{G_{ad}/K_{ad}}$ as the unique dominant equivariant embedding of G_{ad}/K_{ad} which is simple, complete, and without colors. This follows from the results in section 1.4 and the combinatorial criterion for completeness of spherical varieties (cf. [Knop]).

$\overline{G_{ad}/K_{ad}}$ can be realized in many different ways. For $i = 1,\dots,l$, let h_i be a non-zero K_0-fixed vector in $V_{w_0(\overline{\eta}_i)}$; here $V_{w_0(\overline{\eta}_i)}$ is the irreducible G_0-module with highest weight $w_0(\overline{\eta}_i)$ with respect to B_0^-. h_i is unique up to a scalar. Set $h = h_1 + \ldots + h_l$. The wonderful completion of G_0/H_0 is the closure of the orbit of the line $[h]$ in $\mathbb{P}(V_{w_0(\overline{\eta}_1)} \oplus \ldots \oplus V_{w_0(\overline{\eta}_l)})$.

$V_{w_0(\overline{\eta}_1)} \otimes_k \cdots \otimes_k V_{w_0(\overline{\eta}_l)} = V_{w_0(\overline{\eta}_1+\ldots+\overline{\eta}_l)} \oplus W$ where W contains a K_0-fixed vector h_W; set $h_{1,\ldots,l} = h_1 \otimes \ldots \otimes h_l$ and $h' = h_{1,\ldots,l} + h_W$. Then $\overline{G_{ad}/K_{ad}} \cong \overline{G_0[h']} \subseteq \mathbb{P}(V_{w_0(\overline{\eta}_1)} \otimes \ldots \otimes V_{w_0(\overline{\eta}_l)})$, an isomorphism being given by the restriction of the projection $\mathbb{P}(V_{w_0(\overline{\eta}_1+\ldots+\overline{\eta}_l)} \oplus W) \longrightarrow \mathbb{P}(V_{w_0(\overline{\eta}_1+\ldots+\overline{\eta}_l)})$ along $\mathbb{P}(W)$ (see [DP1] §4.1).

Furthermore, the Segre embedding mapping $\mathbb{P}(V_{w_0(\overline{\eta}_1)}) \times \cdots \times \mathbb{P}(V_{w_0(\overline{\eta}_l)})$ into $\mathbb{P}(V_{w_0(\overline{\eta}_1)} \otimes_k \cdots \otimes_k V_{w_0(\overline{\eta}_l)})$ provides an isomorphism between $\overline{G_0[h']}$ and $\overline{G_0([h_1], \ldots, [h_k])}$, whence $\overline{G_0([h_1], \ldots, [h_k])}$ is isomorphic to $\overline{G_{ad}/K_{ad}}$.

Fix an ordered basis of $V_{w_0(\overline{\eta}_1+\ldots+\overline{\eta}_l)}$ consisting, say, of weight vectors, the last one being a highest weight vector for B_0^-. Let $\mathcal{A}$ be the affine subset of $\mathbb{P}(V_{w_0(\overline{\eta}_1+\ldots+\overline{\eta}_l)})$ where the last coordinate is non-zero; $\mathcal{A} \cap \overline{G_0[h']}$ is the affine cell $\mathcal{B}$. Using the isomorphism above, it follows that $\widetilde{\mathcal{A}} \cap \overline{G_0([h_1], \ldots, [h_l])} \cong \mathcal{B}$; here, $\widetilde{\mathcal{A}}$ is defined in a way similar to $\mathcal{A}$: for each $i = 1, \ldots, l$, choose an ordered basis Θ_i of $V_{w_0(\overline{\eta}_i)}$ whose last element is a highest weight vector, and let $\widetilde{\mathcal{A}}$ be the affine subvariety of $\mathbb{P}(V_{w_0(\overline{\eta}_1)}) \times \cdots \times \mathbb{P}(V_{w_0(\overline{\eta}_l)})$ defined by the non-vanishing of the last coordinate in each projective space.

3 Definition of **Env**(G_0/K_0)

3.1 First definition

Let $\mathrm{Env}(G_0/K_0)$ be the affine variety over k (see lemma 2 below) with coordinate ring $k[\mathrm{Env}(G_0/K_0)] = \oplus_{\nu \in \mathcal{L}} k[G_1/H_1]_\nu$, where $\mathcal{L}$ is the $\mathbb{Q}^+$-cone in $(\Lambda(G_0/K_0) \oplus X^*(S_{0K_0}))^{N_0} \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by $\{(\overline{\eta}_i, w_0\overline{\eta}_i), (0, -\overline{\alpha_i})\}_{i=1}^l$, and $k[G_1/H_1]_\nu$ is the isotypic component of $k[G_1/H_1]$ corresponding to the dominant weight ν of B_1. $\Lambda(G_0/K_0) \cong {\mathcal{P}_0}^{K_0} \cong X^*(S_{0K_0})$, and the highest weight semigroup of G_0/K_0 (i.e. the semigroup formed by the dominant integral weights of the isotypic components of G_0/K_0) is $X^+(S_{0K_0})$.

It is also possible to define its coordinate ring by using an idea from [Pop]. Let's put a polyfiltration on $k[G_0/K_0]$ by setting $k[G_0/K_0]_{\leq\lambda} = \oplus_{\mu\leq\lambda} k[G_0/K_0]_\mu$ for $\lambda \in X^*(S_{0K_0}) \cap (\mathbb{Q}^+ \overline{R_0}^+)$, where $\mu \leq \lambda$ means that $\lambda + w_0(\mu) \in \mathbb{Z}^+ \overline{R_0}^+$. The Rees algebra of this polyfiltration is a subalgebra of $k[G_0/K_0][s^{\pm\overline{\eta}_1}, \ldots, s^{\pm\overline{\eta}_l}]$, $s^{\overline{\eta}_1}, \ldots, s^{\overline{\eta}_l}$ being variables algebraically independent over $k[G_0/K_0]$. Then $k[\mathrm{Env}(G_0/K_0)]$ can be identified with this Rees algebra, i.e.

$$k[\mathrm{Env}(G_0/K_0)] \cong \bigoplus_{\lambda \in X^*(S_{0K_0}) \cap (\overline{R_0}^+ \otimes_{\mathbb{Z}} \mathbb{Q})} k[G_0/K_0]_{\leq\lambda} s^\lambda,$$

if we think of s^λ as a character of S_0.

Let A be the S_0/N_0-toric variety $\mathrm{Spec} \oplus_{\mu \in \mathbb{Z}^+\{\overline{\alpha}_1, \ldots, \overline{\alpha_l}\}} k \cdot \chi^\mu$. A is isomorphic to affine space $\mathbb{A}^l$ since the roots $\overline{\alpha_i}$ form a basis for the character lattice of

S_0/N_0 ([Vust2] lemma 3.1).

Lemma 2. *$k[Env(G_0/K_0)]$ is a finitely generated algebra.*

Proof. By corollary 4 in ([Pop]), it is enough to show that $k[\mathrm{Env}(G_0/K_0)]^{U_1}$ is finitely generated, where $U_1 = U_0 \times 1$ (resp. U_0) is a maximal unipotent subgroup of G_1 (resp. G_0). $\oplus_{\overline{\eta}\in X^+(S_{0K_0})} k[G_1/H_1]^{U_1}_{(\overline{\eta},w_0(\overline{\eta}))}$ is isomorphic to $k[G_0/K_0]^{U_0}$, and $\oplus_{\overline{\alpha}\in\mathbb{Z}^+\overline{R_0}^+} k[G_1/H_1]_{(0,-\overline{\alpha})} = k[A]$. Therefore we obtain that $k[\mathrm{Env}(G_0/K_0)]^{U_1} \cong k[A] \otimes_k k[G_0/K_0]^{U_0}$, which is finitely generated since the same holds for $k[G_0/K_0]^{U_0}$. Equivalently, we could have observed simply that $k[\mathrm{Env}(G_0/K_0)]^{U_1}$ is the semigroup algebra of the subsemigroup generated by $\{(\overline{\eta}_i, w_0\overline{\eta}_i), (0, -\overline{\alpha_i})\}_{i=1}^{l}$. □ □

Lemma 3. *$Env(G_0/K_0)$ is a normal variety.*

Proof. According to a theorem of Popov ([Pop]), it is enough to check that $k[\mathrm{Env}(G_0/K_0)]^{U_1}$ is normal; but $k[\mathrm{Env}(G_0/K_0)]^{U_1} \cong k[A] \otimes_k k[G_0/K_0]^{U_0}$ (see lemma 2) and $k[G_0/K_0]^{U_0}$ is a polynomial ring of dimension l according to [Vust2] §3.2. □ □

Lemma 4. *$Env(G_0/K_0)$ is an affine embedding of G_1/H_1.*

Proof. It is enough to prove that the functions in $k[\mathrm{Env}(G_0/K_0)]$ separate the points of G_1/H_1. If $(p_1, s_1)\Delta^{1,-1}(N_0)$ and $(p_2, s_2)\Delta^{1,-1}(N_0)$ are two distinct points of G_1/H_1, $p_i \in G_0/K_0, s_i \in S_{0K_0}$, with $s_1 s_2^{-1} \notin N_0$, then we can separate them using a character $\chi^{-\overline{\alpha}}$. Now if $s_1 s_2^{-1} \in N_0$, it is possible to find a function $f \in k[G_0/K_0]_{\overline{\eta}}$ which separates $p_1 s_1 s_2^{-1}$ and p_2. It follows that $f\chi^{-w_0(\overline{\eta})}((p_1, s_1)\Delta^{1,-1}(N_0)) \neq f\chi^{-w_0(\overline{\eta})}((p_2, s_2)\Delta^{1,-1}(N_0))$. □ □

The three preceding lemmas show that $\mathrm{Env}(G_0/K_0)$ is a spherical variety for G_1 (see the second definition below for more on this).

If we consider G_0 as a symmetric variety of $G_0 \times G_0$ via the involution $(g_1, g_2) \longrightarrow (g_2, g_1)$, then we get the enveloping semigroup of G_0. As a Borel subgroup of $G_0 \times G_0$, we choose $B_0 \times B_0^-$, and its maximal anisotropic torus is $\Delta^{1,-1}(T_0)$.

Let $\theta : T_0 \longrightarrow \Delta^{1,-1}(T_0)$ be the isomorphism $t \mapsto (t, t^{-1})$; θ induces an isomorphism $T_0/T_{0,2} \xrightarrow{\cong} \Delta^{1,-1}(T_0)/\Delta(T_0)$, where $T_{0,2}$ is the subgroup of elements of order 2 of T_0. $X^*(T_0/T_{0,2}) = 2X^*(T_0)$, $X^*(\Delta^{1,-1}(T_0)/\Delta(T_0)) = \{(\nu, -\nu) | \nu \in X^*(T_0)\}$, and $\theta^*(\mu, -\mu) = 2\mu$.

Let τ be the $G_0 \times G_0$-equivariant isomorphism $G_0 \longrightarrow G_0 \times G_0/\Delta(G_0)$ given by $\tau(g) = (g, 1)\Delta(G_0)$. Then $\tau^*(k[G_0 \times G_0/\Delta(G_0)]_{(\mu,-\mu)}) = V_\mu \otimes_k V_{-\mu}$.

According to our definition, $k[\mathrm{Env}(G_0 \times G_0/\Delta(G_0))]$ is equal to

$$\bigoplus_{\nu_2-\nu_1\in R_0^+} k[G_0 \times G_0/\Delta(G_0)]_{(\nu_1,-\nu_1)} \otimes_k k[\Delta^{1,-1}(T_0)/\Delta(T_0)]_{(w_0(\nu_2),-w_0(\nu_2))}.$$

Under the isomorphism

$$\tau \times \theta : G_0 \times (T_0/T_{0,2}) \longrightarrow (G_0 \times G_0/\Delta(G_0)) \times (\Delta^{1,-1}(T_0)/\Delta(T_0)),$$

$k[G_0 \times G_0/\Delta(G_0)]_{(\nu_1,-\nu_1)} \otimes_k k[\Delta^{1,-1}(T_0)/\Delta(T_0)]_{(w_0(\nu_2),-w_0(\nu_2))}$ corresponds to $V_{\nu_1} \otimes_k V_{-\nu_1} \otimes_k k[T_0/T_{0,2}]_{2w_0(\nu_2)}$. The isomorphism $T_0/T_{0,2} \longrightarrow T_0$ given by squaring identifies $k[T_0/T_{0,2}]_{2w_0(\nu_2)}$ with $k[T_0]_{w_0(\nu_2)}$. In conclusion, the decomposition of $k[\mathrm{Env}(G_0 \times G_0/\Delta(G_0))]$ as a $G_0 \times G_0 \times T_0$-module is

$$\bigoplus_{\nu_1,\nu_2 \in X^+(T_0), \nu_2-\nu_1 \in R_0^+} V_{\nu_1} \otimes_k V_{-\nu_1} \otimes_k \chi^{-w_0(\nu_2)},$$

which is how Vinberg had defined the coordinate ring of his enveloping semigroup since $V_{\nu_1} \otimes V_{-\nu_1}$ is isomorphic as a $G_0 \times G_0$-representation to the space of matrix coefficients of the irreducible representation $V_{\nu_1}^*(= V_{-w_0(\nu_1)})$.

We will use later the categorical quotient $\mathrm{Env}(G_0/K_0)/G_0$, which is isomorphic to A. Indeed,

$$k[\mathrm{Env}(G_0/K_0)]^{G_0} = (k[G_0/K_0]^{G_0} \otimes_k k[S_{0K_0}])_{\mathcal{L}} = k \otimes_k k[S_{0K_0}]_{\mathcal{L}} = k[S_{0K_0}]_{\mathcal{L}}.$$

By $(\cdot)_{\mathcal{L}}$, we mean the sum of the isotypic components with highest weights belonging to $\mathcal{L}$.

Let $\pi : \mathrm{Env}(G_0/K_0) \longrightarrow A$ be the quotient morphism. Then the fiber of π over $(1,\ldots,1)$ is G_0/K_0: the same argument as in [Vin] proposition 3 applies, except that in our case we have to use a theorem of Luna ([Luna]) which asserts that a homogeneous space G/L, with G and L reductive, is affinely closed (i.e. it admits only one affine embedding, namely itself) if and only if $[N_G(L) : L]$ is finite.

3.2 Second definition

$\mathrm{Env}(G_0/K_0)$ can also be defined in an equivalent way using the language of section 1. Let E be the G_1-spherical embedding of G_1/H_1 whose colors are all the colors of G_1/H_1 and whose colored cone is the $\mathbb{Q}^+$-cone in $X_*(S_{1H_1}) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by

$$\{(-(\overline{\alpha_i})^\vee, 0)\}_{i=1}^l \cup \{(\widetilde{\eta}_i, -w_0(\widetilde{\eta}_i))\}_{i=1}^l,$$

where the $\widetilde{\eta}_i$ are the indecomposable elements in $\overline{C}_0 \cap X_*(S_{0H_0})$. By lemma 3.1 in [Vust2], the root lattice $\mathbb{Z}\overline{R_0}$ is $X^*(S_0/N_0)$ and its dual is the coweight lattice - the fundamental coweights are those indecomposable elements.

This definition is equivalent to the first one. E is affine because $\mathcal{F}(E) = \mathcal{D}(G_1/H_1)$ ([Knop] Theorem 7.7). $\mathcal{C}(E)^\vee$ denotes the cone dual to $\mathcal{C}(E)$ under the natural pairing $X^*(S_{1H_1}) \times X_*(S_{1H_1}) \longrightarrow k$, and, under the identification in section 1, it sits inside $X^*(S_{1H_1})$. $-\mathcal{C}(E)^\vee \cap X^+(S_{1H_1})$ is the highest weight semigroup of $k[E]$. Indeed, since E is normal, a regular function $f \in k[G_1/H_1]^{(B_1)}$ extends to all of E if and only if $v(f) \geq 0$ $\forall v \in \mathcal{V}(E)$. This means that ($\chi_f$ being the B_1-weight of f)

$$k[E]^{(B_1)} = \{f \in k[G_1/H_1]^{(B_1)} | \chi_f \in -\mathcal{C}(E)^\vee \cap X^+(S_{1H_1})\},$$

hence

$$k[E] = \bigoplus_{\Lambda \in -\mathcal{C}(E)^\vee \cap X^+(S_{1H_1})} k[G_1/H_1]_\Lambda.$$

Notice that in this case $-\mathcal{C}(E)^\vee \subseteq \mathbb{Q}^+$-span of $X^+(S_{1H_1}) \otimes_\mathbb{Z} \mathbb{Q}$. We just have to see now that $-\mathcal{C}(E)^\vee$ is equal to $\mathcal{L}$.

3.3 Action of $\mathbf{Env}(G_0)$

The G_1-action on $\mathrm{Env}(G_0/K_0)$ extends to an action of $G_0 \times T_0$, T_0^σ acting trivially, which decends to an action of the group $G_0 \times T_0/\Delta^{1,-1}(Z_0)$, the group of units of $\mathrm{Env}(G_0)$.

Furthermore, we can extend this to an action of $\mathrm{Env}(G_0)$, the enveloping semigroup of G_0. Since $\mathcal{L} \subseteq \mathcal{L}(\mathrm{Env}(G_0))$, $\mathcal{L}(\mathrm{Env}(G_0))$ being the cone in $(\Lambda(G_0) \oplus X^*(T_0))^{\Delta^{1,-1}(Z_0)} \otimes_\mathbb{Z} \mathbb{Q}$ generated by $\{(\omega_i, w_0(\omega_i)), (0, -\alpha_i)\}_{i=1}^n$, the action homomorphism $k[\mathrm{Env}(X_0)] \longrightarrow k[G_0 \times T_0/\Delta^{1,-1}(Z_0)] \otimes_k k[\mathrm{Env}(X_0)]$ factors through the algebra $k[\mathrm{Env}(G_0)] \otimes_k k[\mathrm{Env}(X_0)]$, which proves our claim.

This extension enjoys the following property. Let E be any affine variety with an action of $\mathrm{Env}(G_0)$, and suppose that we are given a morphism $\phi : G_1/H_1 \longrightarrow E$ which is equivariant with respect to the action of $G_0 \times T_0/\Delta^{1,-1}(Z_0)$; then we can extend ϕ to a morphism $\widetilde{\phi} : \mathrm{Env}(G_0/K_0) \longrightarrow E$ which is $\mathrm{Env}(G_0)$-equivariant. Indeed, the image of the algebra homomorphism $k[E] \longrightarrow k[G_1/H_1]$ must land inside the sum of the G_1-submodules $k[G_1/H_1]_\nu$ with $\nu \in \mathcal{L}(\mathrm{Env}(G_0))$, so it factors through $k[\mathrm{Env}(X_0)]$.

4 Construction of $\mathbf{Env}(G_0/K_0)$ from $\overline{G_{ad}/K_{ad}}$

In this section, we give a geometric construction of $\mathrm{Env}(G_0/K_0)$ from the wonderful embedding $\overline{G_{ad}/K_{ad}}$ of G_{ad}/K_{ad} similar to the one given in [Ritt1] for $\mathrm{Env}(G_0)$. By pulling back the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(V_{w_0(\overline{\eta}_i)})$, we obtain an ample line bundle L_i on $\overline{G_{ad}/K_{ad}} \subseteq \mathbb{P}(V_{w_0(\overline{\eta}_1)}) \times \cdots \times \mathbb{P}(V_{w_0(\overline{\eta}_l)})$.

Let E_0 be the smooth variety $\oplus_{i=1}^l L_i^\vee$. The line bundles $L_i^\vee$ admit a G_0-linearization (it is a general fact, obvious in this specific case, that the action of a simply connected algebraic group on a variety can be lifted to line bundles over it), so we get an action of $G_0 \times S_0$ on E_0 by letting the torus $1 \times S_0$ act linearly on each fiber of $L_i^\vee$ by the character $\overline{\eta}_i$. For $I, J \subseteq \{1, \ldots, l\}$, set $E_{I,J} = \oplus_{i \in I} L_i^\vee|_{X_J}$ ($E_{\phi,J} = X_J =$ zero section of $E_{\{1,\ldots,l\},J}$); these are the closed $G_0 \times S_0$-stable subvarieties of E_0. Let $\mathcal{O}_{I,J}$ be the unique open $G_0 \times S_0$-orbit in $E_{I,J}$. E_0 is a simple $G_0 \times S_0$-spherical variety with unique closed orbit $\mathcal{O}_{\phi,\{1,\ldots,l\}}$.

Let us show that the open orbit $\mathcal{O}_{\{1,\ldots,l\},\phi}$ is isomorphic to G_1/H_1. Over $\mathcal{B}$, the bundles $L_i^\vee$ trivialize, so let f_i be a trivializing section for $L_i^\vee$ over $\mathcal{B}$. Let $p = \sum_{i=1}^l f_i(1 \cdot H_0)$; we want to find the isotropy group of p under the action of $G_0 \times S_0$. Let $(g, s) \in \mathit{Stab}(p)$. Then $gH_0 = H_0 \Longrightarrow g \in H_0$; say $g = s_0 k_0, s_0 \in N_0, k_0 \in K_0$ ($H_0 = (S_0 \cap H_0)K_0$ according to [Rich] §8). It follows that $\overline{\eta}_i(s \cdot s_0) = 1\ \forall i = 1, \ldots, l$, (note that K_0 acts trivially on the fiber of E_0 over $1 \cdot H_0$) hence $s \cdot s_0 \in S_0^\sigma$. $\Longrightarrow s = s_0^{-1}\tilde{s}, \tilde{s} \in S_0^\sigma$. Therefore, $\mathit{Stab}(p) = \{(k_0 \cdot s_0, s_0^{-1}\tilde{s})\} = \Delta^{1,-1}(N_0)(K_0 \times S_0^\sigma) = H_1$, and $\mathcal{O}_{\{1,\ldots,l\},\phi} \cong G_1/H_1$, which proves our claim.

E_0 doesn't have colors because the same is true for $\overline{G_{ad}/K_{ad}}$. The irreducible G_1-stable divisors of E_0 are the $E_{\{1,\dots,l\},j}$, $j = 1, \dots l$, and the $E_{\widehat{\{j\}},\phi}$, $j = 1, \dots, l$, where $\widehat{\{j\}}$ is the complement of $\{j\}$ in $\{1, \dots, l\}$. Let γ_j be the G_1-invariant valuation associated to $E_{\{1,\dots,l\},j}$, and let ϵ_j be the one corresponding to $E_{\widehat{\{j\}},\phi}$.Let x_j be a local equation for $X_j \cap \mathcal{B}$ as in section 3. Set $\widetilde{X}_j = X_j \cap \mathcal{B}$; then x_j becomes a local equation for $E_0|_{\widetilde{X}_j}$. The B_1-weight of x_i is $(w_0(\overline{\alpha_i}), 0)$ (up to reordering the rational functions x_i), hence $\rho(\gamma_j)(-w_0(\overline{\alpha_i}), 0) = \delta_{ij}$.

$E_{\widehat{\{j\}},\phi}$ is the divisor of y_j, which is the restriction to E_0 of the regular function on $\mathbb{P}(V_{w_0(\overline{\eta}_1)}) \times \cdots \times \mathbb{P}(V_{w_0(\overline{\eta}_l)}) \times V_{w_0(\overline{\eta}_1)} \times \cdots \times V_{w_0(\overline{\eta}_l)}$ which sends $(q_1, \dots, q_l, u_1, \dots, u_l)$ to the last coordinate of u_j with respect to the basis Θ_j. It follows that the weight of the B_1-eigenvector y_j is $(-w_0(\overline{\eta}_j), -\overline{\eta}_j)$. We conclude that $\rho(\epsilon_j)(w_0(\overline{\eta}_k), \overline{\eta}_k) = \delta_{jk}$.

Furthermore, $\rho(\gamma_j)(w_0(\overline{\eta}_i), \overline{\eta}_i) = 0, \rho(\epsilon_j)(-w_0(\overline{\alpha_i}), 0) = 0 \ \forall i$, so we may deduce that $\rho(\gamma_i) = (-w_0(\widetilde{\eta}_i), \widetilde{\eta}_i)$ and $\rho(\epsilon_j) = (0, \overline{\alpha_j}^\vee)$.

We have proved the following proposition.

Proposition 1. *E_0 is a simple smooth embedding of G_1/H_1 without colors whose associated cone in $X_*(S_{1H_1})$ is generated by $\{(-w_0(\widetilde{\eta}_i), \widetilde{\eta}_i)\}_{i=1}^l$ and by $\{(0, \overline{\alpha_i}^\vee)\}_{i=1}^l$.*

Let us define E_1 as the variety Spec $\Gamma(E_0, \mathcal{O}_{E_0})$, which is the same as Spec $\oplus_{n_1,\dots,n_l \geq 0} \Gamma(\overline{G_{ad}/K_{ad}}, L_1^{\otimes n_1} \otimes \cdots \otimes L_l^{\otimes n_l})$.

Proposition 2. *E_1 is a simple normal embedding of G_1/H_1 whose colors are all the colors of G_1/H_1. The colored cone of E_1 is the $\mathbb{Q}^+$-cone in $X_*(S_{1H_1})$ generated by $\{(\widetilde{\eta}_i, -w_0(\widetilde{\eta}_i)), (-(\overline{\alpha_i})^\vee, 0), (0, (\overline{\alpha_i})^\vee)\}_{i=1}^l$.*

Proof. E_1 is the normalization of the multi-cone $\widetilde{G_{ad}/K_{ad}}$ in $\prod_{i=1}^l V_{w_0(\overline{\eta}_i)}$ over $\overline{G_{ad}/K_{ad}}$ on which G_1 acts, and this action lifts to E_1. The G_1-morphism $\varphi : E_0 \longrightarrow E_1$ is birational (and proper because so is the morphism $E_1 \longrightarrow \widetilde{G_{ad}/K_{ad}}$), so E_1 is an embedding of G_1/H_1. E_1 is a simple embedding (since integral invariants separate closed orbits, it is a general fact that an affine G_1-variety with a dense G_1-orbit has only one closed orbit), and its colors are all the colors of G_1/H_1.

To find the colored cone of E_1, we simply need the B_1-highest weight semigroup of E_1. By [DP1] §8.3, the decomposition of $k[E_1]$ under the action of G_0 is $k[E_1] = \oplus_{(\gamma,\lambda)\in\mathcal{Q}} V_\gamma^*$ where $\mathcal{Q} = \{(\gamma, \lambda) \in X^-(S_{0K_0}) \oplus X^-(S_{0K_0}) | \lambda - \gamma \in \mathbb{Z}^-\{\overline{\alpha_1}, \dots, \overline{\alpha_l}\}\}$. Here, V_γ is the irreducible representation of G_0 whose highest weight with respect to B_0^- is γ. Therefore, $k[E_1] = \oplus_{(\gamma,\lambda)\in\mathcal{Q}} V_{-\gamma}$, where $V_{-\gamma}$ has highest weight $-\gamma$ with respect to B_0, and thus $k[E_1] = \oplus_{(\mu,\nu)\in\mathcal{Q}} V_\mu$ where now $\mathcal{Q} = \{(\mu, \nu) \in X^+(S_{0K_0}) \oplus X^+(S_{0K_0}) | \nu - \mu \in \mathbb{Z}^+\{\overline{\alpha_1}, \dots, \overline{\alpha_l}\}\}$.

Let $\lambda \in X^-(S_{0K_0})$, say $\lambda = \sum_{i=1}^l n_i w_0(\overline{\eta}_i)$, and set $L_\lambda = L_1^{\otimes n_1} \otimes \cdots \otimes L_l^{\otimes n_l}$. $1 \times S_0$ acts on $\Gamma(\overline{G_{ad}/K_{ad}}, L_\lambda)$ by the character $-w_0(\lambda)$, so the decomposition of $k[E_1]$ under the action of G_1 is

$$k[E_1] = \bigoplus_{(\mu,\nu)\in\mathcal{Q}} V_\mu \otimes_k \chi^{-w_0(\nu)}, \ k[E_1]_{(\mu,w_0(\nu))} = V_\mu \otimes_k \chi^{-w_0(\nu)}.$$

This means that the B_1-highest weight semigroup of E_1 is the intersection of the semigroup generated by $\{(\overline{\eta}_i, w_0\overline{\eta}_i\}_{i=1}^l \cup \{(0, -\overline{\alpha_i})\}_{i=1}^l$ with the semigroup $(\Lambda_+(G_0/K_0) \oplus X^-(S_{0K_0}))^{N_0}$. Dualizing, we get that the colored cone of E_1 inside $X_*(S_{1H_1})$ is the $\mathbb{Q}^+$-cone generated by

$$\{(\widetilde{\eta}_i, -w_0(\widetilde{\eta}_i)), (-(\overline{\alpha_i})^\vee, 0), (0, (\overline{\alpha_i})^\vee)\}_{i=1}^l.$$

□ □

Let A be as in section 3.1, so A is the S_0/N_0-toric variety defined by the cone in $X_*(S_0/N_0)$ generated by $\{\widetilde{\eta}_1, \ldots, \widetilde{\eta}_l\}$ $(= \mathcal{V}(A))$. Let A_1 be the categorical quotient E_1/G_0;

$$A_1 = \text{Spec } k[E_1]^{G_0} = \text{Spec} \bigoplus_{\lambda \in X^+(S_{0K_0}) \cap \mathbb{Z}\{\overline{\alpha_1}, \ldots, \overline{\alpha_l}\}} k \cdot \chi^\lambda,$$

so A_1 is the S_0/N_0-toric variety associated to the cone in $X_*(S_0/N_0)$ consisting of all the coweights $\widetilde{\eta}$ such that $\langle \widetilde{\eta}, \overline{\alpha} \rangle \geq 0$ for all $\overline{\alpha}$ in the intersection of the root lattice with the positive Weyl chamber. The inclusion of the first cone into the second one induces an equivariant morphism $\psi : A \longrightarrow A_1$. Combining this with the quotient $\varphi : E_1 \longrightarrow A_1$, we can consider the fiber product $E_2 = E_1 \times_{A_1} A$.

Proposition 3. *E_2 is an embedding of G_1/H_1 isomorphic to Env(G_0/K_0).*

Proof. The action of G_1 on E_2 is described by $(g, s) \cdot (e, a) = ((g, s)e, sa)$, and the isotropy group of $(1 \cdot H_1, p)$, where $\psi(p) = \varphi(1 \cdot H_1)$, is H_1, so E_2 is an embedding of G_1/H_1.

E_2 is simple because it is affine and contains an open dense orbit under the action of G_1, and the only closed G_1-orbit is $\theta \times_{A_1} 0$, θ being the closed orbit of E_1. It is also normal since, as one can easily check directly, $k[E_2] = k[\text{Env}(G_0/K_0)]$. Of course, this shows that E_2 and Env(G_0/K_0) are isomorphic affine varieties, but we want to give a different proof which is more instructive and uses results from section 1.

We would like to show that the two varieties E_2 and Env(G_0/K_0) share the same combinatorial data. To find $\mathcal{C}(E_2) \cap \mathbb{Z}^+\mathcal{V}(G_1/H_1)$, we apply the result of section 1.5. Let (λ, μ) be a one-parameter subgroup in $X_*(S_{1H_1})$ which is in $\mathcal{V}(G_1/H_1)$. Then $\lim_{t\to 0}(\lambda, \mu)(t)1 \cdot H_1$ exists in E_2 if and only if the limits $\lim_{t\to 0} \pi_j((\lambda, \mu)(t)1 \cdot H_1)$ exist in $\pi_j(E_2)$, where π_j is the projection morphism onto the j^{th} factor. Now $\lim_{t\to 0} \pi_1((\lambda, \mu)(t)1 \cdot H_1)$ exists if and only if $(\lambda, \mu) \in \mathcal{C}(E_1) \cap \overline{C} \cap X_*(S_{1H_1}) = \mathbb{Z}^+\{(\widetilde{\eta}_i, -w_0(\widetilde{\eta}_i)), (-(\overline{\alpha_i})^\vee, 0)\}_{i=1}^l \cap \overline{C} \cap X_*(S_{1H_1}) + \mathbb{Z}^+\{(0, (\overline{\alpha_i})^\vee)\}_{i=1}^l$, and $\lim_{t\to 0} \pi_2((\lambda, \mu)(t)1 \cdot H_1)$ exists if and only if $\mu \in \mathbb{Z}^+\{\widetilde{\eta}_1, \ldots, \widetilde{\eta}_l\}$; the condition $\psi(\pi_1((\lambda, \mu)(t)1 \cdot H_1)) = \varphi(\pi_2((\lambda, \mu)(t)1 \cdot H_1))$ is automatically satisfied since all the morphisms involved are equivariant and equality holds for $t = 1$. Therefore,

$$\mathcal{C}(E_2) \cap \mathbb{Z}^+\mathcal{V}(G_1/H_1) = \mathbb{Z}^+\{(\widetilde{\eta}_i, -w_0(\widetilde{\eta}_i)), (-(\overline{\alpha_i})^\vee, 0)\}_{i=1}^l \cap \overline{C},$$

which is equal to $\mathcal{C}(\text{Env}(G_0/K_0)) \cap \mathbb{Z}^+\mathcal{V}(G_1/H_1)$.

We claim now that the colors of E_2 are all those of G_1/H_1. Let λ_1 be a one-parameter subgroup of $1\times S_0$ such that $\langle\lambda_1,\overline{\alpha_i}\rangle > 0\ \forall i$. Then, if $\tilde{q}$ is a point on the fiber of $L_i^\vee$ over $q \in \overline{G_{ad}/K_{ad}}$, $\lim_{t\to 0}\lambda_1(t)\tilde{q} = \lim_{t\to 0}\overline{\eta}_i(\lambda_1(t))\tilde{q} = q$, q being the projection of $\tilde{q}$ on the zero section of $L_i^\vee$. Therefore, if $\tilde{q}$ is now a point on the multicone $\widetilde{G_{ad}/K_{ad}}$, then $\lim_{t\to 0}\lambda_1(t)\tilde{q} = 0$. It follows that if $D \in \mathcal{D}(G_1/H_1)$ and $\tilde{q}$ is a point on D, we can find a one-parameter subgroup λ_1 of $1\times S_0$ such that $\lim_{t\to 0}\lambda_1(t)\tilde{q} = \theta$ and $\lambda_1(t)\tilde{q} \in D\ \forall t$.

$\varphi\circ\lambda_1$ has image in the open orbit S_0/N_0 of A_1, and we have $\psi(\lambda_1(t)1\cdot N_0) = \varphi(\lambda_1(t)1\cdot H_1)\forall t$. ($A \cong \mathbb{A}^l$ and $1\cdot N_0$ is just the point $(1,\ldots,1)$). $\langle\lambda_1,\overline{\alpha_i}\rangle > 0\ \forall i$ implies that $\lim_{t\to 0}\lambda_1(t)(1,\ldots,1) = (0,\ldots,0)$, so

$$\lim_{t\to 0}(1,\lambda_1(t))(\tilde{q},(1,\ldots,1)) = \theta\times\{0\}.$$

Moreover, $(1,\lambda_1(t))(\tilde{q},(1,\ldots,1)) \in D\ \forall t$, so $\theta\times\{0\}$ is in the closure of D inside E_2. In conclusion, the colors of E_2 are the closures of those of G_1/H_1. □ □

5 Properties of Env(G_0/K_0)

The goal of this section is to establish some properties of Env(G_0/K_0), generalizing those of the enveloping semigroup of a semisimple group.

5.1 Orbit structure of Env(G_0/K_0)

The orbit structure of Env(G_0/K_0) under the action of G_1 is exactly the same as the decomposition of Env(G_0) under the action of $G_0\times T_0/\Delta^{1,-1}(Z_0)$ (or $G_0\times T_0$). Let Σ be the Dynkin diagram of the root system $\overline{R_0}$. For a subset $I\subseteq\{1,\ldots,l\}$, Σ_I denotes the subdiagram corresponding to the roots $\overline{\alpha_i}$ with $i\in I$.

Definition 2 (cf. [Vin]). *A pair (I,J) of subsets $I,J\subseteq\{1,\ldots,l\}$ is said to be essential if no connected component of the complement of J is entirely contained in I.*

Proposition 4. *There exists a bijection between G_1-orbit closures inside the variety Env(G_0/K_0) and essential pairs (I,J) of subsets $I,J\subseteq\{1,\ldots,l\}$.*

Proof. For a simple spherical variety, there exists a bijection between orbit closures and colored faces of its colored cone (see [Knop] for a definition of colored face); therefore, to retrieve Vinberg's parametrization in terms of essential pairs, we simply have to establish a bijection between colored faces of $\mathcal{C}(\mathrm{Env}(G_0/K_0))$ and such pairs exactly as in [Ritt1] §5.3. □ □

5.2 Toric subvarieties

In [Vin], Vinberg considers the closure of the center and of a maximal torus of G inside a given reductive algebraic monoid with group of units G. On the other hand, in [DP2], DeConcini and Procesi use in an essential way the closure of a maximal anisotropic torus inside the wonderful completion of G_{ad}/K_{ad}. Our intention now is to relate this second toric variety to the closure of S_{1H_1} inside $\mathrm{Env}(G_0/K_0)$.

Let $\overline{S_0/N_0}$ be the closure of the embedding of S_0/N_0 in $\overline{G_{ad}/K_{ad}}$ given by the morphism $s \mapsto s \cdot H_0$, $s \in S_0$. $\overline{S_0/N_0}$ is the complete toric variety associated to the fan in $X_*(S_0/N_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ given by the decomposition into Weyl chambers ([DP2] §5.3); in particular, it is non-singular. In fact, the Weyl chamber spanned by our choice of fundamental coweights $\widetilde{\eta}_i$ is just the affine cell $\mathcal{S}_0 = 1 \times \mathbb{A}^l \subseteq \mathcal{B}$. Furthermore, since $\overline{S_0/N_0}$ contains the affine cell $\mathcal{S}_0$, it follows from the description given in section 4 that $\overline{S_0/N_0}$ intersects every G_0-orbit of $\overline{G_{ad}/K_{ad}}$.

We follow the same procedure as in section 4. Let F_0 be the vector bundle $\oplus_{i=1}^{l} L_i^{\vee}|_{\overline{S_0/N_0}}$, and let $F_1 = \mathrm{Spec}\,\Gamma(F_0, \mathcal{O}_{F_0})$. That F_0 is an embedding of S_{1H_1} is clear because the stabilizer of p (see the proof of proposition 1) under the action of S_1 is $S_1 \cap H_1$; since the natural equivariant morphism $F_0 \longrightarrow F_1$ is birationnal, F_1 is also an S_{1H_1}-toric variety.

F_0 is a closed subvariety of E_0, and the proper morphism $\varphi : E_0 \longrightarrow E_1$ maps F_0 onto F_1. In fact, the restriction homomorphism $\Gamma(\overline{G_{ad}/K_{ad}}, L) \longrightarrow \Gamma(\overline{S_0/N_0}, L)$ is surjective for L as in the next paragraph. Therefore, $F_2 = F_1 \times_{A_1} A$ is closed in $\mathrm{Env}(G_0/K_0)$, and it corresponds to the closure of S_{1H_1} inside $\mathrm{Env}(G_0/K_0)$. The next lemma shows that it is a toric variety.

Lemma 5. *F_2 is a normal variety.*

Proof. Fix $n_1, \dots, n_l$, and set $L = L_1^{\otimes n_1} \otimes \cdots \otimes L_l^{\otimes n_l}$; let's find the decomposition of $\Gamma(\overline{S_0/N_0}, L)$ under the action of $S_{0K_0} \times 1$. $\overline{S_0/N_0}$ is covered by affine cells $\mathcal{B}_1, \dots, \mathcal{B}_r$, $\mathcal{B}_i$ corresponding to the i^{th} Weyl chamber inside $X^*(S_0/N_0) \otimes_{\mathbb{Z}} \mathbb{Q}$. (Fix an arbitrary ordering of these chambers so that $\mathcal{B}_1$ corresponds to $\overline{C_0}$, the positive Weyl chamber.) For each i, let $\{\overline{\eta}_i^1, \dots, \overline{\eta}_i^r\}$ be the orbit of $\overline{\eta}_i$ under the action of $\overline{\mathcal{W}_0}$, ordered in such a way that $\overline{\eta}_i^j$ is in the j^{th} Weyl chamber.

L trivializes over $\mathcal{B}_1$, so $\Gamma(\mathcal{B}_1, L) = \oplus_{u \in S} \chi^u$ where S is defined as the set $\{u \in X^*(S_{0K_0}) | u = w_0(\sum_{i=1}^{l} n_i \overline{\eta}_i - \overline{\alpha}), \overline{\alpha} \in \mathbb{Z}^+ \overline{R_0}^+, n_i \geq 0\}$. This follows from the fact that there exists a non-vanishing section over $\mathcal{B}_1$ of weight $-w_0(\sum_{i=1}^{l} n_i \overline{\eta}_i)$, and the others are obtained by multiplying it by the functions $x_j, j = 1, \dots, l$.

Let $w_1, \dots, w_r$ be the elements of $\overline{\mathcal{W}_0}$ such that w_i takes C_0 to the i^{th} Weyl chamber. Then

$$\Gamma(\overline{S_0/N_0}, L) = \cap_{i=1}^{r} \Gamma(L, \mathcal{B}_i) = \bigoplus_{u \in \cap_{i=1}^{r} w_i(S)} k \cdot \chi^u.$$

$\cap_{i=1}^{r} w_i(S)$ consists of the integral points inside a polyhedron in the vector space $X^*(S_{0K_0}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Set $S_{n_1, \dots, n_l} = \cap_{i=1}^{r} w_i(S)$. Then

$$k[F_1] = \bigoplus_{\substack{n_1, \dots, n_l \geq 0 \\ u \in S_{n_1, \dots, n_l}}} k \cdot \chi^u.$$

As an $S_{0K_0} \times S_{0K_0}$-module, the decomposition of $k[F_1]$ is

$$k[F_1] = \bigoplus_{\substack{n_1,\dots,n_l \geq 0 \\ u \in S_{n_1,\dots,n_l}}} k \cdot \chi^u \otimes_k \chi^{n_1\overline{\eta}_1 + \dots + n_l\overline{\eta}_l}.$$

Notice that the weight semigroup of F_1 contains $\{(-w_j\overline{\eta}_i, -\overline{\eta}_i)\}_{i,j=1}^{l,r}$. Moreover it contains also $(0, -\overline{\alpha})$ for $\overline{\alpha} \in \mathbb{Z}\overline{R_0} \cap \overline{C_0}$: if $-\overline{\alpha} = \sum_{k_i \geq 0} -k_i\overline{\eta}_i$; then

$$r(0, -\overline{\alpha}) = \sum_{j=1}^{r}\sum_{i=1}^{l} k_i(-w_j(\overline{\eta}_i), -\overline{\eta}_i)$$

is in the weight semigroup of F_1. But this semigroup is a saturated subsemigroup of the lattice $X^*(S_{1H_1})$ since F_1 is normal, so it contains $(0, -\overline{\alpha})$.

The weight semigroup of $k[F_2]$ is generated by

$$\{(-w_j(\overline{\eta}_i), -\overline{\eta}_i), (0, -\overline{\alpha_i})\}_{i,j=1}^{l,r}.$$

Indeed, if $\kappa = (\kappa', \kappa'')$ belongs to that semigroup, then $\kappa' \in \mathbb{Z}^+\{w_j(\overline{\eta}_i)\}_i^l$ for some (fixed) j, and $\kappa'' - w_j^{-1}(\kappa') \in \mathbb{Z}^+\overline{R_0}^+$.

Set $\Xi_i^j = (w_j(\overline{\eta}_i), \overline{\eta}_i)$, so that $k[F_2] = k[\chi^{\Xi_i^j}, \chi^{(0,\overline{\alpha_i})}]_{i,j=1}^{l,r}$. We claim that the weight semigroup of F_2 is a saturated subsemigroup of $X^*(S_{1H_1})$, which implies that F_2 is normal. Indeed, suppose that $(\lambda, \mu) \in X^*(S_{1H_1})$ and $r(\lambda, \mu) \in \mathbb{Z}^+\{-\Xi_i^j, (0, -\overline{\alpha_i})\}_{i,j=1}^{l,r}$ for some $r \in \mathbb{Z}^+$; then $r\lambda = -w_j\overline{\eta}, r\mu = -\overline{\eta} - \overline{\alpha}$ for some $\overline{\eta} \in X^+(S_{0K_0}), \overline{\alpha} \in \mathbb{Z}^+\overline{R_0}^+$. It follows that $\overline{\eta} = r\overline{\eta}', \lambda = -w_j(\overline{\eta}')$, and $r\mu = -r\overline{\eta}' - \overline{\alpha}$, so $\overline{\alpha} = r(-\mu + \lambda - \lambda - \overline{\eta}')$. $\mu - \lambda \in \mathbb{Z}\overline{R_0}$, and $-\lambda - \overline{\eta}' = w_j(\overline{\eta}') - \overline{\eta}' \in \mathbb{Z}\overline{R_0}$, hence $\overline{\alpha} \in r\overline{R_0}$, $\overline{\alpha} = r\overline{\alpha}'$. We deduce that $\mu = -\overline{\eta}' - \overline{\alpha}'$, so (λ, μ) belongs to the weight semigroup of $k[F_2]$. □ □

From the theory of toric varieties, we know that to each face τ of the cone of F_2 corresponds a distinguished idempotent element x_τ, which is the unique one in the orbit associated to τ. For an arbitrary toric variety $\mathcal{Z}$, we call a point x an idempotent if x is an idempotent for one (hence any) affine toric subvariety of $\mathcal{Z}$ containing x.

Proposition 5. *Any two idempotents of F_2 which are in the same G_1-orbit are conjugate under the action of $\overline{\mathcal{W}_0}$.*

Proof. $\overline{\mathcal{W}_0}$ acts on $\overline{S_0/N_0}$, and combinatorially this action is described by the action of $\overline{\mathcal{W}_0}$ on the Weyl chambers. This action lifts to F_0, hence also to F_1 and F_2. (In the latter case, $\overline{\mathcal{W}_0}$ acts trivially on A and A_1.) Furthermore, $\overline{\mathcal{W}_0}$ permutes the idempotents of $\overline{S_0/N_0}$ in the sense that if $C_1, C_2 = w(C_1), w \in \overline{\mathcal{W}_0}$, are two Weyl chambers corresponding to the affine cells A_{C_1}, A_{C_2} inside $\overline{S_0/N_0}$, and if $x \in A_{C_1}$ is an idempotent, then so is $\omega(x)$.

F_0 is covered by the affine cells $A_C \times \mathbb{A}^l = F_0|_{A_C}$ and $\overline{\mathcal{W}_0}$ permutes these; therefore, the idempotents in F_0 which are in the same orbit under the action of G_1 are conjugate under the action of $\overline{\mathcal{W}_0}$. $\oplus_{i=1}^{l}(L_i^\vee|_{\overline{S_0/N_0}} \setminus \{\text{zero section}\})$ is isomorphic to $F_1 \setminus \{0\}$, so the same is true about idempotents in F_1.

Now suppose $(p_1, q_1), (p_2, q_2) \in F_1 \times A$ are idempotents in the same G_1-orbit. Then q_1 and q_2 are in the same S_0-orbit, so $q_1 = q_2$. From what we know about F_1, it follows that p_1 and p_2 are conjugate under the action of $\overline{\mathcal{W}_0}$. Therefore, the same assertion holds for F_2. □ □

Proposition 6. *Every G_1-orbit of Env(G_0/K_0) meets F_2.*

Proof. Every G_0-orbit of E_0 meets F_0: this follows from the fact that $\overline{S_0/N_0}$ intersects all the G_0-orbit of $\overline{G_{ad}/K_{ad}}$. Since the morphism $E_0 \longrightarrow E_1$ is surjective and it is compatible with $F_0 \longrightarrow F_1$ under the immersions $F_0 \longrightarrow E_0$ and $F_1 \longrightarrow E_1$, the same is true for E_1 and F_1, hence also for E_2 and F_2. □ □

Remark 2. *It is a general result, due to M. Putcha, that in a reductive algebraic monoid M with unit group G any $G \times G$-orbit contains an idempotent which can, furthermore, be chosen in the closure of a maximal torus of G and is then unique up to the action of the Weyl group.*

From the two previous propositions, we conclude that we can retrieve the orbit decomposition of $\text{Env}(G_0/K_0)$ from the orbit structure of F_2:

$$\{G_1 - \text{orbits in Env}(G_0/K_0)\} = \{S_1 - \text{orbits of } F_2\}/\overline{\mathcal{W}_0}.$$

5.3 Abelianization

In [Vin], Vinberg characterizes the enveloping semigroup in terms of a certain universal property among a family of reductive monoids. We want to give a similar characterization of $\text{Env}(G_0/K_0)$ following the same steps. If $\mathcal{X}$ is a G-equivariant affine embedding of a homogeneous space G/L of the reductive group G, $G' = [G, G]$, we call the categorical quotient $\mathcal{X}/G'$ the abelianization of $\mathcal{X}$; it is a toric variety endowed with an action of the torus G/G'. We will generalize this definition to arbitrary simple embeddings, and then will study the properties of the abelianization map in the affine case and determine when it is a flat integral submersion, i.e. when it is dominant, flat, with reduced and irreducible fibers. Such an embedding of G/L is simply called flat. We will consider dominant embeddings of varieties other than G_0/H_0 and G_1/H_1; their classification is similar to the one given in section 1.4, and the reader is refered to [Vust2] for all the general results.

Let $G = G_0 \times \widetilde{T_0}$ be a reductive group with Borel subgroup $B = B_0 \times \widetilde{T_0}$ and maximal torus $T = T_0 \times \widetilde{T_0}$, and let L be a closed subgroup. Let $p_2 : G \longrightarrow \widetilde{T_0}$ be the projection onto the second component(similarly for p_1). The submersion $\varrho : G/L \longrightarrow G_0 \setminus G/L \cong \widetilde{T_0}/p_2(L)$ induces an injection $\varrho^* : X^*(\widetilde{T_0}/p_2(L)) \hookrightarrow \Lambda_+(G/L)$ and a linear map $\varrho_* : \mathcal{Q}(G/L) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow X_*(\widetilde{T_0}/p_2(L)) \otimes_{\mathbb{Z}} \mathbb{Q}$ where $\mathcal{Q}(G/L) = \text{Hom}_{\mathbb{Z}}(\Lambda(G/L), \mathbb{Z})$.

Definition 3. *The abelianization $Ab(E)$ of E is the $\widetilde{T}_0/p_2(L)$-toric variety whose cone in $X_*(\widetilde{T}_0/p_2(L)) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the image of $\mathcal{C}(E)$ under ϱ_*, and the abelianization morphism is the one determined by ϱ_*.*

If E is affine, $Ab(E)$ is just the categorical quotient under the action of G_0, i.e. $Ab(E) = E/G_0$. Indeed, $\varrho_* : \mathcal{C}(E) \longrightarrow \mathcal{C}(Ab(E))$ induces a homomorphism $\varrho^* : \mathcal{C}(Ab(E))^\vee \hookrightarrow \mathcal{C}(E)^\vee$ and $\rho^*(-\mathcal{C}(Ab(E))^\vee) = -\mathcal{C}(E)^\vee \cap (0 \oplus X^*(\widetilde{T}_0/p_2(L)))$, which is the weight semigroup of $k[E]^{G_0}$. Therefore, $k[Ab(E)] \cong k[E]^{G_0}$, and the homomorphism of rings induced by $\varrho_* : \mathcal{C}(E) \longrightarrow \mathcal{C}(Ab(E))$ is the inclusion $k[E]^{G_0} \hookrightarrow k[E]$.

Any symmetric variety arising from an involution of a reductive group with semisimple part G_0 is isomorphic to one coming from a group $G_0 \times \widetilde{T}$, $\widetilde{T}$ a torus, and an involution ς such that $\varsigma(G_0) = G_0$ and $\varsigma(1,t) = (1,t^{-1})$ $\forall t \in \widetilde{T}$. We call such a symmetric variety unmixed.

Let $\mathcal{E}'$ be the set of unmixed symmetric varieties whose semisimple part is G_0/K_0 and which come from an involution of a reductive algebraic group with semisimple part equal to G_0. The semisimple part of $G_0 \times \widetilde{T}_0/L, G_0^\varsigma \times \widetilde{T}_{0,2} \subset L \subset N_{G_0 \times \widetilde{T}_0}(G_0^\varsigma \times \widetilde{T}_{0,2})$, is G_0/L_0, $L_0 = L \cap (G_0 \times 1)$. Let $\mathcal{E}$ be the set of flat embeddings of symmetric varieties isomorphic to elements of $\mathcal{E}'$.

Theorem 3 (cf. [Vin] prop.5)). *For any $E \in \mathcal{E}$ which is an embedding of the symmetric variety G/L, and if $m = n$ (§1.2), any isomorphism φ_0 of the semisimple part of G/L with G_0/K_0 can be extended to an equivariant morphism $\varphi : E \longrightarrow Env(G_0/K_0)$ which is excellent with respect to the abelianization maps.*

(Excellent means that the canonical morphism $E \longrightarrow \mathrm{Env}(X_0) \times_A Ab(E)$ is an isomorphism, $A = Ab(\mathrm{Env}(X_0))$.)

The proof of this proposition will occupy the rest of this subsection. First, we have to find a criterion in terms of colored cones which characterizes flat, simple embeddings. E will denote an embedding of an unmixed symmetric variety $G/L \in \mathcal{E}'$ for $G = G_0 \times \widetilde{T}_0$ arising from an involution ς, $K = G^\varsigma = K_0 \times \widetilde{T}_{0,2}$, and $S = S_0 \times \widetilde{T}_0$ will be a maximal anisotropic torus of G inside the ς-stable maximal torus $T = T_0 \times \widetilde{T}_0$, so that $\mathcal{C}(E) \subseteq X_*(S_L)$. Let L_1 be the finite group $L \cap S/K \cap S$, so that $S_L \cong (S_{0K_0} \times \widetilde{T}_0/\widetilde{T}_{0,2})/L_1$ and $X^*(S_L) \cong (X^*(S_{0K_0}) \oplus X^*(\widetilde{T}_0/\widetilde{T}_{0,2}))^{L_1}$.

As suggested by Vinberg, we define a preorder on $-\mathcal{C}(E)^\vee \cap X^+(S_L)$ by $\nu_1 \geq \nu_2$ if $\nu_1 - \nu_2 \in (0 \oplus X^*(\widetilde{T}_0/p_2(L))) \cap (\mathcal{C}(E)^\vee)$. (We cannot obtain a partial order because, unlike in the case of toric varieties, it does not seem possible in general to reduce to the case when $\mathcal{C}(E)^\vee \cap (X^*(\widetilde{T}_0/p_2(L)) \otimes_{\mathbb{Z}} \mathbb{Q})$ contains no linear subspaces.) Let $\mathcal{M}$ be the set of minimal elements, ν_1 being minimal if $\nu_2 \leq \nu_1 \Longrightarrow \nu_1 \leq \nu_2$.

Proposition 7. *$Ab : E \longrightarrow Ab(E)$ is flat and its fibers are reduced and irreducible if and only if there exists a homomorphism $h_* : X_*(\widetilde{T}_0/\widetilde{T}_{0,2}) \longrightarrow X_*(S_0 \times 1_L)$ and a cone $\Delta \subseteq X_*(\widetilde{T}_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\mathcal{C}(E)$ is of the form*

$\{(\lambda_1, \lambda_2) \in X_*(S_L) | \lambda_1 + h_*(\lambda_2) \in$ *span of* $\mathcal{F}(E), \lambda_2 \in \Delta\}$, *and* h^* *satisfies* $\chi^{h^*(\nu)}(p_2(x)) = \chi^{-\nu}(p_1(x))\ \forall x \in L \cap S, \forall \nu \in X^*(S_0 \times 1_L)$.

Corollary 1. *The abelianization morphism of* $Env(X_0)$ *is a flat integral submersion.*

Proof. In this case, Δ is the cone generated by $-w_0(\widetilde{\eta}_i), i = 1, \ldots, l$ and $h_* : X_*(S_{0K_0}) \longrightarrow X_*(S_{0K_0})$ is w_0. □ □

The previous proposition is a consequence of the next three results.

Lemma 6. *The following statements are equivalent:*

1. $Ab : E \longrightarrow Ab(E)$ *is flat.*
2. $k[E] \cong k[Ab(E)] \otimes_k k[G/L]_{\mathcal{M}_1}$, *where* $k[G/L]_{\mathcal{M}_1} = \oplus_{\mu \in \mathcal{M}_1} k[G/L]_\mu$ *and* $\mathcal{M}_1$ *is a set of representatives of the cosets in* $\mathcal{M}$ *of the group* $(-\mathcal{C}(E)^\vee) \cap \mathcal{C}(E)^\vee \cap X^*(\widetilde{T}_0/p_2(L))$.

Proof. 2) $\Longrightarrow$ 1)is clear, so let us turn to the other implication. The essential point here is that the $k[Ab(E)]$-submodule $k[E]^{U_2}$ of $k[E]$ admits a complement which is also a $k[Ab(E)]$-module. The rest of the proof is as in [Vin]; it consists of showing that if $\nu_1, \nu_2 \in \mathcal{M}, \chi_1, \chi_2 \in (-\mathcal{C}(E)^\vee) \cap X^*(\widetilde{T}_0/p_2(L))$ and $\nu_1 + \chi_1 = \nu_2 + \chi_2$, then $\nu_1 \geq \nu_2$ and $\nu_2 \geq \nu_1$. □ □

Lemma 7. *The fibers of* $Ab : E \longrightarrow Ab(E)$ *are reduced and irreducible if and only if* $\mathcal{M}$ *is a subsemigroup of* $(-\mathcal{C}(E)^\vee) \cap X^+(S_L)$.

Proof. Let e be the distinguished idempotent in the unique closed orbit in $Ab(E)$. It is sufficient to determine when the fiber $Ab^{-1}(e)$ is reduced and irreducible, and for this we can argue as in [Vin] §4. □ □

We will need also the following lemma whose proof is similar to the one of proposition 12 in [Ritt2].

Lemma 8. *Let* G/L *be a symmetric variety. Let* τ *be a polyhedral cone contained in* $\mathbb{Q}^+\mathcal{V}(G/L)$ *such that* $\tau + \mathbb{Q}^+\rho(\mathcal{D}(G/L))$ *is strictly convex. Then there exists a unique subset* $\widetilde{\mathcal{F}} \subset \rho(\mathcal{D}(G/L))$ *and a set of colors* $\mathcal{F} \subset \mathcal{D}(G/L)$ *such that* $\widetilde{\mathcal{F}} = \rho(\mathcal{F})$ *and the colored cone* $(\tau + \mathbb{Q}^+\widetilde{\mathcal{F}}, \mathcal{F})$ *corresponds to an affine embedding of* G/L.

Proof of proposition 7. Let us translate the results above into the language of colored cones. We assume first that E is flat. Then we get a group epimorphism $\mathcal{M} - \mathcal{M} \longrightarrow X^*(S_0 \times 1_L)$ (it is surjective since the dominant morphism $G/L \longrightarrow E$ is an embedding), so we can find a homomorphism $X^*(S_0 \times 1_L) \longrightarrow \mathcal{M} - \mathcal{M}$ which is a right inverse. This inverse is of the form $\nu \longrightarrow (\nu, h^*(\nu))$, where h^* is a group homomorphism $X^*(S_0 \times 1_L) \longrightarrow X^*(\widetilde{T}_0/\widetilde{T}_{0,2})$, and $\nu \in X^+(S_0 \times 1_L) \Longrightarrow h^*(\nu) \in \mathcal{M}$; since $(\nu, h^*(\nu)) \in X^*(S_L)$, $\chi^{h^*(\nu)}(p_2(x))$ and $\chi^{-\nu}(p_1(x))$ are equal for all $x \in L \cap S$. Setting $\Delta^\vee = -\mathcal{C}(E)^\vee \cap (X^*(\widetilde{T}_0) \otimes_{\mathbb{Z}} \mathbb{Q})$, we conclude that $-(\mathcal{C}(E) + \rho(\mathcal{D}(G/L)))^\vee = \{(\nu, \mu) \in X^*(S_L) | \mu - h^*(\nu) \in -\Delta^\vee, \nu \in X^+(S_0 \times 1_L)\}$.

Consider the $\mathbb{Q}^+$-cone $\tau = \{(-h_*(\widetilde{\eta}), \widetilde{\eta}) | \widetilde{\eta} \in \Delta\}$ in $\mathbb{Q}^+\mathcal{V}(G/L)$, Δ being the dual of $\Delta^\vee$ in $X_*(\widetilde{T}_0/\widetilde{T}_{0,2})$. Then $(\mathcal{C}(E) + \mathbb{Q}^+\rho(\mathcal{D}(G/L)))^\vee = (\tau + \mathbb{Q}^+\rho(\mathcal{D}(G/L)))^\vee$, hence $\mathcal{C}(E) + \mathbb{Q}^+\rho(\mathcal{D}(G/L)) = \tau + \mathbb{Q}^+\rho(\mathcal{D}(G/L))$. $\mathcal{C}(E) + \mathbb{Q}^+\rho(\mathcal{D}(G/L))$ is strictly convex since its dual is the highest weight semigroup of E, which generates $X^*(S_L)$ as a group. According to lemma 8, there exists $\mathcal{F} \subseteq \mathcal{D}(G/L)$ such that $(\tau + \mathbb{Q}^+\rho(\mathcal{F}), \mathcal{F})$ is the colored cone of an affine embedding $\widetilde{E}$ of G/L. The $\mathbb{Q}^+$-span of the highest weight semigroup of $\widetilde{E}$ is $-(\tau + \mathbb{Q}^+\rho(\mathcal{D}(G/L)))^\vee$, so $\widetilde{E} \cong E$ and $\mathcal{F} = \mathcal{F}(E)$.

Conversely, if $\mathcal{C}(E)$ is of the form given in proposition 7, then the second of the two equivalent statements in each of lemma 6 and 7 holds. □ □

We are now in a position to prove theorem 3.

Proof. Let $E \in \mathcal{E}$. The notation related to E is borrowed from the proof of proposition 7. We can assume that φ_0 is the identity. The homomorphism h_* can be extended to a homomorphism $\widetilde{h_*} : X_*(S_L) \longrightarrow X_*(S_{1H_1})$: indeed, by our assumption on h^* and the fact that $\chi^\nu|_{N_0} = \chi^{w_0(\nu)}|_{N_0}$, the composite of $id \times (h^* \circ w_0)$ with the homomorphism $X^*(S_{1H_1}) \longrightarrow X^*(S_{1K_1})$ maps to $X^*(S_L)$, and we let $\widetilde{h_*}$ be its adjoint.

We claim that $\widetilde{h_*}(\mathcal{C}(E)) \subseteq \mathcal{C}(\mathrm{Env}(G_0/K_0))$. Combined with the fact that $\widetilde{h_*}(\mathcal{F}(E)) \subseteq \mathcal{F}(\mathrm{Env}(G_0/K_0))$, this shows that E admits a morphism to the variety $\mathrm{Env}(G_0/K_0)$ (see [Knop]); that this morphism is excellent can be deduced as in [Vin]. From the proof of lemma 11, we know that $V^2_{\overline{\eta}_i}$ contains $V_{2\overline{\eta}_i - \overline{\alpha_i}}$ as an irreducible component. Therefore, $(2\overline{\eta}_i, h^*(2\overline{\eta}_i))$ and $(2\overline{\eta}_i - \overline{\alpha_i}, h^*(2\overline{\eta}_i))$ both belong to the highest weight semigroup of $k[E]$. It follows that $\widetilde{h_*}(\overline{\alpha_i}) \in -\Delta^\vee$.

If $\widetilde{\eta} \in \Delta$, then $\langle \widetilde{\eta}, -h^*(\overline{\alpha_i}) \rangle \geq 0 \implies \langle -h_*(\widetilde{\eta}), \overline{\alpha_i} \rangle \geq 0$, so $-h_*(\widetilde{\eta}) \in X_+(S_{0K_0})$. As a consequence, we conclude that $\widetilde{h_*}(-h_*(\widetilde{\eta}), \widetilde{\eta})$, which equals $(-h_*(\widetilde{\eta}), w_0(h_*(\widetilde{\eta})))$, is in the $\mathbb{Z}^+$-span of $\{(\widetilde{\eta}_i, -w_0(\widetilde{\eta}_i))\}_{i=1}^l$; this proves our claim. □ □

6 Construction of $\overline{G_{ad}/K_{ad}}$ from Env(G_0/K_0)

The wonderful completion of G_{ad}/K_{ad} can be realized as a geometric quotient of an open subvariety of $\mathrm{Env}(G_0/K_0)$ when $m = n$ (§1.2), which we will assume throughout this section; in the case when G_0/K_0 is a semisimple algebraic group, this was done by Vinberg ([Vin]) and our approach his similar to his.

The S_0/N_0-orbits of A ($\cong \mathbb{A}^l$) are parametrized by subsets of $\{1, \ldots, l\}$ in the obvious way. We denote by S_I the orbit corresponding to $I \subseteq \{1, \ldots, l\}$. More precisely, $S_{\{1,\ldots,\hat{\jmath},\ldots,l\}}$ has codimension one, and S_I is the open orbit in $\cap_{j \notin I} \overline{S_{\{1,\ldots,\hat{\jmath},\ldots,l\}}}$. For $I \subseteq \{1, \ldots, l\}$, let $\mathcal{O}_I$ be the unique G_1-orbit in $Ab^{-1}(\overline{S_I})$ which is open in $Ab^{-1}(\overline{S_I})$.

Theorem 4. *Let Σ be the open G_1-stable subvariety $\cup_I \mathcal{O}_I$ of $Env(G_0/K_0)$. Then there exists a geometric quotient $\Sigma/1 \times S_0$, and it is isomorphic to the wonderful completion of G_{ad}/K_{ad}.*

Lemma 9. Σ *is simple.*

Proof. If $I \neq \phi$, consider $S_I \subset \overline{S_I} \subset A$ and $Ab^{-1}(\overline{S_I})$. ($\subset$ denotes a strict inclusion.) $\mathcal{O}_I$ is open in $Ab^{-1}(S_I)$, and $\overline{\mathcal{O}_I} = Ab^{-1}(\overline{S_I}) \supset \mathcal{O}_I$. Choose $k \in I$; then $S_{I\setminus\{k\}} \subseteq \overline{S_I}$, so $\mathcal{O}_{I\setminus\{k\}} \subseteq \overline{\mathcal{O}_I} \cap \Sigma$, whence $\mathcal{O}_{I\setminus\{k\}}$ is in the closure of $\mathcal{O}_I$ inside Σ. Therefore, $\mathcal{O}_\phi$ is the only closed orbit in Σ. □ □

Lemma 10. Σ *has no colors.*

Proof. $\overline{\mathcal{O}_\phi} = Ab^{-1}(0)$ and the ideal of functions vanishing on this fiber is

$$\bigoplus_{\nu \in \Lambda_+(G_1/H_1)\setminus \mathcal{M}} k[\mathrm{Env}(G_0/K_0)]_\nu .$$

$\mathcal{M}$ is the subsemigroup of $\Lambda_+(G_1/H_1)$ generated by $\{(\overline{\eta}_i, w_0(\overline{\eta}_i))\}_{i=1}^l$. In particular, since $(\overline{\eta}_i, w_0(\overline{\eta}_i)) \in \mathcal{M}$, $f_i^1 \not\equiv 0$ on $\mathcal{O}_\phi$, where f_i^1 is a (fixed) choice of a highest weight vector in $k[\mathrm{Env}(G_0/K_0)]_{(\overline{\eta}_i, w_0(\overline{\eta}_i))}$ with respect to B_1. However, we claim that each f_i^1 is identically zero on (at least) one color of G_1/H_1, which will complete the proof.

Let $\pi : G_0 \longrightarrow G_0/K_0$ be the quotient morphism, and let $\widetilde{D}_i$ be a color of G_0/K_0. According to lemma 3.4 in [Vust2], if we let $\tilde{f}_i$ be a generator of the ideal of $\pi^{-1}(\widetilde{D}_i)$ in $k[G_0]$ (G_0 is simply connected, so its divisor class group is trivial), $1 \leq i \leq q$, q being the cardinality of $\mathcal{D}(G_0/K_0)$, then we can divide these $\tilde{f}_i$ in such a way that, up to reordering, $\tilde{f}_i \in k[G_0/K_0]$ for $1 \leq i \leq q - 2r(K_0)$ ($r(K_0)$ being the rank of the character group of K_0), and for each $q - 2r(K_0) < i \leq q - r(K_0)$, $\tilde{f}_i$ is an eigenvector under right multiplication by K_0, and there exists $\tilde{f}_{i+r(K_0)}$ such that $\tilde{f}_i \tilde{f}_{i+r(K_0)}$ is invariant under K_0; furthermore, we can take f_i^1 to be $\tilde{f}_i$ if $1 \leq i \leq q - 2r(K_0)$ and to be $\tilde{f}_i \tilde{f}_{i+r(K_0)}$ if $q - 2r(K_0) < i \leq q - r(K_0)$.

$f_i^1 = \tilde{f}_i \otimes_k \chi^{-w_0(\overline{\eta}_i)}$ is a regular function on $G_0/K_0 \times S_{0K_0}$ and it vanishes on the divisor $D_i = \widetilde{D_i} \times S_{0K_0}$. Furthermore, f_i^1 descends to a regular function on G_1/H_1, and its divisor of zeros contains a color of G_1/H_1. □ □

Consider the B_1-stable affine subvariety $\mathcal{B}_\Sigma = \Sigma \setminus \cup_{D \in \mathcal{D}(G_1/H_1)} \overline{D}$ [Knop]. Let $\Omega_i = k[G_1/H_1]^*_{(\overline{\eta}_i, w_0(\overline{\eta}_i))}$, $\Omega = \oplus_{i=1}^l \Omega_i$. For each i, choose a basis $f_i^1, \ldots, f_i^{n_i}$ of the irreducible G_1-module $k[G_1/H_1]_{(\overline{\eta}_i, w_0(\overline{\eta}_i))}$ consisting of eigenvectors of T_1 with f_i^1 as above. Let $\{f_i^{j,*}\}_{j=1}^{n_i}$ be the dual basis. We consider the equivariant morphism $\psi : \mathrm{Env}(G_0/K_0) \longrightarrow \Omega$ given by $\psi(x) = \sum_{i,j} f_i^j(x) f_i^{j,*}$ for $x \in \mathrm{Env}(G_0/K_0)$. Set $\Omega_i' = \Omega_i \setminus \{0\}, \Omega' = \oplus_{i=1}^l \Omega_i'$, and $\Omega_i'' = \{\sum_{j=1}^{n_i} a_i^j f_i^{j,*} \in \Omega_i | a_i^1 \neq 0\} = \{v \in \Omega_i | f_i^1(v) \neq 0\}, \Omega'' = \oplus_{i=1}^l \Omega_i''$.

Lemma 11. *$\psi|_{\mathcal{B}_\Sigma}$ is a closed immersion into Ω''.*

Proof. The complement of $\mathcal{B}_\Sigma$ in $\mathrm{Env}(G_0/K_0)$ consists of the closures inside $\mathrm{Env}(G_0/K_0)$ of the colors of G_1/H_1 because $\mathcal{B}_\Sigma$ meets every G_1-stable prime divisor. Therefore $k[\mathcal{B}_\Sigma] = k[\mathrm{Env}(G_0/K_0)][(f_1^1)^{-1}, \ldots, (f_l^1)^{-1}]$. We only have to verify that the characters $\chi^{-w_0(\overline{\alpha_i})}$ of S_0 are in the algebra generated by the f_i^j and the $(f_i^1)^{-1}$.

Let $V_{\overline{\eta}}$ be the irreducible representation of G_0 with highest weight $\overline{\eta}$. Under our assumptions, $\overline{\eta}_i = (\omega_i - \sigma(\omega_i))$, and $\sigma(\omega_i) = -\omega_k$ for some k. (We assume here that i and k are fixed, $i, k \leq m$, and we do not exclude the case $i = k$.) The square of $V_{\overline{\eta}_i}$ contains the irreducible representation $V_{2\overline{\eta}_i - \overline{\alpha_i}}$: this can be proved after reducing to a similar problem for a reductive group of rank ≤ 2, namely the reductive subgroup of G corresponding to the roots in $\mathbb{Z}\{\alpha_i, \alpha_k\}$. It follows that $k[G_0/K_0]_{(2\overline{\eta}_i - \overline{\alpha_i})}$ is a submodule of the product $k[G_0/K_0]_{\overline{\eta}_i} k[G_0/K_0]_{\overline{\eta}_i}$. Note that $2\overline{\eta}_i - \overline{\alpha_i}$ is a dominant weight, so we can write $2\overline{\eta}_i - \overline{\alpha_i} = \sum_{j=1}^{l} c_j \overline{\eta}_j$. Therefore the highest weight vector $(f_1^1)^{c_1} \cdots (f_l^1)^{c_l} \chi^{-w_0(\overline{\alpha_i})}$ of $V_{2\overline{\eta}_i - \overline{\alpha_i}} \otimes_k \chi^{-2w_0(\overline{\eta}_i)}$ is contained in the subalgebra of $k[G_1/H_1]$ which is generated by the functions in $k[G_1/H_1]_{(\overline{\eta}_i, w_0(\overline{\eta}_i))}$. □

We are now able to prove the following proposition.

Proposition 8. *$\psi|_\Sigma$ is a closed immersion into Ω'.*

Proof. Since ψ is equivariant, it maps Σ isomorphically onto a closed subvariety of $G_1 \cdot \Omega''$. Let's prove that $G_1 \cdot \Omega'' = \Omega'$. Fix i, and let $\xi \in \Omega'_i$. Since f_i^1 is a highest weight vector, the span of the vectors in its G_1-orbit is Ω_i^*, so $\exists g \in G_1$ such that $(g \cdot f_i^1)(\xi) \neq 0$. Thus $f_i^1(g^{-1}\xi) \neq 0$ and $\xi \in g\Omega''_i$. Now if $(\xi_1, \ldots, \xi_l) \in \Omega'$, we can pick a $g \in G_1$ such that $(\xi_1, \ldots, \xi_l) \in g\Omega''$. □

Finally, we can prove the main result of this section.

Proof of theorem 4. $\psi(sx) = \sum_{i=1}^{l} (w_0(\overline{\eta}_i)(s) \sum_j f_i^j(x)(f_i^j)^*)$ where $s \in 1 \times S_0$, $x \in \Sigma$; this means that via ψ the action of $1 \times S_0$ on Σ becomes simply the restriction of the linear action on Ω' given by multiplication by $w_0(\overline{\eta}_i)(s)$ on the i^{th}-direct summand, and a geometric quotient for this action is $\mathbb{P}(\Omega_1) \times \cdots \times \mathbb{P}(\Omega_l)$. Hence $\Sigma/1 \times S_0$ is a projective variety. Moreover, $G_1/H_1/1 \times S_0 = G_0/N_0K_0 = G_{ad}/K_{ad}$ since $H_0 = N_0K_0$ ([Rich]).

To see that $\Sigma/1 \times S_0$ is the wonderful completion of G_{ad}/K_{ad}, notice that only one orbit is closed because Σ is simple (lemma 9), and $\Sigma/1 \times S_0$ has no colors since the same is true for Σ (lemma 10) and $1 \times S_0 \subseteq B_1$. □ □

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