

Hyperfine Splitting in Non-Relativistic Bound States

by

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Abstract

We study the mass difference between the spin singlet and spin triplet states of positronium and heavy quarkonium, an effect which is referred to as the hyperfine splitting. For positronium, a bound state of an electron and a positron, we analyze the one-loop nonrelativistic effective Hamiltonian in d -dimensions, which we parametrize as $d = 4 - 2\epsilon$. Our result constitutes an important part of the analysis in [1], which studies positronium's hyperfine splitting to $\mathcal{O}(\alpha^7 m_e)$, and substantially reduces the overall theoretical uncertainty. This is crucial for comparing high precision predictions of quantum electrodynamics with the results of modern experiments.

For quarkonium, a non-relativistic flavourless quark-antiquark bound-state, we set up a matching procedure between the perturbative analysis of the short-distance interactions and the nonperturbative lattice analysis of the long-distance effects. In particular, our result is used in [2], and it corrects an error in the previous matching calculation of Ref. [3], which was subsequently used in the analyses [4] and [5]. Combined with the one-loop perturbative lattice calculation, our result brings theory and experiment into agreement and effectively solves the η_b mass puzzle.

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List of Abbreviations

List of commonly used abbreviations

HFS	Hyperfine Splitting
QED	Quantum Electrodynamics
QCD	Quantum Chromodynamics
QFT	Quantum Field Theory
EFT	Effective Field Theory
NR	Non Relativistic
QM	Quantum mechanics
LO	Leading Order
NLO	Next to Leading Order
LL	Leading Log
UV	Ultra Violet
IR	Infra Red

Chapter 1

Introduction: Non-Relativistic Bound States

1.1 Positronium

Positronium (Ps) is a bound state that consists of an electron and a positron (anti-electron). At leading order we can approximate the energy levels by solving a Schrodinger equation with a Coulomb potential, somewhat similar to the solution of the hydrogen atom. In Ps however, the two masses are identical and the system must be treated in the centre of mass frame; there is no stationary nucleus in this case. This leads to spectral lines that have half the value of the equivalent hydrogen lines in leading order.

In its own right, Ps is a very important system to study because it is a purely leptonic system. Since strong interactions in this system are only present through hard fermion loop calculations, the effects are suppressed by the mass of the electron over the typical hadronic mass scale all squared. Weak interactions are similarly suppressed by the electron mass over the mass of the W and Z bosons. As a result it provides an ideal system for performing very high precision tests of Quantum Electrodynamics (QED), which is now one of the most accurately tested theories in all of physics.

Positronium can assume two different spin configurations depending on whether the combined spin state of the electron-positron pair are symmetric or antisymmetric. The spin-one configuration is known as orthopositronium (o-Ps), while the spin-zero configuration is called parapositronium (p-Ps). In this work we will be interested in studying the effect of hyperfine splitting (HFS) on the positronium bound state. HFS is defined as the difference in energy between the o-Ps and p-Ps states. In hydrogen the effects of hyperfine splitting are suppressed, when compared to fine structure corrections, by a factor of the electron mass over the proton mass. This is not so in Ps, and as a result hyperfine corrections become comparable to fine structure corrections, making it a much more important effect to study in detail.

In either spin-configuration Ps is unstable, decaying into two (p-Ps) or three (o-Ps) photons in times on the order of $\sim 10^{-10}$ s and $\sim 10^{-7}$ s respectively. Consequently, excited states are so short lived as to be negligible when compared to the ground state configuration ($1S$ -state). This means that HFS becomes a leading effect to consider when taking measurements of positronium masses. Precise measurements of this effect have yielded [6, 7]

$$\Delta\nu^{exp} = 203.3875(16) \text{ GHz}, \quad (1.1)$$

$$\Delta\nu^{exp} = 203.38910(74) \text{ GHz}. \quad (1.2)$$

There is a lot of effort right now to increase the accuracy of these measurements and a recent result, claiming less systematic uncertainty from thermalization, has produced a value substantially larger [8]:

$$\Delta\nu^{exp} = 203.3941(16)_{\text{stat.}}(11)_{\text{syst.}} \text{ GHz}. \quad (1.3)$$

On the theoretical side, the present knowledge may be summarized as:

$$\begin{aligned}
\Delta\nu^{\text{th}} = \Delta\nu^{LO} & \left\{ 1 - \frac{\alpha}{\pi} \left(\frac{32}{21} + \frac{6}{7} \ln 2 \right) - \frac{5}{14} \alpha^2 \ln \alpha \right. \\
& + \left(\frac{\alpha}{\pi} \right)^2 \left[\frac{1367}{378} - \frac{5197}{2016} \pi^2 + \left(\frac{6}{7} + \frac{221}{84} \pi^2 \right) \ln 2 \right. \\
& \left. \left. - \frac{159}{56} \zeta(3) \right] - \frac{3}{2} \frac{\alpha^3}{\pi} \ln^2 \alpha + \left(-\frac{62}{15} + \frac{68}{7} \ln 2 \right) \frac{\alpha^3}{\pi} \ln \alpha \right. \\
& \left. + D \left(\frac{\alpha}{\pi} \right)^3 \right\}, \tag{1.4}
\end{aligned}$$

The leading order (LO) term, which has been taken out as an overall multiplicative factor, gets contributions from both scattering and virtual pair annihilation and is calculated to be

$$\Delta\nu^{LO} = \left(\left[\frac{1}{3} \right]_{\text{scat.}} + \left[\frac{1}{4} \right]_{\text{ann.}} \right) \alpha^4 m_e = \frac{7}{12} \alpha^4 m_e. \tag{1.5}$$

In the above, α is the electromagnetic coupling (the fine structure constant), m_e is the electron/positron mass, and in the first equality we have separated out the contributions coming from scattering and annihilation effects. The first-order correction was calculated in [9]. The logarithmically enhanced $\alpha^6 \ln(\alpha)$ term was found in [10, 11]. The nonlogarithmic $\mathcal{O}(\alpha^6)$ term includes the pure radiative contribution [12], the three-, two- and one-photon annihilation contributions [13, 14, 15, 16], the non-annihilation radiative recoil contribution [17, 18], and the pure recoil correction computed numerically in [19, 20] and analytically in [21, 22]. In $\mathcal{O}(\alpha^7)$, only the leading double-logarithmic [23] and the single-logarithmic terms [24, 25, 26] are known, while the nonlogarithmic coefficient D in (1.4) is not yet available. Including all the terms known so far, we obtain a value that is above the experimental values (1.1) and (1.2) by 2.6 and 3.5 standard deviations, respectively

$$\Delta\nu^{\text{th}} = 203.391\,69(41) \text{ GHz}. \tag{1.6}$$

At the same time, it is only 1.2 standard deviations below the most recent result (1.3). In this thesis we compute a significant piece of the coefficient D , coming from the one-photon annihilation channel.

1.2 Quarkonium

The theoretical study of flavourless nonrelativistic heavy quark-antiquark bound states, or *quarkonium*, is amongst the earliest applications of perturbative quantum chromodynamics [27]. Quarkonium is fundamentally different in comparison to positronium, its QED analogue, due to the peculiar property that QCD enters a strong coupling regime around the scale $\Lambda_{QCD} \approx 200 MeV$ [28]. On the other hand, since the Bohr radius of quarkonium is smaller than the confinement radius ($r \sim \Lambda_{QCD}^{-1}$) where the strong interactions become nonperturbative, QCD perturbation theory can be applied for the analysis of states with low quantum numbers. This makes heavy quark-antiquark systems an ideal laboratory for determining fundamental parameters of QCD, such as the strong coupling constant α_s and the heavy-quark masses m_Q . However, in some cases perturbative QCD fails to accurately describe the experimental data. A famous example is the so-called. “ η_b -mass puzzle”, which currently attracts a lot of attention from the experimental and theoretical physics communities. There is a significant discrepancy between the mass of the recently discovered η_b meson, i.e. the lowest energy spin singlet bound state of bottomonium ($b\bar{b}$), and the perturbative QCD predictions for hyperfine splitting (HFS). The resolution of this puzzle could shed new light on the dynamics of strong interactions.

For bottomonium $b\bar{b}$, the HFS is given by the mass difference between the spin-singlet η_b meson and the spin triplet $\Upsilon(1S)$ meson. The η_b meson has been observed by Babar and Belle collaboration in the radiative decays of the excited Υ states. Very high accuracy of Υ -spectroscopy allows for the determination of HFS with only a few MeV error. The results are given in [29] and [30], and are respectively

$$E_{hfs}^{exp} = 66.1_{-4.8}^{+4.9}(stat) \pm 2.0(syst) MeV, \quad (1.7)$$

$$E_{hfs}^{exp} = 57.9 \pm 2.3 MeV. \quad (1.8)$$

The most accurate theoretical prediction for HFS includes the complete first-order corrections in α_s as well as the resummation of all-order next-to-leading logarithmic corrections of the form $\alpha_s^{n+1} \ln^n \alpha_s$. Numerically it gives [31]

$$E_{hfs}^{QCD} = 41 \pm 11(th)_{-s}^{+9}(\delta\alpha_s)MeV, \quad (1.9)$$

where “th” stands for the errors that come from higher-order perturbative corrections as well as any nonperturbative effects. The term $\delta\alpha_s$ represents the inherent uncertainty in $\alpha_s(M_Z) = 0.118 \pm 0.003$. This result is about two standard deviations lower than the experimentally measured values.

One explanation for the above discrepancy is that perturbation theory becomes inapplicable when the momentum is on the order of the binding energy $q \sim \alpha_s^2 m_b$. The strong interaction is characterized by the corresponding running coupling at this scale, and the effective expansion parameter $\alpha_s(\alpha_s^2 m_b) \sim 1$ can hardly be considered small there, thus perturbation theory effectively breaks down. One way to get control over this problem is to use numerical lattice simulations of QCD which are not based on the expansion in α_s . However, existing lattice results do not properly take into account the contributions of the “hard” virtual momentum, on the order of heavy quark mass $q \sim m_b$, which is cut off by the lattice spacing $a \gg 1/m_b$. Fortunately the hard region contribution can be more reliably calculated within perturbation theory since the corresponding expansion parameter there is $\alpha_s(m_b) \sim 1/5$.

The issue is now: how do we make perturbative calculations of the hard region contribution consistent with the lattice simulations? This is accomplished through systematic separation of hard and soft effects. The procedure that is used is called “matching”, wherein the hard contribution is given by the difference between the full QCD perturbative result, and the same order contribution contained in the effective NRQCD lattice perturbation theory.

1.3 Outline

The plan of the thesis is as follows: In the next chapter we introduce the basic theory of nonrelativistic hydrogen-like bound states, where we derive the leading order approximation for HFS within the framework of non-relativistic quantum mechanics (NRQM).

In chapter 3, we set up a natural framework for doing bound-state calculations within relativistic quantum theory. Firstly, we show that not only do we recover the leading order HFS result of chapter 2, but systematically incorporate all other non-radiation based effects. We also show how the inclusion of electron-positron annihilation modifies the leading order HFS result for positronium. Finally, the leading order HFS for quarkonium is found within the framework of QCD.

In chapter 4 we discuss the general approach for the inclusion of radiative effects in the form of perturbative corrections. In particular, we demonstrate that the perturbation theory for our bound states must be developed about the nonrelativistic Coulomb approximation, rather than free electron and positron states. We show how this can be systematically performed within the framework of nonrelativistic effective field theory, which separates the contributions from the various regions of momentum space. Finally, we discuss the numerical effect of the one-photon annihilation contribution on the HFS of positronium.

In chapter 5, we apply straightforward generalizations of the effective field theory approach to the theory of QCD in order to separate out the effects of the various contributing regions of momentum. We then set up the formal matching procedure between full QCD and effective NRQCD lattice theory, and perform the calculations necessary for the matching. Following this, we briefly discuss the renormalization group (RG) within the context of QCD and explain how it can be used to obtain a RG-improved result for the HFS of heavy quark bound states.

Chapter 2

Basic Properties of NR Bound-States

2.1 Energy Levels at Leading Order

The procedure for solving the hydrogen atom can readily be applied to many other two-particle systems like positronium and quarkonium. We will present a very brief overview of the subject matter and then proceed to apply it, within a relativistic framework, to positronium and quarkonium in the next chapter. The unfamiliar reader may consult any introductory book on quantum mechanics for a more thorough treatment of what follows (see for example [32]).

The quantum mechanical Hamiltonian for the hydrogen atom has the standard form with a Coulomb potential

$$\hat{H} = -\frac{\hbar^2}{2m_e}\nabla^2 + V_C(r), \quad V_C(r) \equiv -\frac{\alpha}{r}, \quad (2.1)$$

where $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r^2}r - \frac{1}{\hbar^2 r^2}\hat{\mathbf{L}}^2$, and m_e is the mass of the electron. $\hat{\mathbf{L}}^2$ is the operator in Hilbert space that, acting on an eigenstate, gives the square of the total angular-momentum. By writing the Hamiltonian in this way, we have made explicit that the angular momentum operator $\hat{\mathbf{L}}^2$ commutes with

\hat{H} , and can therefore be simultaneously diagonalized. Put another way, the Hamiltonian is spherically symmetric and thus admits, as its angular solution, the eigenfunctions of the angular-momentum operator. These solutions are the famous spherical harmonics $Y_m^l(\theta, \phi)$,

$$\hat{L}^2 Y_m^l(\theta, \phi) = l(l+1) Y_m^l(\theta, \phi) \Rightarrow \Psi(r, \theta, \phi) \equiv \phi(r) Y_m^l(\theta, \phi). \quad (2.2)$$

Doing the separation of variables in (2.2) and solving, gives the hydrogen wavefunctions

$$\Psi_{nlm}(r, \theta, \phi) = \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-l-1)!}{2n[(n+1)!]^3}} \left(\frac{2r}{na_0}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na_0}\right) Y_m^l(\theta, \phi) e^{-\frac{r}{na_0}}, \quad (2.3)$$

where L_{q-p}^p are the associated Laguerre polynomials. In (2.3), a_0 is the most probable separation of the electron from the atom's centre in its ground state, and is known as the Bohr radius $a_0 = \frac{1}{\alpha m_e} \approx 53 \text{ pm}$.

The energy levels of Hydrogen in terms of the quantum number n are

$$E_n = -\frac{\alpha^2}{2n^2} m_e \approx -\frac{13.6eV}{n^2}. \quad (2.4)$$

For the case of positronium and quarkonium, both masses involved in the Hamiltonian are equal. As such, there can be no approximation of a stationary nucleus. We thus treat this two body problem by reducing it to a one particle problem with a reduced mass $m_r \equiv \frac{m_1 m_2}{m_1 + m_2}$, circling around the system's center of mass. For positronium and quarkonium, the reduced masses are respectively $\frac{1}{2}m_e$ and $\frac{1}{2}m_q$ (m_q being the mass of the heavy quark).

Note that one of the consequences of using a reduced mass in solving the Schrodinger equation, is that the ‘‘Bohr’’ radius that appears in the wavefunctions (2.3), is twice as large for positronium ($a_0^{Ps} = 2a_0$)¹. If we now substitute into (2.4), we can get the leading order expression for positronium's

¹Quarkonium's Bohr radius has another alteration which we will get to in section 3.4

energy spectrum

$$E_n^{e\bar{e}} = -\frac{\alpha^2}{4n^2} m_e. \quad (2.5)$$

Quarkonium on the other hand interacts via the strong force, and thus $\alpha \rightarrow \alpha_s$. But, in addition we must also include a colour factor which in its essence is nothing but a group theory factor that arises because QCD is based on a non-abelian gauge theory. Therefore I will put off the presentation of its leading order energies until it can be expressed in its natural frame as a piece of a relativistic scattering amplitude (see section 3.4).

2.2 Overview of Spin Related Perturbations

2.2.1 Spin-Orbit Coupling

Any charged particle with spin angular-momentum acts like a magnetic dipole. If an electron (spin- $\frac{1}{2}$) is immersed in a magnetic field, it will feel a torque acting on it which tends to align its spin orientation($\hat{\mu}$), parallel to the field. This torque can be written as $\vec{\tau} = \vec{\mu} \times \vec{B} = \vec{r} \times \vec{F}$, which has the solution $\vec{F} = \nabla_r(\vec{\mu} \cdot \vec{B}(r))$. Integrating the expression provides us with an energy $H = -\vec{\mu} \cdot \vec{B}$.

Next, we know from classical electrodynamics that any moving charge sets up a magnetic field. In Hydrogen we tend to think about the electron as circling around the stationary nucleus. From the (non-inertial) frame of the electron, the proton appears to be circling around *it*, and thus sets up a magnetic field proportional to its apparent orbital angular-momentum¹.

The magnetic moment of the electron is proportional to its spin, thus producing the familiar term $\vec{\sigma} \cdot \vec{B}$ from the Pauli-equation (a non-relativistic limit for the Dirac-equation in the presence of an electromagnetic field A^μ). This effect can be calculated as a perturbation to the non-relativistic Hamiltonian,

¹ $B = \frac{\mu_0 I}{2r} = \frac{\mu_0 e v}{4\pi r^2}$. Then $L = m|\vec{r} \times \vec{v}| \sim mvr \Rightarrow B \sim \frac{\mu_0}{4\pi} \frac{e}{mr^3} L$.

the so-called spin-orbit interaction(S.O.)

$$\delta H_{s.o.} = \frac{\mu_0}{8\pi} \frac{e^2}{m_e^2 r^3} \mathbf{S} \cdot \mathbf{L} = \frac{\frac{1}{2} \hbar c}{(m_e c)^2} \left(\frac{\alpha}{r^3} \right) \mathbf{S} \cdot \mathbf{L}. \quad (2.6)$$

As can be seen from the above perturbation, the Hamiltonian no longer commutes individually with \vec{L} and \vec{S} , therefore the operator eigenvalues are no longer separately conserved. This non-commutation will also persist in the Dirac Hamiltonian as well (see sec. 3.2).

2.2.2 Spin-Spin Coupling

The previous section was intended to act as a segue into the interaction of interest in this paper. Thus we will first discuss the physical concepts involved in spin-spin coupling through the viewpoint of non-relativistic quantum mechanics. We will then proceed to discuss the interaction in its natural place as an aspect of quantum field theory (QFT), in the next chapter.

The main idea, starting with hydrogen, is that the nucleus itself constitutes a magnetic dipole and so sets up another magnetic field that is unrelated to the relative motion of the two particles,

$$\mathbf{B} = \frac{\mu_0}{4\pi r^3} [(3\boldsymbol{\mu}_p \cdot \hat{r})\hat{r} - \boldsymbol{\mu}_p] + \frac{2\mu_0}{3} \boldsymbol{\mu}_p \delta^3(\mathbf{r}). \quad (2.7)$$

This gives yet another magnetic field for $\boldsymbol{\mu}_e$ to couple to. The gyromagnetic ratios are²

$$\boldsymbol{\mu}_p = \frac{g_p e}{2m_p} \mathbf{S}_p \quad \text{and} \quad \boldsymbol{\mu}_e = -\frac{e}{m_e} \mathbf{S}_e, \quad (2.8)$$

so that the perturbation takes the form

$$\delta H_{s.s.} = \frac{g_p e^2}{m_e m_p c^2} \left[\frac{3(\mathbf{S}_e \cdot \hat{r})(\mathbf{S}_p \cdot \hat{r})}{8\pi r^3} - \frac{\mathbf{S}_e \cdot \mathbf{S}_p}{8\pi r^3} + \frac{1}{3} \mathbf{S}_e \cdot \mathbf{S}_p \delta^3(\mathbf{r}) \right]. \quad (2.9)$$

²The proton's ratio is not simple like the electron's because it is a composite structure.

In any state where the orbital angular momentum is zero ($l = 0$), the expectation value of the first two terms cancel each other³. We are then only left with the term proportional to the delta function. The correction to the energy levels is then given by first order perturbation theory

$$E_{s.s.} = \langle \psi_{100} | \delta H_{s.s.} | \psi_{100} \rangle = \frac{g_p e^2}{3m_e m_p c^2} \langle \mathbf{S}_e \cdot \mathbf{S}_p \rangle | \psi_{100}(0) |^2. \quad (2.10)$$

In the case of hydrogen, the factor of $1/m_p$ is responsible for the HFS contribution being *much less* than the fine structure terms like spin-orbit coupling, owing to the proton mass being $m_p \sim 1800 m_e$. This will not be the case in particles like quarkonium or positronium, and as such becomes a much more important effect to study.

Now, similarly to spin-orbit coupling, we see that the addition of spin-spin coupling to the Hamiltonian makes it so that individual spins are no longer conserved. We are thus left with the problem of finding some spin related quantum numbers that *are* conserved and characterize the system. We can start by writing it out in terms of total angular momentum

$$\mathbf{S} = \mathbf{S}_e + \mathbf{S}_p \Rightarrow \mathbf{S}_e \cdot \mathbf{S}_p = \frac{1}{2}(\mathbf{S}^2 - \mathbf{S}_e^2 - \mathbf{S}_p^2). \quad (2.11)$$

Both particles are spin-1/2, thus $\mathbf{S}_e^2 = \mathbf{S}_p^2 = s(s+1) = 3/4$. This still leaves the problem of finding out what the quantity \mathbf{S}^2 is.

In classical physics we are capable of just adding any number of vectors together by introducing a coordinate system and projecting the vectors onto each axis, or onto a set of appropriate basis vectors, and then adding all the components separately. In the case of quantum mechanics, we can know *at most* the total magnitude of the angular momentum vector, and the value of one of its chosen components (typically the z-component). The other components

³ $\int (\mathbf{a} \cdot \hat{\mathbf{r}})(\mathbf{b} \cdot \hat{\mathbf{r}}) d\cos\theta d\phi = \frac{4\pi}{3} \mathbf{a} \cdot \mathbf{b}$, which will cancel the (integrated) second term in (2.9).

(say L_x and L_y) do not commute with L_z and thus we can't simultaneously diagonalize these components. This leaves us with an obvious problem: How *do* we add angular momentum vectors in quantum mechanics?

In the ground state, the total orbital angular momentum is zero and we are interested to know how the total spin configuration of the two-particle system looks. For fermions, which have spin-1/2, the z-component can have only two states ($m_s = \pm 1/2$). We typically call them spin-up and spin-down, and represent them $|\uparrow\rangle$ and $|\downarrow\rangle$ respectively.

To get the total magnitude of two *indeterminate* vectors we start by introduce raising and lowering operators $S^+ = S_{(1)}^+ + S_{(2)}^+$ and $S^- = S_{(1)}^- + S_{(2)}^-$ which act linearly on all possible states :

$$|\uparrow\rangle|\uparrow\rangle, |\uparrow\rangle|\downarrow\rangle, \dots \quad . \quad (2.12)$$

The result is that the total value of spin ranges from $|s_1 - s_2| \rightarrow |s_1 + s_2|$ in integer steps. Specifically, we get three states with total angular momentum equal to one ${}^3S_1 \equiv |s=1; m_s=0, \pm 1\rangle$, called the triplet. We also get one state with total angular-momentum equal to zero ${}^1S_0 \equiv |s=0; m_s=0\rangle$ called the singlet. This can be written symbolically as a set of orthonormal kets:

$$\left\{ \begin{array}{l} |1, 1\rangle = |\uparrow\uparrow\rangle \\ |1, 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1, -1\rangle = |\downarrow\downarrow\rangle \end{array} \right\} \quad s = 1 \quad , \quad (2.13)$$

$$\left\{ |0, 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \right\} \quad s = 0 \quad . \quad (2.14)$$

The hyperfine splitting occurs because the spin-spin perturbation takes on a

value which depends on the overall spin configuration (i.e. singlet or triplet).

$$\mathbf{S}_e \cdot \mathbf{S}_p = \frac{1}{2}(\mathbf{S}^2 - \mathbf{S}_e^2 - \mathbf{S}_p^2) = \frac{1}{2}\mathbf{S}^2 - \frac{3}{4}, \quad (2.15)$$

where $\mathbf{S}^2 = 0$ for the singlet, and $\mathbf{S}^2 = 2$ for the triplet. Therefore we get $\mathbf{S}_e \cdot \mathbf{S}_p = -3/4$ for the singlet, and $\mathbf{S}_e \cdot \mathbf{S}_p = 1/4$ for the triplet. Then if we take the difference between the two states in hydrogen, we get the famous 21cm line

$$E_{hfs} = \frac{4g_p}{3m_p m_e^2 c^2 a^4} = 5.88 \times 10^{-6} \text{eV}. \quad (2.16)$$

The leading HFS for positronium and quarkonium follow in a simple way from this, but we prefer to present these values in the next chapter within a relativistic framework, and show that it agrees with the details presented above in the NRQM framework.

Chapter 3

Hyperfine Splitting at Leading Order

It was mentioned in the last section that spin-spin coupling has its natural place in QFT, and this statement is in fact true of spin in general. The Schrodinger equation makes no mention of spin at all, being a differential operator acting on a scalar function. Historically, it had to be added to the theory in an ad-hoc manner. The Dirac equation is a relativistic wave equation for spin- $\frac{1}{2}$ particles, and marks the natural start for a discussion leading to the relativistic scattering expressions of interest. Of course, a thorough treatment of the subject matter is far too lengthy to fully expound in this paper. We will present it in a way that will put the necessary ideas on the table, and will assume a basic familiarity with the Dirac equation and the Feynman diagram representation of perturbation theory. This will give the reader the relevant formalism required to understand the work presented throughout.

3.1 The Dirac Equation and Spin

The Schrodinger equation is non-relativistic by nature. This is immediately obvious from the fact that it is first order in time derivatives but second order

in spatial derivatives. Any relativistically covariant quantum theory must treat time and space on equal footing, and reduce to the Schrodinger equation in the appropriate non-relativistic limit. The Dirac equation in momentum space, where the γ^μ are the familiar gamma matrices, and m is a fermion mass, is

$$(\not{p} - m)\psi = 0, \quad \not{p} \equiv p_\mu \gamma^\mu. \quad (3.1)$$

If we multiply this by γ^0 , and go to position space, we get

$$i \frac{\partial \psi}{\partial t} = \hat{H}_D \psi \quad : \quad \hat{H}_D \equiv \gamma^0 (i \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m) = -i \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m, \quad (3.2)$$

where, following convention, we have written $\gamma^0 = \beta$ and $\boldsymbol{\alpha} = -\gamma^0 \boldsymbol{\gamma}$. The relation (3.2) defines the free Dirac (or relativistic) Hamiltonian \hat{H}_D . The Dirac equation is a matrix equation and as such $\psi(x) = \psi_\alpha(x)$, must have four components. We know from NR quantum mechanics that fermions like the electron and the quark have only two spin degrees of freedom. The reason for a four-spinor is that relativistic quantum theory introduces antiparticles into the works. Roughly speaking, two spin degrees of freedom are for the fermion, and two are for the antifermion. This is all consistent then, only if two degrees vanish in the fermion's rest frame. In the classical regime where $E \gg |\vec{p}|$, so that $p \approx \langle m, \vec{0} \rangle$,

$$(\not{p} - m) \psi \Rightarrow m(\gamma^0 - 1) \psi = 0. \quad (3.3)$$

We choose a basis in which γ^0 is diagonal

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.4)$$

Then we plug this into (3.3) and get

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0. \quad (3.5)$$

Two of the degrees of freedom vanish as predicted, and $\gamma^0 - 1$ acts like a projection operator that picks out the NR physics¹. In the rest frame of the particle, these will represent the two spin state solutions for the fermion, and we can write them (up to a normalization factor)

$$u_1, u_2 = \begin{pmatrix} \uparrow \\ 0 \end{pmatrix}, \begin{pmatrix} \downarrow \\ 0 \end{pmatrix} \quad (3.6)$$

For an unspecified four-momentum, the free Dirac equation has four linearly independent plane-wave solutions:

$$u_r(p) e^{-ip \cdot x}, \quad v_s(p) e^{+ip \cdot x} \quad r, s = 1, 2. \quad (3.7)$$

We note that because the new four spinor representation is formed from a direct sum of two irreducible (and inequivalent) spinor representations, we get a representation of the spin operator by diagonally stacking the two dimensional representations of the spin operator (ubiquitously chosen as the pauli matrices). We then obtain the spin operator for Dirac theory: $\Sigma = \sigma \otimes \mathbf{1}_{2 \times 2}$.

Let us now give precise definitions (representations) to the spinors and matrices we will be using in this paper. The presentation from here on will follow closely the section on the Breit equation given in Landau and Lifshitz [33]. A useful spinor for our purposes is:

¹Indeed $(\gamma^0 - 1)^2 = -2(\gamma^0 - 1)$, which is a projector up to normalization.

$$u_s = \sqrt{2m} \begin{pmatrix} \left(1 - \frac{\mathbf{p}^2}{8m^2c^2}\right)w_s \\ \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2mc}\right)w_s \end{pmatrix}, \quad (3.8)$$

where the w 's are 2-spinors. This reduces to the rest frame spinor in the limit that momentum goes to zero, with the exception that a relativistic spinor normalization is used ($\bar{u}u = 2m$).

The antiparticle spinors must satisfy $\bar{v}v = -2m$, which is easily seen from an application of charge conjugation $\psi_c = i\gamma^2\psi^*$:

$$\bar{v}v = u^T\gamma^{2\dagger}\gamma^0\gamma^2u_s^* = u^T\gamma^0\gamma^2\gamma^2u_s^* = -\bar{u}u. \quad (3.9)$$

To get the second equality we used $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$; to get the last equality we took the transpose, since it is just a number after all. These spinors are consistent with the Dirac representation for the gamma matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}. \quad (3.10)$$

Note that for arbitrary momentum the spinors are energy-eigenstates, but are *not* spin-eigenstates. That is u_r and v_s are 4-spinors that satisfy $(\not{p} - m)u_r = 0$, and $(\not{p} + m)v_s = 0$; but $[H_D, \boldsymbol{\Sigma}] \neq 0$. This should be clear since the Dirac equation introduces spin-orbit coupling explicitly in the basic spinor and gamma matrix definitions. Recall from NRQM that the ad-hoc addition of spin-spin coupling fouled up commutation relations between \hat{H} and \mathbf{S} , and we can now see that this is an implicit property of our relativistic wave equation. As one would expect though, the *total* angular momentum *is* conserved $[H_D, \boldsymbol{\Sigma} + \mathbf{L}] = 0$.

3.2 Scattering Interaction at LO

In non relativistic scattering theory, the potential $V(r)$ can be obtained from the Fourier transform of the amplitude for scattering $\Gamma = \langle\psi_i|V|\psi_f\rangle$. In a

typical scattering experiment, we are interested in the probability amplitude that a particle that is initially far away from a stationary source, will scatter off the source and end up in the asymptotic state ψ_f . In the initial and final states the particles are essentially free. The Born approximation then reads

$$\Gamma_{Born} \propto \int d^3r e^{-i\mathbf{r}\cdot(\mathbf{p}-\mathbf{p}')} V(r). \quad (3.11)$$

Classically we can speak of particle interactions due to a potential only up to order $1/c^2$ because at order $1/c^3$, the effects of radiation begin to show up. We will discuss how to include the effects of radiation in a later section, but for now our program will be as follows:

- Calculate the scattering amplitude as formulated in relativistic QFT (at tree level).
- Expand the amplitude (i.e. the spinors and propagators) to order $1/c^2$.
- Take the Fourier transform in order to recover what should be the equivalent potential $V(r)$ according to the Born approximation.
- Proceed as before and use this potential as a perturbation to the Coulomb Hamiltonian.

Starting with Ps, the first order scattering amplitude for two fermions inter-

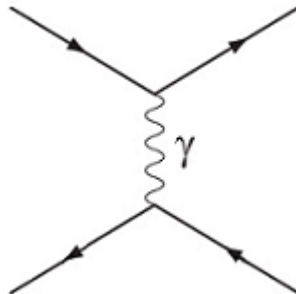


Figure 3.1: Tree Level Scattering

acting electromagnetically and having masses m_1 and m_2 , can be written:

$$M_{scatt.} = -e^2 [\bar{u}_r(p'_1)\gamma^\mu u_s(p_1)] D_{\mu\nu}(k^2) [\bar{v}_t(p_2)\gamma^\nu v_w(p'_2)], \quad (3.12)$$

where $D_{\mu\nu}(k^2)$ is the gauge-propagator. The gauge boson for QED is the photon with coupling e , whilst for QCD it is the gluon (with coupling g_s). For now, we do the calculation for QED, and leave the QCD one-gluon scattering calculation to the end where it will be modified by $\alpha \rightarrow \alpha_s$ and we will have to include a colour factor as well. We will follow the custom when working in the NR limit, and use the Coulomb gauge

$$D_{00} = -\frac{1}{\mathbf{k}^2}, \quad D_{ij} = \frac{1}{\mathbf{k}^2 - \omega_k^2/c^2} \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \right). \quad (3.13)$$

For now we are explicitly keeping factors of the speed of light ($c \neq 1$). The order in $1/c$ then organizes the terms, allowing us to see at which stage of the game new physics appears which is not present in the Schrodinger equation.

Let's check the consistency of all this and neglect terms of order $1/c$ and higher (which is the same as setting $c \rightarrow \infty$). The spinors reduce to those of NRQM and we simply get the leading part of the D_{00} term

$$M_{scatt.} = (2m_1)(-2m_2)(w_r^\dagger w_s) V(k) (w_t^\dagger w_w), \quad V(k) = -\frac{e^2}{\mathbf{k}^2}. \quad (3.14)$$

$V(k)$ is the Fourier transform of the Coulomb potential where

$$\mathbf{k} = \mathbf{p}'_1 - \mathbf{p}_1 = \mathbf{p}_2 - \mathbf{p}'_2, \quad (3.15)$$

just as we suspected for the complete NR-limit. To be consistent with the order of the relativistic expansion, we must also include the next term in the expansion of the relativistic kinetic energy in our Hamiltonian. The Hamiltonian operator is concerned with kinematical degrees of freedom, so we define

the free Hamiltonian (with no potential present) as

$$\hat{H}^{(0)} \psi = (E - m_1 c^2 - m_2 c^2) \psi, \quad \text{with } H^{(0)} = \sum_{i=1}^2 \frac{\hat{\mathbf{p}}_i^2}{2m_i} - \frac{\hat{\mathbf{p}}_i^4}{8m_i^3 c^2}. \quad (3.16)$$

Let us now write out our scattering amplitude to the required accuracy. We use a concise notation where the labels 1,2 and 1',2' represent both momentum and spin indices. For example we write $u_1 \equiv u_{\lambda_1}(p_1)$.

$$\begin{aligned} -M_{scatt.} &= e^2 [\bar{u}_{1'} \gamma^0 u_1] [\bar{v}_2 \gamma^0 v_{2'}] D_{00} + e^2 [\bar{u}_{1'} \gamma^i u_1] [\bar{v}_2 \gamma^j v_{2'}] D_{ij} \\ &= e^2 [u_{1'}^\dagger, u_1] [v_2^\dagger, v_{2'}] D_{00} + e^2 [\bar{u}_{1'} \gamma^i u_1] [\bar{v}_2 \gamma^j v_{2'}] D_{ij} \end{aligned} \quad (3.17)$$

We insert the expressions for the spinors and gamma matrices [eqn.'s (3.8),(3.10)], and with some massaging get

$$u_{1'}^\dagger, u_1 = (2m_1) w_{1'}^* \left[1 - \frac{\mathbf{k}^2}{8m_1 c^2} + \frac{i\boldsymbol{\sigma}_1 \cdot \mathbf{k} \times \mathbf{p}_1}{4m_1 c^2} \right] w_1, \quad (3.18)$$

$$\bar{u}_{1'} \boldsymbol{\gamma} u_1 = (1/c) w_{1'}^* [i\boldsymbol{\sigma}_1 \times \mathbf{k} + 2\mathbf{p}_1 + \mathbf{k}] w_1. \quad (3.19)$$

The terms arising from the anti-particle spinors are very similar (we set $1 \rightarrow 2$ and $\mathbf{k} \rightarrow -\mathbf{k}$). If we carry out the multiplication and simplify, we get the tree level result

$$M_{scatt.} = (2m_1)(-2m_2) w_{1'}^\dagger w_2^\dagger V(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}) w_1 w_{2'}. \quad (3.20)$$

The potential in momentum-space, with higher order terms in $1/c$ now included, becomes:

$$\begin{aligned}
V(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}) = -e^2 \left[\frac{1}{\mathbf{k}^2} - \frac{1}{8m_1c^2} - \frac{1}{8m_2c^2} + \frac{(\mathbf{k} \cdot \mathbf{p}_1)(\mathbf{k} \cdot \mathbf{p}_2)}{m_1m_2\mathbf{k}^4} - \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1m_2\mathbf{k}^2} \right. \\
+ \frac{i\boldsymbol{\sigma}_1 \cdot \mathbf{k} \times \mathbf{p}_1}{4m_1^2c^2\mathbf{k}^2} - \frac{i\boldsymbol{\sigma}_2 \cdot \mathbf{k} \times \mathbf{p}_2}{4m_2^2c^2\mathbf{k}^2} - \frac{i\boldsymbol{\sigma}_1 \cdot \mathbf{k} \times \mathbf{p}_2}{2m_1m_2c^2\mathbf{k}^2} \\
\left. + \frac{i\boldsymbol{\sigma}_2 \cdot \mathbf{k} \times \mathbf{p}_1}{2m_1m_2c^2\mathbf{k}^2} + \frac{(\boldsymbol{\sigma}_1 \cdot \mathbf{k})(\boldsymbol{\sigma}_2 \cdot \mathbf{k})}{4m_1m_2c^2\mathbf{k}^2} - \frac{(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)}{4m_1m_2c^2} \right]. \quad (3.21)
\end{aligned}$$

We are of course only interested in the spin-spin interaction and thus can limit our attention to only those terms containing *both* spin operators ($\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$), which upon examination is only the last two terms. Performing a Fourier transform on these two terms we get [33]

$$V_{spin} = -\frac{e^2}{(16\pi)m_1m_2c^2} \left[\frac{\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{r^3} - \frac{3(\boldsymbol{\sigma}_1 \cdot \mathbf{r})(\boldsymbol{\sigma}_2 \cdot \mathbf{r})}{r^5} - \frac{8\pi}{3}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \delta^3(\mathbf{r}) \right]. \quad (3.22)$$

Putting in $\mathbf{S}_i = \frac{1}{2}\boldsymbol{\sigma}_i$, we obtain precisely the potential that we got from the perturbation theory of NRQM. It is quite satisfying to see that not only have we generated the interaction of interest in the relativistic formalism, but we have systematically included all other contributions to the potential in this order.

Now, we can just plug the non-vanishing piece into the expression for first order perturbation theory as before and get

$$E_{s.s.} = \frac{8\pi\alpha}{3m_e^2c} \langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle |\psi_{100}(0)|^2. \quad (3.23)$$

Now, using the result that $|\psi_{100}(0)|^2 = 1/\pi(2a_0)^3$, the scattering amplitude gives:

$$E_{s.s.}^{(S)} = \frac{\alpha^4 m_e c^2}{3} \left\{ \begin{array}{l} +1/4 \text{ (triplet)} \\ -3/4 \text{ (singlet)} \end{array} \right\}, \quad (3.24)$$

where the superscript emphasizes that this is the contribution from the scat-

tering diagram only; there is another contribution as well to which we now turn.

3.3 Annihilation of Positronium

To obtain the complete leading order result for positronium's HFS we need to include, in addition to the scattering interaction above, an annihilation contribution. Annihilation is a process which has no classical analogue, but in the positronium bound state there is always the possibility that the electron-positron pair will annihilate (momentarily) into a single photon before returning to the original particle pair. Photons are particles of spin one and as such we must conclude that only a pair that is in the spin triplet configuration can undergo single photon annihilation. As a result, we see that only the energy levels of this configuration will be affected by such an interaction. It is also worth noting that quarkonium gets no contribution from single gluon annihilation for reasons to be explained in the next section.

The amplitude that is attributed to this process is similar in appearance to the scattering amplitude but requires some simple modifications. We relabel a bit and use p_-, p_+ and p'_-, p'_+ to label the incoming and outgoing momentum respectively. We also repurpose the variable k to now be

$$k \equiv p_- + p_+ = p'_- + p'_+. \quad (3.25)$$

With these redefinitions, the amplitude for photon annihilation takes the form

$$M_{ann.} = [\bar{v}(p_+)\gamma^\mu u(p_-)] D_{\mu\nu}(k^2) [\bar{u}(p'_-)\gamma^\nu v(p'_+)]. \quad (3.26)$$

These amplitudes are of course gauge invariant, and we can switch to the usual Feynman gauge at our leisure

$$D_{\mu\nu} = \frac{1}{k^2} g_{\mu\nu}. \quad (3.27)$$

Now, since the particles are nearly on shell $p_i^0 = mc + \mathcal{O}(\alpha^2)$, we can write $k^0 = (p_+^0 + p_-^0)^2 \approx 4m_e^2 c^2 \gg \mathbf{k}^2$. The propagator is therefore simply

$$D_{\mu\nu} \approx \frac{1}{4m_e^2 c^2} g_{\mu\nu}, \quad (3.28)$$

i.e. is of order $1/c^2$ already. Thus we need only keep spinors to first order for this calculation

$$u(p_-) = \sqrt{2m_e} \begin{pmatrix} w_- \\ 0 \end{pmatrix}, \quad (3.29)$$

with corresponding positron spinor

$$v(p_+) = \sqrt{2m_e} \begin{pmatrix} 0 \\ w_+ \end{pmatrix} = \sqrt{2m_e} \begin{pmatrix} 0 \\ -i\sigma_2 w_-^* \end{pmatrix}. \quad (3.30)$$

With this simplification we know that for instance

$$[\bar{v}(p_+) \gamma^0 u(p_-)] = [\bar{u}(p'_-) \gamma^0 v(p'_+)] = 0, \quad (3.31)$$

$$[\bar{v}(p_+) \boldsymbol{\gamma} u(p_-)] = [v^*(p_+) \boldsymbol{\alpha} u(p_-)] = 2m_e (w_+^* \boldsymbol{\sigma} w_-). \quad (3.32)$$

The amplitude then becomes

$$M_{ann.} = -\frac{e^2}{4m_e^2 c^2} (2m_e)^2 (w_+^* \boldsymbol{\sigma} w_-) (w_-'^* \boldsymbol{\sigma} w_+'). \quad (3.33)$$

In the end we need electron (positron) spinors with electron (positron) spinors. To accomplish this we use the little known identity [33]

$$(w_+^* \boldsymbol{\sigma} w_-) (w_-'^* \boldsymbol{\sigma} w_+') = \frac{3}{2} (w_-'^* w_-) (w_+^* w_+') + \frac{1}{2} (w_-'^* \boldsymbol{\sigma} w_-) (w_+^* \boldsymbol{\sigma} w_+'). \quad (3.34)$$

All said and done then, we have that

$$M_{ann.} = -(2m_e)^2 w_-'^* w_+'^* \left[\frac{e^2}{8m_e^2 c^2} (3 + \boldsymbol{\sigma}_+ \cdot \boldsymbol{\sigma}_-) \right] w_- w_+. \quad (3.35)$$

This then leads to a contribution that can simply be added to the HFS already found (since $E_{hfs} \equiv E_{S_3^1} - E_{S_1^0}$)

$$\begin{aligned}
E_{hfs} &= E_{s.s.}^{(S)} + E_{s.s.}^{(A)} = \frac{1}{3}\alpha^4 m_e c^2 + \frac{1}{4}\alpha^4 m_e c^2 \\
&= \frac{7}{12}\alpha^4 m_e c^2 \\
&= 8.2 \times 10^{-4} \text{ eV}.
\end{aligned} \tag{3.36}$$

Notice that this is approximately one-hundred times the HFS in Hydrogen. In terms of frequency we get

$$\Delta\nu^{LO} = 204.4 \text{ GHz} \tag{3.37}$$

3.4 Colour Factors and Quarkonium

Strong interactions, as formulated in QCD, require quarks to carry one of three different colour charges. The established notation is to label these by the three colours Red, Green, and Blue (R,G,B) as well as their anti-colour equivalents ($\bar{R}, \bar{G}, \bar{B}$). We now turn to the subject of modifying the tree level scattering amplitude in order to accommodate $SU(3)_c$ colour group factors. In order to avoid a full blown explanation of non-abelian gauge theories, which would require us to divert into Lagrangian densities which have highly non-trivial covariant derivative properties, we will settle here for a heuristic discussion.

We must first of all attribute to each incoming and outgoing quark line a factor indicating one of three different possible colours. If you like we can even give them a specific representation (basis) such as the familiar one used in Euclidean space (sometimes written as e_1, e_2, e_3)

$$|R\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |B\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |G\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.38}$$

Next, we also have to deal with colour factors for the gluons, which are the gauge-bosons (i.e. force carriers) for QCD. Quarks can change colour after interacting with a gluon, so although a quark-gluon vertex looks very similar to the fundamental vertex of QED, the gluons are themselves “charged” entities which carry one unit of colour and one of anti-colour. For example a quark may change from red to blue in such an interaction; colour conservation then demands that the gluon carry away the red unit of the incoming quark as well as a unit of anti-blue to cancel the colour of the outgoing quark. More generally we say that the colour state of the quark changes from $|i\rangle \rightarrow |j\rangle$, and the gluon carries away the appropriate units of colour and anticolour.



Figure 3.2: Quark-Gluon Vertex $-igT_{ij}^a$, $a = 1\dots 8$.

In the language of group theory, the quarks and antiquarks are in the fundamental (three dimensional) representation of $SU(3)$. We must therefore combine two such representations, one for the quarks and one for the antiquarks, to get all the possible gluon states. Group theory gives us the combination $3 \otimes \bar{3} = 8 \oplus 1$ for the gluons. In other words, the gluons can combine into a colour octet and a colour singlet. Since all free particles must be colour singlets, and since no free singlet gluon has ever been observed, we do not include colour singlet gluons in our theory. We now see why no annihilation effects occur in quarkonium at LO, because the quark singlet state cannot annihilate into a colour octet. In keeping with the explicit representation for the quarks above, we could for example form some gluon matrices $T_{ij}^a \rightarrow |a\rangle_{ij}$, which are

in what's called the adjoint representation:

$$\begin{aligned}
|1\rangle_{ij} &= \frac{1}{2} (|R\rangle_i \otimes |\bar{B}\rangle_j + |B\rangle_i \otimes |\bar{R}\rangle_j) \\
|2\rangle_{ij} &= -\frac{i}{2} (|R\rangle_i \otimes |\bar{B}\rangle_j - |B\rangle_i \otimes |\bar{R}\rangle_j) \\
|3\rangle_{ij} &= \frac{1}{2} (|R\rangle_i \otimes |\bar{R}\rangle_j - |B\rangle_i \otimes |\bar{B}\rangle_j) \\
&\vdots \\
|8\rangle_{ij} &= \frac{1}{2\sqrt{3}} (|R\rangle_i \otimes |\bar{R}\rangle_j + |B\rangle_i \otimes |\bar{B}\rangle_j - 2|G\rangle_i \otimes |\bar{G}\rangle_j). \tag{3.39}
\end{aligned}$$

As you can see they are all Hermitian matrices, and as quantum mechanical operators, they act on the quark states. Apart from the factors of $1/2$ in front, these are the so-called Gell-man matrices (λ_{ij}^a). You can think of them as the analogous 3×3 matrices for $SU(3)$ that the Pauli-matrices are for $SU(2)$. By the arguments above, these carry precisely the information of the incoming and outgoing quarks, and that is all that is needed at a vertex. Explicitly;

$$\begin{aligned}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda^2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
\lambda^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda^8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{3.40}
\end{aligned}$$

Let us now tackle the tree level scattering amplitude, which is represented by the diagram:

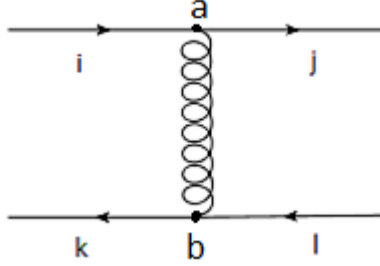


Figure 3.3: Colour Factor at Tree level.

The respective vertices give us colour factors T_{ij}^a and T_{kl}^b , but since the gluon propagator goes straight from one vertex to the other we can safely put in a factor of δ^{ab} before we sum over all possibilities¹. Then, since any free particle must be a colour singlet, our meson had better also be in the singlet configuration

$$\frac{1}{\sqrt{3}}(|R\bar{R}\rangle + |B\bar{B}\rangle + |G\bar{G}\rangle), \quad (3.41)$$

or there's not much point in all this. Thus the incoming quarks $|i\rangle$ and $|k\rangle$ form one singlet, while the outgoing $|j\rangle$ and $|l\rangle$ form another. In other words (c.f. (3.38) and (3.41))

$$|q\bar{q}\rangle_{in} = |q\bar{q}\rangle_{out} = \frac{1}{\sqrt{3}} \sum_{i=1}^3 |e_i\rangle |e_i\rangle, \quad (3.42)$$

which amounts to an overall factor of $\frac{1}{3} \sum_{i,j} \delta_{ik} \delta_{jl}$ (c.f. figure 3.3). Therefore if we consider the matrix element for this scattering event, and sum over all

¹Again this is rather heuristic, if the lack of rigour bothers you there are many excellent textbooks that derive the Feynman rules explicitly from the QCD Lagrangian. See for instance [28]

possible gluon exchanges:

$$\begin{aligned} \sum_{a,b} \delta^{ab} \langle q | T^a | q \rangle \langle \bar{q} | T^b | \bar{q} \rangle &= \frac{1}{3} \sum_{a,b} \sum_{i,j} \delta^{ab} T_{ij}^a T_{ji}^b \\ &= \frac{1}{12} \sum_{a,b} \delta^{ab} \text{Tr}\{\lambda^a \lambda^b\}. \end{aligned} \quad (3.43)$$

The trace then gives us another factor of $2\delta^{ab}$, which can be checked by explicit multiplication. Thus the colour factor works out to be

$$C_F = \frac{1}{6} \sum_{a=1}^8 \delta^{aa} = \frac{4}{3}. \quad (3.44)$$

So what does this result tell us exactly? Well we have already shown how the QED scattering amplitude at leading order reproduces the Coulomb potential. If we therefore tack onto this result the appropriate colour factor C_F , then

$$V_s = -\frac{C_F \alpha_s}{r}. \quad (3.45)$$

If we then plug this potential into the Schrodinger equation and solve, we get the leading energy spectrum

$$E_n^{(0)} = -\frac{C_F^2 \alpha_s^2}{4n^2} m_q. \quad (3.46)$$

The expression for quarkonium's ‘‘Bohr’’ radius is then given by the relation $a_0^{q\bar{q}} = (\frac{1}{2} C_F \alpha_s m_q c)^{-1}$. Therefore

$$|\psi_{100}^{q\bar{q}}(0)|^2 = \left(\frac{C_F}{2}\right)^3 \frac{1}{\pi a_0^3}, \quad (3.47)$$

and the final formula for the HFS in quarkonium at leading order is (c.f. (3.24))

$$E_{hfs}^{q\bar{q}} = \frac{C_F^4 \alpha_s^4 m_q c^2}{3}. \quad (3.48)$$

Chapter 4

Radiative Corrections To HFS in Positronium

4.1 Introduction

The bound state of positronium is nonrelativistic by nature, as is evident by comparing the LO Coulomb energies with the mass of the lepton pair. The behaviour of positronium is described by complicated multiscale dynamics characterized by four different regions of momentum $k = (k_0, \mathbf{k})$:

1. The hard region, where k_0 and \mathbf{k} scale like the electron mass m_e .
2. The soft region, where k_0 and \mathbf{k} scale like $m_e v \sim m_e \alpha$.
3. The potential region, where k_0 scales like $m_e v^2 \sim m_e \alpha^2$ and \mathbf{k} scales like $m_e v \sim m_e \alpha$.
4. The ultrasoft region, where k_0 and \mathbf{k} scale like $m_e v^2 \sim m_e \alpha^2$.

The nonrelativistic nature of the bound system means that the electron and positron velocities (v) are on the order of the fine structure constant α , which can be seen by taking the expectation value of v^2 in the ground state $\Psi_{100}(r)$. Bound states appear as poles in the scattering amplitudes as will be shown in

more detail in sections 3.6 and 3.7 on Green's functions. For positronium, the energies are known to be $P_0 = 2m_e + E_b$, where the binding energy is $E_b = -\frac{1}{4} \frac{m_e \alpha^2}{n^2}$ to lowest order approximation. This of course means that the e^+e^- scattering amplitude has a pole for each $n = 1, 2, \dots$, just below the threshold for pair production ($E_{thresh} \equiv 2m_e$). But the only way for there to be poles is for the perturbative expansion of the scattering process to diverge.

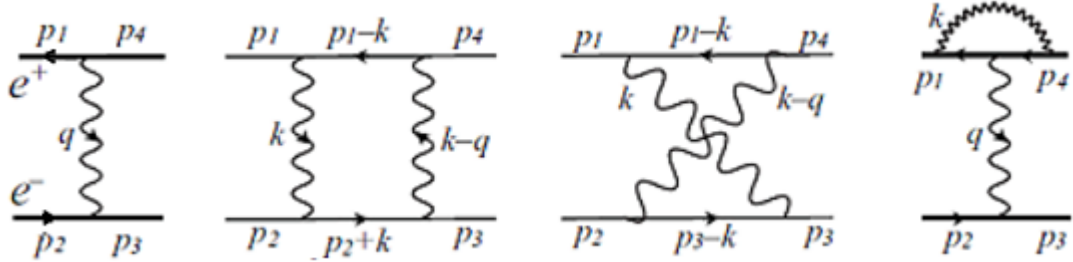


Figure 4.1: The Born diagram, and some first order correction diagrams to e^+e^- scattering

Let us examine the order in α that the ladder diagrams exhibit in the potential region (i.e. in the region of dynamics exhibited by the bound system). The Born diagram in the above contains two vertices that give an α contribution to the numerator and $\mathbf{q}^2 \sim \alpha^2$ in the denominator from the photon propagator, which gives a total overall factor $1/\alpha$. The box diagram gives α^2 from the vertices and α^{-2} for each of the photon propagators. Additionally, the electron and positron propagators are off shell on the order $k_0 \sim q_0 \sim \alpha^2$. Finally the loop momenta give $dk^0 d^3\mathbf{k} \sim (\alpha^2)(\alpha^3)$, and the total is again $\sim 1/\alpha$. Inspection of a general ladder diagram with n photon propagators, shows we get

1. α^n from vertices
2. α^{-2n} from photon propagators
3. $\alpha^{-4(n-1)}$ from fermion propagators
4. $\alpha^{5(n-1)}$ from loop momenta

This gives a total power $(5n - 5) - (4n - 4) - 2n + n = -1$, for all ladder diagrams. Thus all ladder diagrams contribute on the same order as the born diagram in the potential region, which allows the series to diverge. In the soft region however, the energy scales only as $m_e v \sim m_e \alpha$, and the fermion propagators now only contribute a power of $-2(n - 1)$, and loop momenta contribute $4(n - 1)$ for the loops, giving a total of α^{n-2} from the diagrams. Thus we see that as we move to higher regions of momentum (soft and hard), their relative contributions to the perturbative series will be convergent. To further understand the structure of the diagrams, we can see that in the ladder diagram we have the situation where the integration contour is pinched between two poles in the complex plane. This can be seen by taking $p_2 \sim p_1 \sim (m_e, \mathbf{0}) + \mathcal{O}(\alpha)$ and noting that in the potential region, $k_0^2 \sim \mathcal{O}(\alpha^2)$, we have

$$M_{box} \propto \frac{1}{((p_2 + k)^2 - m^2 + i\epsilon)((p_1 - k)^2 - m^2 + i\epsilon)} \sim \frac{-4m_e}{(k^0 - \frac{\mathbf{k}^2}{2m_e} + i\epsilon)(k^0 + \frac{\mathbf{k}^2}{2m_e} - i\epsilon)}.$$

In the crossed box though for instance, we would have both poles in the region $Im(k_0) > 0$, and we can close the contour from below, which avoids the singularities altogether. A similar situation occurs in the vertex, which only makes contributions to the hard momentum region. In general, all but the ladder diagrams will scale with higher orders of α than the Born diagram. As we will see in the section on Green's functions, it is precisely these diagrams which form the largest contribution to the potential in the Hamiltonian formulation, viz. the coulomb term, which must be taken into account in all orders.

We can already start to get a feeling, from this rudimentary examination, that it would be beneficial for us to be able to separate out the contributions from the different regions. In order to accomplish this goal in a systematic way, the method of nonrelativistic effective theories has been developed [34]. We now pass to the discussion of these effective theories where QED in this language would be called the “full” theory or full QED.

4.2 Effective Field Theory

Effective Field Theory (EFT) is exactly what it sounds like, it is a theory that can accurately describe the physics at a particular length and/or energy scale. For instance the De Broglie wavelength of a baseball is about one tenth of a Planck length; the fact then that it technically behaves as a quantum mechanical wave doesn't matter. Similarly, even if you have Aroldis Chapman pitch that baseball at a record 170 km/h, the expressions for the classical and relativistic kinetic energy only differ by about one part in 10^{13} . Thus we can safely say that even by the most conservative estimate, baseball is governed by classical mechanics.

I wish to begin the discussion of effective field theories with a great example of a field theory model that predicted its own demise. The theory of weak interactions was originally modelled by Enrico Fermi as a contact interaction (i.e. contained no mediators). A four particle (2-in, 2-out) scattering event in this framework, say neutrino scattering, would look like

$$iM^{(1)} = G [\bar{\psi} \gamma^\mu (1 - \gamma^5) \psi] [\bar{\psi} \gamma_\mu (1 - \gamma^5) \psi] \quad (4.1)$$

at leading order; that is, it will just be some number dependent on the neutrino mass and incoming momenta, times the coupling G . The next order diagram is quadratically divergent

$$iM^{(2)} \sim G^2 \int d^4p \left(\frac{1}{p}\right) \left(\frac{1}{p}\right) \quad (4.2)$$

So we impose a cutoff Λ at some energy scale that is much larger than the masses and momenta of interest, and this allows us to write the amplitude schematically as

$$iM \sim G + \Lambda^2 G^2 + \mathcal{O}(G^3) \quad (4.3)$$

This is true provided that $\Lambda^2 \gg m_\nu^2, p_i^2$; but we can already see that it has to

break down once $\Lambda^2 \sim \frac{1}{G}$, so that the second order term becomes comparable to the first. In other words, new physics has to appear once the physical parameters of the interaction such as the center of mass energy go like $E^2 \sim \frac{1}{G}$. Indeed new mediator particles are produced at this energy, the W s (and later the Z) bosons. The reparation of this cutoff dependence came with the unification of electromagnetic and weak interactions, and the correct first order diagram is actually

$$iM^{(1)} = \left(\frac{g_w}{2\sqrt{2}} \right)^2 \frac{-g_{\mu\nu} + q_\mu q_\nu / m_W^2}{q^2 - m_W^2 + i\epsilon} [\bar{\psi} \gamma^\mu (1 - \gamma^5) \psi] [\bar{\psi} \gamma^\nu (1 - \gamma^5) \psi]. \quad (4.4)$$

Evidently then, from the limit $q^2 \ll m_W^2$, we know that

$$G = \frac{g_w^2}{8m_W^2}. \quad (4.5)$$

Now, our goal in using effective theories in the work presented here, is to create a natural framework which will allow us to go from relativistic field theories to NR quantum mechanics. To do this we must first introduce the QED Lagrangian density, which reads

$$\mathcal{L}_{QED} = \bar{\Psi} (i\not{D} - m) \Psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \quad (4.6)$$

The gauge-covariant derivative is defined in terms of the four-vector potential $A^\mu = \langle \phi, \mathbf{A} \rangle$, by the relationship

$$\begin{aligned} iD^\mu &\equiv i\partial^\mu - eA^\mu = \langle (i\partial_t - e\phi), (-i\nabla - e\mathbf{A}) \rangle \\ &= \langle (E - V), (\mathbf{p} - e\mathbf{A}) \rangle, \end{aligned} \quad (4.7)$$

where we have replaced the scalar-potential $\phi(r)$ by the electric potential $V(r) = e\phi(r)$. The anti-symmetric electromagnetic-field tensor $F^{\mu\nu}$, can be

written out in terms of the potential as well

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (4.8)$$

Explicitly, the electric and magnetic fields are $E_i \equiv cF^{0i}$, and $B_i \equiv -\frac{1}{2}\epsilon_{ijk}F^{jk}$, which we can use to create the gauge-invariant Lorentz-scalar $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(\mathbf{E}^2/c^2 - \mathbf{B}^2)$. It is also sometimes instructive to write the field tensor as a matrix

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}. \quad (4.9)$$

4.2.1 NRQED and pNRQED

We now pass to NRQED by performing an expansion in inverse rest energy $1/m_e c^2$. This manifests the inherent NR nature of the system and brings it to a Lagrangian containing the electron and positron as separate fields, each a 2-spinor. In order to perform this expansion, we pass to a frame where the kinetic energy of the particles is very small in comparison to their rest masses, and we take out the dominant time-dependent term explicitly as $\Psi \rightarrow e^{-im_e c^2 t} \tilde{\Psi}$. We also change units to the so-called natural units in which $c \rightarrow 1$. The Dirac equation in the presence of an electromagnetic field A^μ then becomes

$$[i\not{D} + m_e(\gamma^0 - 1)]\tilde{\Psi} = 0. \quad (4.10)$$

If we now write out the 4-spinor in terms of its 2-spinors

$$\begin{pmatrix} iD_t & -i\boldsymbol{\sigma} \cdot \mathbf{D} \\ i\boldsymbol{\sigma} \cdot \mathbf{D} & -iD_t - 2m_e \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = 0, \quad (4.11)$$

we get two separate equations

$$iD_t\psi - i\boldsymbol{\sigma} \cdot \mathbf{D}\chi = 0, \quad (4.12)$$

$$(iD_t + 2m_e)\chi - i\boldsymbol{\sigma} \cdot \mathbf{D}\psi = 0. \quad (4.13)$$

This can then be formally solved to give

$$\chi = \left(\frac{1}{iD_t + 2m_e} \right) i\boldsymbol{\sigma} \cdot \mathbf{D}\psi. \quad (4.14)$$

We now back-substitute this result into our QED Lagrangian, which for the ψ piece becomes

$$\mathcal{L}_\psi = \psi^\dagger \left(iD_t - i\boldsymbol{\sigma} \cdot \mathbf{D} \frac{1}{iD_t + 2m_e} i\boldsymbol{\sigma} \cdot \mathbf{D} \right) \psi \quad (4.15)$$

The operator in the second term is then expanded in powers of $1/m_e$ to the desired order

$$\frac{1}{iD_t + 2m_e} = \frac{1}{2m_e} \left[1 - \frac{iD_t}{2m_e} + O(1/m_e^2) \right] \quad (4.16)$$

which when inserted into \mathcal{L}_ψ becomes

$$\mathcal{L}_\psi = \psi^\dagger \left(iD_t + \frac{(\boldsymbol{\sigma} \cdot \mathbf{D})^2}{2m_e} - \frac{1}{4m_e^2} (\boldsymbol{\sigma} \cdot \mathbf{D}) iD_t (\boldsymbol{\sigma} \cdot \mathbf{D}) \right) \psi + O(1/m_e^2) \quad (4.17)$$

The first term in the expansion is fairly straightforward and more importantly confirms the important fact that the magnetic moment of the electron (at leading order) is indeed $\mu_S = \frac{1}{2m_e}$; we show its derivation explicitly. Start with the identity

$$\begin{aligned} \sigma_i \sigma_j &= \frac{1}{2} [\sigma_i, \sigma_j] + \frac{1}{2} \{ \sigma_i, \sigma_j \} \\ &= \delta_{ij} + i\epsilon_{ijk} \sigma_k, \end{aligned} \quad (4.18)$$

so that $(\boldsymbol{\sigma} \cdot \mathbf{D})^2$ becomes

$$\sigma_i \sigma_j D_i D_j = \mathbf{D}^2 + i \boldsymbol{\sigma} \cdot (\mathbf{D} \times \mathbf{D}). \quad (4.19)$$

If the operator components of $\mathbf{D} = \mathbf{p} - e\mathbf{A}$ commuted, then the second term would vanish. As it is

$$\begin{aligned} (\mathbf{D} \times \mathbf{D})_k &= \epsilon_{ijk} [D_i, D_j] \\ &= i \epsilon_{ijk} ([\partial_i, A_j] + [A_i, \partial_j]) \\ &= i \epsilon_{ijk} (\partial_i A_j - \partial_j A_i), \end{aligned} \quad (4.20)$$

which follows from

$$\begin{aligned} [\partial_i, A_j] f &= \partial_i (A_j f) - A_j (\partial_i f) \\ &= (\partial_i A_j) f + A_j (\partial_i f) - A_j (\partial_i f) \\ &= (\partial_i A_j) f. \end{aligned} \quad (4.21)$$

Thus for the first term we get

$$(\boldsymbol{\sigma} \cdot \mathbf{D})^2 = \mathbf{D}^2 - \boldsymbol{\sigma} \cdot (\nabla \times \mathbf{A}) = \mathbf{D}^2 - \boldsymbol{\sigma} \cdot \mathbf{B}, \quad (4.22)$$

and we get the pleasing result that

$$\frac{(\boldsymbol{\sigma} \cdot \mathbf{D})^2}{2m_e} = \frac{\mathbf{D}^2}{2m_e} - \frac{\boldsymbol{\sigma} \cdot \mathbf{B}}{2m_e}. \quad (4.23)$$

Thus to order $1/m_e$ we recover the famous Pauli-equation for spin-1/2 particles

$$\hat{H} |\psi\rangle = \left[\frac{1}{2m_e} (\mathbf{p} - e\mathbf{A})^2 - \frac{e}{2m_e} \boldsymbol{\sigma} \cdot \mathbf{B} + V(r) \right] |\psi\rangle = i \frac{\partial}{\partial t} |\psi\rangle. \quad (4.24)$$

To get the next order in (4.17) involves $(\boldsymbol{\sigma} \cdot \mathbf{D}) i D_t (\boldsymbol{\sigma} \cdot \mathbf{D})$, and is a bit more involved. The interested reader can start by commuting D_t past the D_i 's in a symmetric way¹, and by noticing that we can write

$$[D_t, \mathbf{D}] = \mathbf{E}, \quad (4.25)$$

which follows from the definition of the field tensor. All said and done then, we get

$$\begin{aligned} \mathcal{L}_{NRQED} = \psi^\dagger & \left(i D_t + \frac{\mathbf{D}^2}{2m_e} + \frac{\mathbf{D}^4}{8m_e^3} - c_F \frac{e \boldsymbol{\sigma} \cdot \mathbf{B}}{2m_e} + c_D \frac{e(\mathbf{D} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{D})}{8m_e^2} + \right. \\ & \left. c_S \frac{ie \boldsymbol{\sigma} \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D})}{8m_e^2} + \mathcal{O}(1/m_e^3) \right) \psi + (\psi \rightarrow \chi) \\ & + d_\sigma \frac{\alpha}{m_e^2} \psi^\dagger \boldsymbol{\sigma} \psi \chi^\dagger \boldsymbol{\sigma} \chi + \dots \end{aligned} \quad (4.26)$$

where $\psi \rightarrow \chi$ just indicates that a contribution, equivalent to the first term, comes from the positron field (and of course $e \rightarrow -e$). The first three terms are kinetic and receive no corrections. The (Wilson) coefficients c_i , $i = \{F, D, S\}$, which stand for Fermi, Darwin, and Spin-orbit respectively, do get corrections from higher order loop calculations. The final term represents a four fermion contact operator (which vanishes in the Born approximation) and the ellipses indicate other contact terms that do not affect the splitting. We have arranged it in such a fashion that in leading order $c_i = 1$, and corrections can be incorporated naturally as a series in α . We can for instance write

$$c_i = c_i^{(0)} + \alpha c_i^{(1)} + \dots \quad (4.27)$$

They characterize the contribution from the hard modes that have been inte-

¹that is to say, write it:

$$(\boldsymbol{\sigma} \cdot \mathbf{D}) D_t (\boldsymbol{\sigma} \cdot \mathbf{D}) = \frac{1}{2} (\boldsymbol{\sigma} \cdot \mathbf{D}) \boldsymbol{\sigma} \cdot \left\{ \mathbf{D} D_t + [D_t, \mathbf{D}] \right\} + \frac{1}{2} \left\{ D_t \mathbf{D} + [\mathbf{D}, D_t] \right\} \cdot \boldsymbol{\sigma} (\boldsymbol{\sigma} \cdot \mathbf{D})$$

grated out. In order to obtain these coefficients, a matching procedure must be used wherein select green's functions from the full theory and effective theory are set equal to each other at some momentum (e.g. $q_{match} = \mu_0$). Full theory graphs are expanded in $1/m_e$ (before evaluation of the graphs). The effective theory knows nothing about the hard modes, and so a general matching condition looks schematically like

$$c_i = \text{Full graph}(\mu_0) - \text{Eff. graph}(\mu_0). \quad (4.28)$$

Now, we obtained the NRQED Lagrangian by formally integrating out the hard modes $k \sim m$, which then appear as corrections to the Wilson coefficients. But, we have not yet made contact with a Schrodinger-like picture. Essentially the problem is that there are still dynamical degrees of freedom with soft momentum, and consequentially integrals are still complicated by having multiple scales (albeit one less). As an illustration of what we mean, consider the 4-fermion Green function with two photon exchanges in NRQED

$$ie^4 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q-p)^2 + i\eta} \frac{1}{(q-p')^2 + i\eta} \frac{i}{q^0 + \frac{1}{2}E - \frac{\mathbf{q}^2}{2m} + i\eta} \frac{i}{-q^0 + \frac{1}{2}E - \frac{\mathbf{q}^2}{2m} + i\eta} \quad (4.29)$$

We essentially have two contributing regions, so we separate the above integral as $I_p + I_s$. In the first region, where $E \sim p^0 \sim p'^0 \sim q^0 \sim mv^2$ and $|\mathbf{p}| \sim |\mathbf{p}'| \sim |\mathbf{q}| \sim mv$, we can essentially neglect zero-components in the photon propagators

$$I_p \sim \int \frac{d^4q}{(2\pi)^4} \frac{1}{(\mathbf{q}-\mathbf{p})^2 + i\eta} \frac{1}{(\mathbf{q}-\mathbf{p}')^2 + i\eta} \frac{1}{q^0 + \frac{1}{2}E - \frac{\mathbf{q}^2}{2m} + i\eta} \frac{1}{q^0 - (\frac{1}{2}E - \frac{\mathbf{q}^2}{2m} + i\eta)}. \quad (4.30)$$

If we now perform the contour integral over the q^0 and notice that the photon propagators are reduced to Coulombic ones, we can rewrite it as

$$I_p \sim \int \frac{d^3\mathbf{q}}{(2\pi)^3} \tilde{V}_C(\mathbf{p}, \mathbf{q}) \frac{1}{E - \mathbf{q}^2/m + i\eta} \tilde{V}_C(\mathbf{p}', \mathbf{q}). \quad (4.31)$$

We already saw that the tree level scattering diagram produced precisely the Coulomb potential (c.f. section 3.2). What we see here, is merely the second order iteration of the Coulomb potential, and it arises in just the region we would expect based on our analysis in section 4.1. Later on, in the section on Green functions, we will see how this diagram and all higher order ladder diagrams produce every iteration of the Coulomb potential.

In the second region where $q^0 \sim \mathbf{q} \sim mv$ (soft momentum), we can then replace the fermion propagators with the static approximation

$$I_s \sim \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q-p)^2 + i\eta} \frac{1}{(q-p')^2 + i\eta} \frac{i}{q^0 + i\eta} \frac{i}{-q^0 + i\eta}, \quad (4.32)$$

where it is understood that the poles from the potential contribution are to be omitted, or more formally, they are subtracted off. We are thus left with separate soft and potential additive contributions. This point will be very important when we discuss the method of expansion by regions later on in this chapter.

4.3 The Hamiltonian of pNRQED

For doing non-relativistic physics, the NRQED Lagrangian has a significant advantage over the full QED Lagrangian because it emphasizes the importance of the NR-operators by representing all possible operators as a series in $1/m_e$. It also allows the relevant radiative corrections (from the hard modes) to be incorporated through multiplicative coefficients (Wilson coefficients) which are themselves written as a series expansion in α . In this sense NRQED is a double series expansion in both $1/m_e$ and α . It is however not yet optimal because, as we just saw, photons still have soft degrees of freedom and therefore we cannot

give a proper homogeneous power counting prescription.

Thus, if in addition to the hard modes we also integrate out the effects of the soft momenta, by performing another matching procedure (expanding NRQED diagrams in $E/k, \mathbf{p}/k$), we arrive at what is called potential NRQED (pNRQED) [35]. The culmination of this methodology creates a natural connection between a quantum field theory and its appropriate Schrodinger equation from NRQM. The hard and soft photon fields do not show up in this framework and are, as previously stated, “integrated out”. Their contributions do however show up as additional operators in the Hamiltonian and as radiative corrections to the operators present already in the Hamiltonian of the system. These interactions describe the evolution of the NR electron-positron pair at the level of an instantaneous potential. The corrections are in terms of both the coupling constant α , and electron/positron velocity v , to the leading Coulomb approximation. In addition to potential terms, there are also ultrasoft effects that cannot be described by instantaneous potentials, which will be discussed briefly later on.

The effective Hamiltonian valid to N³LO, can be written [36]

$$\begin{aligned}
H = (2\pi)^3 \delta(\mathbf{q}) & \left(\frac{\mathbf{p}^2}{m} - \frac{\mathbf{p}^4}{4m^3} \right) + C_c(\alpha) V_C(\mathbf{q}) + C_{1/m}(\alpha) V_{1/m}(\mathbf{q}) + \frac{\pi\alpha}{m^2} \\
& \times \left[C_\delta(\alpha) + C_p(\alpha) \frac{\mathbf{p}^2 + \mathbf{p}'^2}{2\mathbf{q}^2} + C_{S^2}(\alpha) \mathbf{S}^2 + C_\lambda(\alpha) \Lambda(\mathbf{p}, \mathbf{q}) + C_t(\alpha) T(\mathbf{q}) \right].
\end{aligned} \tag{4.33}$$

In the above expression, \mathbf{p} and \mathbf{p}' are the spatial momenta of the incoming and outgoing electron/positron respectively, written in centre of mass coordinates.

The operators, in order of appearance are:

$$\begin{aligned}
V_C(\mathbf{q}) &= -\frac{4\pi\alpha}{\mathbf{q}^2}, \quad V_{1/m}(\mathbf{q}) = \frac{2\pi^2\alpha}{m\mathbf{q}}, \quad \mathbf{S} = \frac{\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2}{2}, \\
\Lambda(\mathbf{p}, \mathbf{q}) &= i \frac{\mathbf{S} \cdot (\mathbf{p} \times \mathbf{q})}{\mathbf{q}^2}, \quad T(\mathbf{q}) = \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - 3 \frac{(\mathbf{q} \cdot \boldsymbol{\sigma}_1)(\mathbf{q} \cdot \boldsymbol{\sigma}_2)}{\mathbf{q}^2}.
\end{aligned} \tag{4.34}$$

where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the spin operators of the electron and positron respectively. The $C_i(\alpha)$ are similar to the Wilson coefficients found in the hard matching, but they now have contributions from both the hard *and* soft modes. They too can be written as a power series in α .

We are interested in the situation of the spin-triplet configuration, but we will be performing the calculations of the potential in d -dimensional spacetime. Note that when performing calculations, this prescription does not limit d to take on an integer value and we represent it by $d = 4 - 2\epsilon$, where epsilon is only taken to zero at the end of the calculation. We therefore need to perform a projection onto the spin-triplet configuration, keeping this in mind. In particular we use $\text{Tr}(\sigma_i\sigma_i) = 2(d - 1)$, and we treat the commutator $[\sigma_i, \sigma_j]$ as an irreducible entity. By irreducible we mean that we cannot use the epsilon tensor, and so in this scheme $[\sigma_i, \sigma_j] \neq 2i\epsilon_{ijk}\sigma_k$. If we perform this spin projection, we get the following form for the potential (i.e. the piece neglecting kinetic terms) in the Hamiltonian above

$$\delta V = -\frac{4\pi\alpha}{\mathbf{q}^2} \left[C_C + C_{1/m} \frac{\pi^2 |\mathbf{q}|}{m} + C_{1/m^2} \frac{\mathbf{q}^2}{m^2} + C_p \frac{\mathbf{p}^2 + \mathbf{p}'^2}{2m^2} \right]. \quad (4.35)$$

We have done the calculation and confirmed that the *Born* result for the coefficients, correct to order ϵ^2 , is indeed given by (c.f. [37])

$$C_p^{(0)} = 1, \quad C_{1/m}^{(0)} = 1, \quad C_{1/m^2}^{(0)} = -\frac{4 - \epsilon - 2\epsilon^2}{6 - 4\epsilon}. \quad (4.36)$$

The details of their calculation appear at the beginning of appendix A. In this thesis we calculate the NLO (in α) contribution to the coefficient C_{1/m^2} , and to order ϵ , the necessity of which will become clear later on. We can see from the form that this contribution makes to (4.35), that in position space, the correction becomes a delta function. When perturbation theory is applied to this particular perturbation, we will therefore get something proportional to

the wavefunction at the origin. Indeed, one of the main objectives of this thesis is to provide the correction to the Coulombic wavefunctions at the origin.

Now, since the Lagrangians of these effective theories are only equal to the full Lagrangian in the limit where all terms in the expansions are kept, they will only be representative of a certain piece of the total phase space. This means that some formal cutoffs have to be introduced in order to ensure that only the relevant regions of momentum space are considered in the diagrams that are used in the matching procedure. These cutoffs introduce additional unphysical scales into the problem, which is precisely what we were trying to avoid by using effective theories. The dependence on the scales is certainly *substantially* simpler than the fully relativistic diagrams, but still, the matching procedure between the full and effective theories can get rather involved. By far, the more efficient method to use is based on what is called “expansion by regions”, and we discuss this in the next section.

4.4 Dimensional Regularization and Expansion by Regions

The main results of this thesis produce a potential, valid to one loop order, that includes the effects of the dimensionality of the space-time in which the Feynman diagrams are performed. This is accomplished by systematically performing expansions in the unspecified space-time dimension “ d ”, via an extension of the dimensional regularization (DR) framework. Additionally, the method of expansion by regions (EBR), which we mentioned at the end of the last section, leans very heavily on a systematic application of DR, and as such we must first give a full account of the procedures involved therein, before we can continue on to discuss the methods of EBR.

4.4.1 Dimensional Regularization

In higher order perturbative calculations, it is well known that divergences appear. Before we can get meaningful results, we must eliminate these spurious infinities via a renormalization prescription which we will discuss fully for the on-shell case in the section on renormalization. But, even before a renormalization scheme can be chosen, we must find a way to regulate (parametrize) the infinities. This can be accomplished in a number of ways. One way, still commonly taught in QFT courses, is the Pauli-Villars regularization. In this scheme, one introduces regulators into the definition of the propagators. It is worthwhile considering a brief example of this method as it is with similar techniques, as we mentioned in the previous section, that parts of the NR bound state effects were originally calculated. Additionally, it will facilitate the discussion of Quarkonium in the next chapter when we consider how to piece together the perturbative and lattice approaches. Consider then a one loop correction that appears in massless scalar field theory¹, which has the schematic structure

$$I_{\phi^4} \propto \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + i\eta)((p-k)^2 + i\eta)} \quad (4.37)$$

This integral clearly diverges in the ultraviolet region of the integration variable where $I_{\phi^4} \sim \int \frac{d^4k}{k^4}$, and it does so logarithmically. It also diverges in the infrared region as $k \rightarrow 0$. In this case we parametrize the infinities by altering the scalar propagator

$$\frac{i}{p^2} \rightarrow \frac{i}{p^2 - \lambda^2} - \frac{i}{p^2 - \Lambda^2}, \quad (4.38)$$

which is restored to its original definition in the limit that $\lambda \rightarrow 0$ and $\Lambda \rightarrow \infty$.

¹It is quite common to use scalar field theory for explaining many aspects of regularization and renormalization (see later on), because it avoids the discussion of some of the unnecessary complications of gauge field theories, such as gauge symmetry and spin.

Physically speaking, we can quasi think of this as adding a small positive mass (λ) to the theory to make it infrared finite, and parametrizing some interaction distance $x \sim \frac{1}{\Lambda}$ below which the theory does not accurately describe the physical phenomena. Using the modified propagator we get a result containing a term like $\ln(\Lambda/m)$, and a suitable renormalization can now be performed. It should be noted however that applying this regularization scheme does not preserve gauge invariance in QED, because the photon mass is not zero in the calculation.

A method that *is* gauge-invariant, is that of dimensional regularization. In this method, the integrals are carried out in d -dimensional space-time and the proper theory is restored in the limit that $d \rightarrow 4$. If we write $d = 4 - 2\epsilon$, then the divergences appear as poles in ϵ as we set $\epsilon \rightarrow 0$. The integration measure is changed accordingly to

$$\frac{d^4k}{(2\pi)^4} \rightarrow \frac{d^d k}{(2\pi)^d} = \frac{d\Omega_d}{(2\pi)^d} dk k^{d-1}. \quad (4.39)$$

Vector spaces are designed to have a finite integer dimensionality, or at least a countable infinity of integer dimensions (as in the Hilbert spaces of QM); but a non-integer dimension? It is therefore far from clear that the above prescription will give well defined results. The proofs pertaining to which integration properties hold true in non-integer dimensions and give the correct known results when the limit of integer integer dimensions is restored, can get rather involved. As such, we do not address all these things here, but instead refer the reader to such works as [39]. Now, let us consider the same example integral as we did before in (4.37)

$$\begin{aligned} I_{\phi^4} &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + i\eta)((p-k)^2 + i\eta)} \\ &= \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(d/2 - 1)^2}{\Gamma(d - 2)} \Gamma(2 - d/2) (p^2)^{2-d/2} \\ &= \frac{i}{(4\pi)^2} \left(\frac{1}{\epsilon} - \ln(p^2) - \gamma_E + \ln(4\pi) + 2 \right) \end{aligned} \quad (4.40)$$

where $\gamma_E \approx 0.5772$ is the Euler-Mascheroni constant. Now, one might immediately object that the above has a logarithmic term containing a dimensionful quantity, and rightly so. The reason for this is because we did not account for the fact that the mass dimension of the coupling constant of a QFT actually *depends* on the dimensionality of the space-time. If we wish to use this prescription, then we must introduce a unit of mass into the integration measure $(\nu_0^2)^{2-d/2}$, which simply becomes unity as $\epsilon \rightarrow 0$.

$$\frac{d^4 k}{(2\pi)^4} \rightarrow \frac{(\nu_0^2)^{2-d/2} d^d k}{(2\pi)^d} \quad (4.41)$$

$$I_{\phi^4} \rightarrow \frac{i}{(4\pi)^2} \left(\frac{1}{\epsilon} - \ln(p^2/\nu_0^2) + \dots \right) \quad (4.42)$$

Next, we must address the obvious issue of the Dirac algebra. For starters, we note that now we have the relationship $g_{\mu\nu}g^{\mu\nu} = d$. If we then take the fundamental definition of the Dirac-algebra to still hold true $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, we then have to introduce d such gamma matrices $(\gamma^0, \gamma^1, \dots, \gamma^{d-1})$. With this, the familiar contraction identities from four dimensional space-time get altered to reflect this, and new terms not present before appear. We will need a number of new identities

$$\begin{aligned} \gamma^\mu \gamma_\mu &= d \\ \gamma^\mu \gamma^\nu \gamma_\mu &= -(d-2)\gamma^\nu \\ \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= 4g^{\nu\rho} - (4-d)\gamma^\nu \gamma^\rho \\ \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu &= -2\gamma^\sigma \gamma^\rho \gamma^\nu + (4-d)\gamma^\nu \gamma^\rho \gamma^\sigma \end{aligned} \quad (4.43)$$

all of which can be proved by repeated use of the anti-commutator. With that, we have completed our discussion of DR, which although rather brief, is nonetheless sufficient to move on and discuss EBR.

4.4.2 Expansion by Regions

In the interest of clarity we will introduce expansion by regions by way of a simple “toy” example [40]. This should serve to illustrate the key points involved, and the interested reader is referred to the appendix of this thesis for details regarding its implementation in real-world loop calculations. Consider then the integral

$$F(q, m, \epsilon) = \int_0^\infty \frac{k^{-\epsilon} dk}{(k+m)(k+q)}, \quad (4.44)$$

where $k^{-\epsilon}$ will here serve a purpose very similar to the DR mass scale that we needed to introduce in order to avoid dimensionful logs in the previous section. Now let us give this integral some scaling properties like $0 < m \ll q$. The application of EBR to this integral is of course going to be gross overkill as it can easily be evaluated in its present form

$$F(q, m, \epsilon) = -\frac{\pi}{\sin(\pi\epsilon)} \frac{q^{-\epsilon} - m^{-\epsilon}}{q - m} \rightarrow \frac{\ln(q/m)}{q - m} \quad (\text{as } \epsilon \rightarrow 0). \quad (4.45)$$

But, in order to see the congruity of the method, we need just such an example. Now, start by expanding the integral in a “power series” in m/k

$$F_{k\text{-large}} = \int_0^\infty \frac{k^{-\epsilon} dk}{k(k+q)} + \dots \quad (4.46)$$

Obviously this doesn’t give the correct leading m behaviour because such an expansion is only valid when $|k| > m$, and in carrying out the integral, k will take on all values. Nonetheless, let us stubbornly plow ahead anyways shall we? The whole series gives

$$F_{k\text{-large}} \sim \sum_{n=0}^{\infty} (-1)^n m^n \int_0^\infty \frac{k^{-(n+\epsilon+1)}}{k+q} dk \quad (4.47)$$

$$= \frac{\pi}{q^{1+\epsilon} \sin(\pi\epsilon)} \sum_{n=0}^{\infty} \left(\frac{m}{q}\right)^n \quad (4.48)$$

Let's make another huge mistake¹, and for some reason assume $k \sim m \ll q$ so that we can perform another series expansion, but this time in $1/q$

$$F_{k\text{-small}} \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{q^{n+1}} \int_0^{\infty} \frac{k^{n-\epsilon}}{k+m} dk \quad (4.49)$$

$$= \frac{-\pi}{qm^\epsilon \sin(\pi\epsilon)} \sum_{n=0}^{\infty} \left(\frac{m}{q}\right)^n. \quad (4.50)$$

Now add these two ridiculous answers together

$$\begin{aligned} F_{k\text{-large}} + F_{k\text{-small}} &= \frac{\pi}{q \sin(\pi\epsilon)} \left(\frac{1}{q^\epsilon} - \frac{1}{m^\epsilon} \right) \sum_{n=0}^{\infty} \left(\frac{m}{q}\right)^n \\ &= -\frac{\pi}{\sin(\pi\epsilon)} \frac{q^{-\epsilon} - m^{-\epsilon}}{q - m}. \end{aligned} \quad (4.51)$$

We got the correct result! It seems that by some mathematical mystery, the complete answer is actually given by $F \sim F_{k\text{-large}} + F_{k\text{-small}}$. Okay...but *why* did that work? The whole method was pure nonsense wasn't it? Let's do the calculation more correctly then; in the first integral the expansion failed because it was only valid for $|k| > m$, so let us introduce some scale $m < \Lambda < q$ and make it the lower limit in the first expansion. In the second expansion the series fails once $k > q$, but since $\Lambda < q$, let's just integrate up to the limit Λ . We then have just a simple split integral

$$\begin{aligned} F(q, m, \epsilon) &= \int_0^{\Lambda} \frac{k^{-\epsilon} dk}{(k+m)(k+q)} + \int_{\Lambda}^{\infty} \frac{k^{-\epsilon} dk}{(k+m)(k+q)} \\ &= \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{q^{n+1}} \int_0^{\Lambda} \frac{k^{n-\epsilon}}{k+m} dk + m^n \int_{\Lambda}^{\infty} \frac{k^{-(n+\epsilon+1)}}{k+q} dk \right] \end{aligned} \quad (4.52)$$

We already know that if we extend these limits to the whole range $0 \dots \infty$ that we recover the correct answer, let us therefore define the leftover piece and

¹There is indeed a method to my madness I assure you.

examine how it cancels. We get

$$F_{\text{left}} = \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{q^{n+1}} \int_{\Lambda}^{\infty} \frac{k^{n-\epsilon}}{k+m} dk + m^n \int_0^{\Lambda} \frac{k^{-(n+\epsilon+1)}}{k+q} dk \right], \quad (4.53)$$

but *now*, because of the limits in F_{left} , we can expand the first term in m/k and the second in k/q . It is interesting (and lucky for us) to note then that we get exact cancellation order by order. Consider for instance the sum of the leading terms

$$\begin{aligned} F_{\text{left}}^{(0)} &= \frac{1}{q} \int_0^{\Lambda} \frac{1}{k^{1+\epsilon}} dk + \frac{1}{q} \int_{\Lambda}^{\infty} \frac{1}{k^{1+\epsilon}} dk \\ &= \frac{1}{q \Lambda^{\epsilon}} \left(\frac{1}{\epsilon} - \frac{1}{\epsilon} \right). \end{aligned} \quad (4.54)$$

The crucial thing in performing these calculations is well separated scales. Notice also that to get convergence, we had to analytically continue to the region $\epsilon < 0$ in the first integral, and $\epsilon > 0$ in the second. Well that's it, we can now pass to its implementation in general loop calculations.

Expansion by regions is a technique for the asymptotic expansion of loop integrals with several momentum scales [40], [41]. It systematically expands Feynman diagrams in any limit of momenta and masses, and the general procedure is as follows:

1. Separate the momentum integral into regions where the loop momentum is characterized by one of the scales in the problem.
2. In each region perform a Taylor expansion in the parameters that are considered small there.
3. Extend the integration limits of each region to the whole virtual momentum space and use dimensional regularization for both IR and UV divergences.
4. Set to zero any scaleless integrals.

The sum of the contributions of all the regions then recovers the full result to any order required. The regions to be considered here in our case are those listed at the start of the chapter. When we apply this method to the diagrams of our calculation, they are evaluated in the full theory of QED. For instance, all diagrams with hard contributions are performed by first performing an expansion in q^2/m^2 . In the end, the results from this procedure must exactly coincide with those of the effective field theory approach. Indeed the only difference in their methodology is at what level the expansions are carried out. In the EFT approach, the expansion in relevant parameters is carried out at the level of the Lagrangian, and thus it provides a modified set of Feynman rules when perturbation theory is constructed from it. In the expansion by regions method, the expansion in the same parameters is carried out at the level of the diagrams instead.

Now, since we know that the method of expansion by regions will reproduce the contribution from the hard modes exactly, it is then possible to use the Feynman rules from NRQED to reproduce the soft contribution, with the *caveat* that the integration is carried out over all momentum space and not restricted by any regulators. The beauty of this method is that we get automatic matching, and there is no need for unphysical matching parameters!

4.5 Renormalization

We start our discussion of renormalization with a crucial remark common to any interacting field theory: The parameters that show up in a Lagrangian density such as mass and charge are only what we call their *bare* values. We call them this because there is no reason to suspect that once the electron¹ is actually interacting with the EM field, that its charge and mass *should* remain the same. Indeed, from general arguments of special relativity, we

¹Any electrically charged particle will do of course, but we might as well use an example that is particularly relevant to the overall discussion.

know that the energy from the surrounding EM field *must* change the value of the electron's mass. The charge on the other hand, is altered from the fact that the photons making up the field are constantly creating and destroying electron-positron pairs from the vacuum, which produce a shielding effect. The take away from this is that things like the bare mass m_0 and bare charge e_0 are not even *observable* quantities.

Ok, so what does this actually mean in practical calculations, and why bring it up here? Well, as was mentioned in previous sections, quantum field theories have a habit of popping out infinities once higher order perturbative calculations are considered. We just discussed a couple of methods for regulating these infinities so that we aren't stuck writing our integrals as $I \rightarrow \infty$, but we are still stuck with the problem of what to *do* with them. While one might be a little troubled by the appearance of divergent integrals (and rightfully so), the solution to the problem is in fact the most natural thing in the world. Essentially we *demand* that the things we are evaluating are what we say they are. That is, we demand that the electron's charge and mass take on the values that are measured in experiments.

The last remark is best explained by simply doing it, so let us begin with an example from scalar field theory. The bare mass (m_0) will appear in any Green's function as the pole of the propagator

$$D_F = \frac{i}{p^2 - m_0^2 + i\eta}. \quad (4.55)$$

As we said, quantum corrections will in general make changes to the mass that appears there. There is however a result in QFT which states that the exact two point function is equal to a one particle Feynman propagator with a pole occurring at the value of the particle's *physical* mass, plus a branch cut that includes contributions from multi-particle states². This result is derived

²Also, if any two particle bound states are possible, they will show up as additional isolated poles just below the threshold for pair production (see earlier discussions).

separately for each QFT without even specifying the nature of the interaction; see for instance [28] for a more detailed discussion. In our scalar field example then, we have

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \frac{iZ}{p^2 - m^2 + i\eta} + \text{multi-p states}, \quad (4.56)$$

where Z is the value of the residue of the propagator pole, and is generally called the field strength (or sometimes the wave function) renormalization.

4.5.1 Photon Propagator and Charge Renormalization

We start the discussion by looking into the way in which the photon propagator is modified by higher order calculations. Let us define $i\Pi^{\mu\nu}(q^2)$ to be the sum of all one-particle-irreducible(1PI) diagrams:



$$\mu \text{ wavy } \overset{q}{\text{---}} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \nu = i\Pi^{\mu\nu}$$

Figure 4.2: All 1-particle irreducible insertions into the photon propagator

By one particle irreducible, we mean all diagrams that cannot be separated into two or more sub-diagrams simply by cutting a photon line. Now, the only tensors that can appear in this are $g^{\mu\nu}$ and $q^\mu q^\nu$, thus $\Pi^{\mu\nu}(q^2) = \Pi(q^2)g^{\mu\nu} + \Omega(q^2)q^\mu q^\nu$. Then the Ward identity $q_\mu \Pi^{\mu\nu} = 0$, restricts the coefficients:

$$\begin{aligned} \Pi(q^2)q^\nu + \Omega(q^2)q^2q^\nu = 0 &\Rightarrow \Omega = -\frac{\Pi}{q^2} \\ \Rightarrow \Pi^{\mu\nu}(q^2) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\Pi(q^2). \end{aligned} \quad (4.57)$$

Now by definition, all one particle irreducible insertions exhaust the list of diagrams in such a way that the only possibility remaining for the *full* photon propagator is the following series.

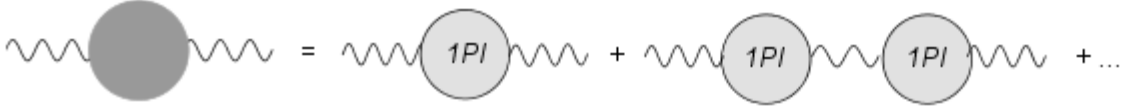


Figure 4.3: The exact photon propagator

The above series of diagrams actually forms a geometric series. To see this, we start by writing out the right hand side of the graphical series (figure 4.3), as

$$\begin{aligned} & \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2} \left(g^{\rho\sigma} - \frac{q_\rho q_\sigma}{q^2} \right) i\Pi(q^2) \frac{-ig_{\sigma\nu}}{q^2} + \dots \\ &= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2} \Delta_\nu^\rho \frac{\Pi(q^2)}{q^2} + \frac{-ig_{\mu\rho}}{q^2} \Delta_\sigma^\rho \Delta_\nu^\sigma \left(\frac{\Pi(q^2)}{q^2} \right)^2, \end{aligned} \quad (4.58)$$

where $\Delta_\nu^\rho \equiv g_\nu^\rho - \frac{q^\rho q_\nu}{q^2}$. Luckily for us, Δ_b^a in the above expression is *idempotent* in the sense that

$$\begin{aligned} \Delta_\sigma^\rho \Delta_\nu^\sigma &= (\delta_\sigma^\rho - q^\rho q_\sigma/q^2) (\delta_\nu^\sigma - q^\sigma q_\nu/q^2) \\ &= \delta_\nu^\rho - q^\rho q_\nu/q^2 \\ &= \Delta_\nu^\rho, \end{aligned} \quad (4.59)$$

because from this we can write (4.58) in a much simpler form

$$\begin{aligned} D_{\mu\nu} &= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2} \Delta_\nu^\rho \left(\Pi(q^2) + [\Pi(q^2)]^2 + \dots \right) \\ &= \frac{-i}{q^2(1 - \Pi(q^2))} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) - \frac{iq_\mu q_\nu}{q^4}. \end{aligned} \quad (4.60)$$

Additionally, we can also omit all terms with $q_\mu q_\nu$, because they will not contribute to any S-matrix element. This is again due to the Ward identity in the form of current conservation. The photon propagator will always be next to the conserved current operator s^μ , whose conservation equation in momentum

space reads $q_\mu s^\mu = 0$. So finally, we have that

$$D_{\mu\nu} = \frac{-ig_{\mu\nu}}{q^2(1 - \Pi(q^2))}. \quad (4.61)$$

Therefore, by the arguments above, we have shown that the photon remains massless to all orders provided that $\Pi(q^2 = 0)$ is regular.

Now we already mentioned that the bare charge, which appears at either end of the photon propagator when attached to a vertex, is not even an observable quantity. The thing is, we have to give an actual definition of the electric charge in terms of physically measurable objects. We could define it in such a way that we recover the charge measured in the limit that a Coulomb potential $V_c = -\frac{\alpha}{r}$ is recovered. However, we can make this argument a little more formal by defining it as the value of the electric charge that is measured at large distances (i.e. where $q \rightarrow 0$).

Now, in practice, we know that the function $\Pi(0)$ is a divergent quantity, and therefore we have no hope of recovering the Coulomb potential like we did in chapter 3. So, following the discussion above, we *define* the physical charge as $e = \sqrt{Z_3}e_0$, where we have multiplied the bare charge by the residue of the propagator

$$Z_3 = \frac{1}{1 - \Pi(0)}. \quad (4.62)$$

This then assures that everything is as it should be,

$$\begin{aligned} \frac{-ig^{\mu\nu}}{q^2} \left(\frac{e_0^2}{1 - \Pi(q^2)} \right) &= \frac{-ig^{\mu\nu}}{q^2} \left(\frac{e^2(1 - \Pi(0))}{1 - \Pi(q^2)} \right) \\ &= \frac{-ig^{\mu\nu}}{q^2} \left(\frac{e^2}{1 - [\Pi(q^2) - \Pi(0)]} \right), \end{aligned} \quad (4.63)$$

where the second equality is true to the order considered. The above procedure is an example of what is called the on-shell renormalization scheme. In general it is defined by requiring that the renormalized propagators have their pole at the physical mass, and additionally that the residue of the pole is one.

One might object at this point that we don't yet know whether other diagrams will produce additional modifications to the on-shell charge. In fact, as we discuss in the next section, the combined effects of the electron propagator renormalization and vertex correction exactly cancel.

4.5.2 The Electron Propagator and Vertex Correction

The two-point function for Dirac fields is

$$\frac{iZ_2(\not{p} + m)}{p^2 - m^2 + i\epsilon} + \dots, \quad (4.64)$$

where again, the omitted terms are representative of the multi-particle contributions. Of course, at leading order the two-point function is just the free field propagator

$$S_F = \frac{i(\not{p} + m_0)}{p^2 - m_0^2 + i\epsilon}, \quad (4.65)$$

so let's see how higher order corrections will alter the appearance of this propagator. For starters we expect that since the leading order expression contains a pole at the bare mass m_0 , the exact function should have a pole at $m = m_0 + \mathcal{O}(\alpha)$. Let us start by defining $-i\Sigma(p)$ to be the sum of all one-particle irreducible (1PI) diagrams which, similar to the case of the photon propagator, is just the sum of all diagrams that cannot be split into two sub-diagrams by cutting somewhere along an electron line.



Figure 4.4: The exact electron propagator

This time, the geometric series gives

$$\begin{aligned} & \frac{i(\not{p} + m_0)}{p^2 - m_0^2} + \frac{i(\not{p} + m_0)}{p^2 - m_0^2} (-i\Sigma) \frac{i(\not{p} + m_0)}{p^2 - m_0^2} + \dots \\ & = \frac{i}{\not{p} - m_0 - \Sigma(p)}, \end{aligned} \quad (4.66)$$

The physical mass is then found from the condition

$$[\not{p} - m_0 - \Sigma(\not{p})]_{\not{p}=m} = 0. \quad (4.67)$$

To get the field-strength renormalization (i.e. the residue of the pole) we perform an expansion around the on-shell point

$$\Sigma(\not{p}) = \Sigma(\not{p}=m) + (\not{p} - m)\Sigma'(\not{p}=m), \quad (4.68)$$

from which it follows that

$$S_F(p) = \frac{iZ_2}{\not{p} - m} \quad (4.69)$$

and

$$Z_2^{-1} = 1 - \Sigma'(\not{p}=m) \quad (4.70)$$

In the appendix we perform the calculation of the NLO correction to the electron propagator and show that it takes the form

$$\Sigma_2 = \frac{\alpha}{4\pi} [A(p^2)m + B(p^2)\not{p}]. \quad (4.71)$$

To the order desired then, we can write the field strength renormalization

$$\begin{aligned} Z_2 &= 1 + \Sigma'(\not{p}=m) \\ &= 1 + \frac{\alpha}{4\pi} \left[B(m^2) + 2m^2 \left(\frac{\partial A}{\partial p^2} + \frac{\partial B}{\partial p^2} \right)_{p^2=m^2} \right]. \end{aligned} \quad (4.72)$$

In the same way as with the photon propagator, we can always absorb the field renormalization into the coupling as $e = Z_2 e_0 = (1 + \delta Z_2) e_0$.

Let us now pass to the discussion of the vertex correction, which has the structure

$$\bar{u}(p')[-ie\Gamma^\mu]u(p) = \bar{u}(p')[-ie(\gamma^\mu + \mathcal{O}(\alpha))]u(p). \quad (4.73)$$

We can use Lorentz invariance and the Ward identity ($q_\mu\Gamma^\mu = 0$) to constrain the form of Γ^μ to be

$$\Gamma^\mu = \gamma^\mu A(q^2) + (p'^\mu + p^\mu)B(q^2). \quad (4.74)$$

Next, we apply the Gordon identity to our expression, which states that

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p+p')^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p). \quad (4.75)$$

This allows us to remove the term proportional to p^μ and p'^μ . With a simple relabelling of the coefficient functions then, we can rewrite the overall vertex

$$\bar{u}(p')\Gamma^\mu u(p) = \bar{u}(p')\left[F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p). \quad (4.76)$$

F_2 is known as the Pauli form-factor and is of order α at leading order; it gives rise to the so-called anomalous magnetic moment. On the other hand, we already know that F_1 (the Dirac form-factor) is just 1 at leading order, so we will find it convenient to define $F_1(q^2) = 1 + \delta F_1(q^2)$.

Let us now follow an argument similar to that which we gave for the photon propagator. We know that if we are to recover the correct low energy behaviour to leading order, we must have

$$\Gamma^\mu(q^2 = 0) = \gamma^\mu. \quad (4.77)$$

The above is another example of an on-shell renormalization condition. This immediately implies that $F_1(q^2=0) = 1$ to all orders. The correction to F_1 was calculated in the appendix, but it certainly did not go to zero at $q^2 = 0$. Worse

still, we have yet to include the effect of the electron self energy correction δZ_2 . In total we actually have

$$\Gamma^\mu(q^2 = 0) \rightarrow (1 + \delta Z_2 + \delta F_1(0))\gamma^\mu. \quad (4.78)$$

Obviously we have a big problem here unless some cancellation occurs. Luckily this is precisely what happens, but it is not miraculous, it is guaranteed to all orders by an application of the Ward-Takahashi identity (for a proof see section 7.4 of [28]). Therefore we should really define the on shell Dirac-Form factor as

$$\tilde{F}_1(q^2) = 1 + \delta Z_2 + \delta F_1(q^2). \quad (4.79)$$

In the next section we will explicitly show that the implied cancellation does occur, and more importantly does so to $\mathcal{O}(\epsilon)$.

4.6 Calculation of C_{1/m^2} at NLO

The contributions to the NLO result arise from higher order diagrams in the perturbative expansion. All the details of their calculation also appear with the Born result in Appendix A, and give the result to order epsilon in the dimensionality. The way in which the graphs contribute to the potential is a little different for each diagram, and we will elucidate that procedure here by pulling all needed results from the appendix. For instance, the propagator corrections as well as the vertex corrections, while presented as individual pieces, are to be included in the full scattering amplitude as insertions into the tree diagram. The box diagrams constitute full results in and of themselves. The full one-loop diagrams relevant for the scattering amplitude are¹:

¹“Permutations”, simply means permuting the leg on which the fermion self energy appears.

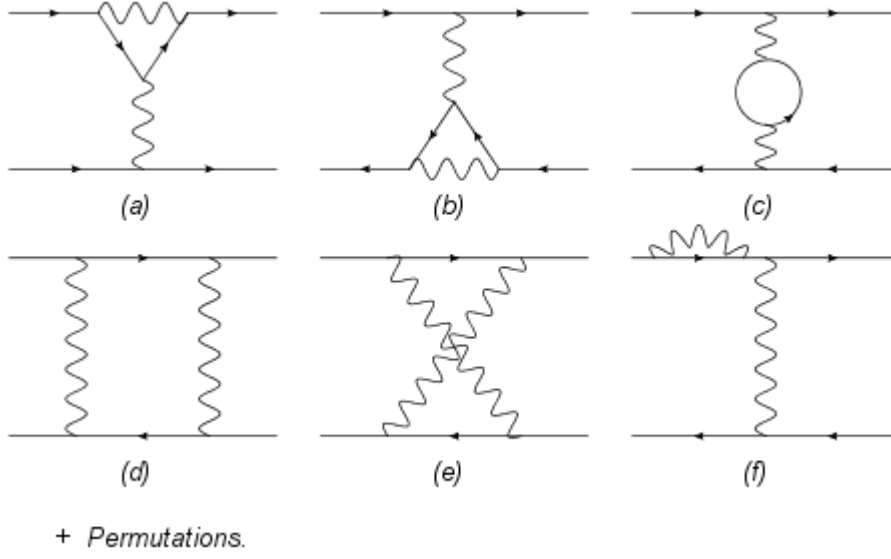


Figure 4.5: One loop scattering diagrams

We organize the series for the Wilson coefficient in the Hamiltonian according to

$$C_{1/m^2} = C_{1/m^2}^{(0)} + \left(\frac{\alpha}{4\pi}\right) C_{1/m^2}^{(1)} + \left(\frac{\alpha}{4\pi}\right)^2 C_{1/m^2}^{(2)} + O(\alpha^3). \quad (4.80)$$

Let us start by determining the contribution that the vertex function makes to the potential. We have the amplitude for diagram (a) as

$$\begin{aligned} M &= \frac{4\pi e^2}{\mathbf{q}^2} \left[\bar{u}(p'_1) \Gamma^\mu u(p_1) \right] \left[\bar{v}(p_2) \gamma_\mu v(p'_2) \right] \\ &= \frac{4\pi e^2}{\mathbf{q}^2} \left[\bar{u}(p'_1) \left(F_1(q^2) \gamma^\mu + F_2(q^2) \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right) u(p_1) \right] \left[\bar{v}(p_2) \gamma_\mu v(p'_2) \right] \end{aligned} \quad (4.81)$$

The second order values for both the Dirac (F_1) and Pauli (F_2) form factors

$$\delta F_1(q^2) = \frac{\alpha}{\pi} \left[\frac{3}{4\epsilon} + 1 - \frac{q^2}{m^2} \left(\frac{1}{6\epsilon} + \frac{1}{8} \right) \right] + \frac{\alpha}{\pi} \left[2 + \frac{\pi^2}{16} - \frac{q^2}{m^2} \left(\frac{1}{2} + \frac{\pi^2}{72} \right) \right] \epsilon, \quad (4.82)$$

$$F_2(q^2) = \frac{\alpha}{2\pi} \left[1 + \frac{1}{6} \frac{q^2}{m^2} + \left(4 + \frac{5}{6} \frac{q^2}{m^2} \right) \epsilon \right]. \quad (4.83)$$

Now, the total on-shell Dirac form factor is defined as

$$\tilde{F}_1 = 1 + \delta Z_2 + \delta F_1, \quad (4.84)$$

where Z_2 is defined by (4.72) and the functions in (A.44), giving

$$\delta Z_2 = -\frac{\alpha}{\pi} \left[\frac{3}{4\epsilon} + 1 + \epsilon \left(2 + \frac{\pi^2}{16} \right) \right] \quad (4.85)$$

which when inserted into the above, ensures that $\tilde{F}_1(q^2=0) = 1$. Notice that this cancellation occurs even to order ϵ as we claimed. We stress that its explicit calculation provides a very important check on our results. Putting it all together we get

$$\tilde{F}_1(q^2) = 1 - \frac{\alpha}{\pi} \left(\frac{q^2}{m^2} \right) \left[\frac{1}{6\epsilon} + \frac{1}{8} + \left(\frac{1}{2} + \frac{\pi^2}{72} \right) \epsilon \right]. \quad (4.86)$$

Now let us recall the leading result (4.35), which upon multiplying by \tilde{F}_1 gives

$$\tilde{F}_1(q^2) \delta V_{1/m^2}^{(0)} = -\frac{4\pi\alpha}{\mathbf{q}^2} \left[-\left(\frac{4 - \epsilon - 2\epsilon^2}{6 - 4\epsilon} \right) \frac{\mathbf{q}^2}{m^2} + \frac{\alpha}{\pi} \left(\frac{\mathbf{q}^2}{m^2} \right) \left(\frac{1}{6\epsilon} + \frac{1}{8} + \left(\frac{1}{2} + \frac{\pi^2}{72} \right) \epsilon \right) \right], \quad (4.87)$$

that is to say that it gives back precisely the tree result, plus a correction term proportional to α . If we fit the contribution into the scheme (4.80), and remember to multiply by two because figure 3.2b gives an identical piece, we get

$$C_{1/m^2, F_1}^{(1)} = \left(\frac{4}{3\epsilon} + 1 + \left(4 + \frac{\pi^2}{9} \right) \epsilon \right). \quad (4.88)$$

Still considering the vertex, we now calculate the contribution coming from the Pauli form factor. We must therefore consider a new type of contribution which did not appear in leading order, i.e.

$$\left[\bar{u}(p'_1) \left(\frac{i\sigma^{\mu\nu} q_\nu}{2m} \right) u(p_1) \right] \left[\bar{v}(p_2) \gamma_\mu v(p'_2) \right]. \quad (4.89)$$

The calculation of the spinor algebra is of significant enough size that we have relegated it to the appendix, and we here only quote the result which consisted of spin dependent and spin independent contributions

$$\begin{aligned}
V_{Pauli} &= -\left(\frac{4\pi\alpha}{\mathbf{q}^2}\right) 2F_2(q^2) \left(\frac{-\mathbf{q}^2}{m^2}\right) \left(\frac{1+\epsilon-2\epsilon^2}{6-4\epsilon} + \frac{1}{4}\right) \\
&= -\left(\frac{4\pi\alpha}{\mathbf{q}^2}\right) \left(\frac{-\mathbf{q}^2}{m^2}\right) \frac{\alpha}{\pi} \left(1+4\epsilon+O(q^2/m^2)\right) \left(\frac{5}{12} + \frac{5}{18}\epsilon + O(\epsilon^2)\right) \\
&= -\left(\frac{4\pi\alpha}{\mathbf{q}^2}\right) \left(\frac{-\mathbf{q}^2}{m^2}\right) \frac{\alpha}{4\pi} \left(\frac{5}{3} + \frac{70}{9}\epsilon\right). \tag{4.90}
\end{aligned}$$

It follows that

$$C_{1/m^2, F_2}^{(1)} = -\left(\frac{5}{3} + \frac{70}{9}\epsilon\right), \tag{4.91}$$

which will be added to $C_{F_1, 1/m^2}^{(1)}$ along with all other contributions at the end.

We now move on to discuss the contribution arising from the photon propagator correction. In the on shell scheme we get the renormalized photon propagator to one-loop order

$$D_{\mu\nu}^R = \frac{-ig_{\mu\nu}}{q^2} \left(\frac{e^2}{1 - [\Pi_2(q^2) - \Pi_2(0)]} \right). \tag{4.92}$$

The value of the denominator can be found from expression (A.33)

$$\begin{aligned}
\Pi_2(q^2) - \Pi_2(0) &= \frac{-2\alpha}{\pi} \int_0^1 dx (1-x)x \ln \left(\frac{m^2}{m^2 - (1-x)xq^2} \right) \\
&\cong \frac{2\alpha}{\pi} \int_0^1 dx (1-x)^2 x^2 \left(\frac{q^2}{m^2} \right) + \mathcal{O}(q^4/m^4) \\
&= -\frac{\alpha}{15\pi} \left(\frac{\mathbf{q}^2}{m^2} \right), \tag{4.93}
\end{aligned}$$

where the second equality follows from expanding in q^2/m^2 to the desired order. We note here that there is no extra ϵ -dependence (i.e. dependence on the dimensionality), as it cancelled in this particular case. The contribution to the coefficient from the photon propagator thus reads

$$C_{1/m^2, \gamma}^{(1)} = \frac{4}{15}. \quad (4.94)$$

The last contribution to the hard region of the calculation comes from the hard boxes. If we add the contributions from the planar (A.128), and crossed box (A.112), we get

$$\begin{aligned} M_{Boxes} &= \frac{\alpha^2}{m^2} (1 - 2\epsilon) (1 + \epsilon - 2\epsilon^2) + \frac{\alpha^2}{m^2} \left(\frac{1}{\epsilon} + \frac{2}{3} + \frac{47}{9}\epsilon + \frac{\pi^2}{12}\epsilon \right) \\ &= \frac{\alpha^2}{m^2} \left(\frac{1}{\epsilon} - \frac{1}{3} + \frac{56}{9}\epsilon + \frac{\pi^2}{12}\epsilon \right). \end{aligned} \quad (4.95)$$

Since there is a $1/4\pi$ in the definition of the first order coefficient already, we just multiply this result by -1 and get

$$C_{1/m^2, hb}^{(1)} = - \left(\frac{1}{\epsilon} - \frac{1}{3} + \frac{56}{9}\epsilon + \frac{\pi^2}{12}\epsilon \right) \quad (4.96)$$

We can now add all the contributions from the hard region $\{C_{1/m^2, i}^{(1)}\}$, to get

$$C_{1/m^2}^{(1)} = \frac{1}{3\epsilon} - \frac{1}{15} + \epsilon \left(-10 + \frac{\pi^2}{36} \right) \quad (4.97)$$

$$= \frac{1}{3\epsilon} - \frac{1}{15} + \epsilon c_{1/m^2}^{(1, \epsilon)} \quad (4.98)$$

This will contribute to the wavefunction at the origin. The soft contribution to the coefficient comes entirely from the two soft boxes and the double vertex graphs. If we add them together, we get

$$C_{\mathbf{q}}^{(1)} = \frac{7}{3\epsilon} - \frac{1}{3} + \epsilon \left(2 - \frac{7}{36}\pi^2 \right) \quad (4.99)$$

$$= \frac{7}{3\epsilon} - \frac{1}{3} + \epsilon c_{\mathbf{q}}^{(1, \epsilon)} \quad (4.100)$$

The above contains an implicit logarithm $\ln \mathbf{q}^2/\mu^2$ with the same coefficient as the pole, where μ is the mass term resulting from the use of dimensional regularization¹. As such, the Fourier transform of the log is no longer a simple delta

function. The remaining divergences in these coefficients will be cancelled by the ultrasoft contribution ($k^0 \sim \mathbf{k} \sim mv^2$), which arises from the electric dipole interaction $H_{us} = \mathbf{E} \cdot \mathbf{r}$, where E and r are the electric field and charge separation respectively. Therein, real dynamical photons are emitted, propagate, and are subsequently reabsorbed by the bound state.

4.7 The method of Green's functions

Let us take as our starting point the Schrodinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = H(t)\Psi. \quad (4.101)$$

We now split the Hamiltonian into two pieces. The first piece is one for which the full solution is known, and the other is a perturbation.

$$H = H_0 + \nu \delta H \quad . \quad (4.102)$$

Ideally in the above, we should have some parameter $\nu \ll 1$, but this is not always the case. We can also expand the wave-function and eigen-energies in a series expansion in the parameter ν ,

$$\psi(x) = \sum_n \nu^n \psi^{(n)}(x), \quad E_n = \sum_m \nu^m E_n^{(m)} \quad (4.103)$$

where the leading terms $\psi^{(0)}(x)$ and $E_n^{(0)}$ are the exact result of the unperturbed system, and corrections are systematically included as will now be shown.

This brings us to the theory of Green's functions. The Green's function is the solution to the Schrodinger equation with a delta function as a source term:

$$(H - E)\psi(x) = 0 \quad \rightarrow \quad (H_x - E)G(x, y) = \delta(x - y) \quad (4.104)$$

¹There is of course a benign $\ln m^2/\mu^2$ term in the hard contribution

The spectral representation of the green's function for a discrete eigen-spectrum is:

$$G(x, y) = \sum_n \frac{\psi_n^*(y)\psi_n(x)}{E_n - E}, \quad (4.105)$$

which has poles in the complex energy plane at $E = E_n$. This can easily be modified to include systems with continuous spectra as well, by taking the continuum limit of the sum into an integral equivalent. Thus it is understood that the sum includes an integral over the scattering states as well, if applicable. The perfect example of this is the free field green's functions that are used to form the propagators of covariant perturbation theory. To prove (4.105), we take it as true and show consistency.

$$(H_x - E)G(x, y) = \sum_n \frac{\psi_n^*(y)(H_x - E)\psi_n(x)}{E_n - E},$$

but the hamiltonian acting on ψ_n will give E_n , thus canceling the denominator and giving

$$\sum_n \psi_n^*(y)\psi_n(x) \quad (4.106)$$

To see that this is equivalent to a delta function $\delta(x - y)$, we multiply by $\int f(y)dy$

$$\sum_n \underbrace{\int dy \psi_n^*(y)f(y)}_{c_n} \psi_n(x) = f(x)$$

$\Rightarrow \sum_n c_n \psi_n(x) = f(x)$, i.e. the answer to integrating a delta function. Thus (4.105) is a representation for $G(x, y)$.

So why is this useful? It becomes useful in perturbation theory when the perturbation is an inhomogeneous term containing ψ . Take G_0 as the Green's function for the unperturbed Hamiltonian.

$$\begin{aligned}
H &= H_0 + \nu\delta H, & (H_0(x) - E)G_0(x, y) &= \delta(x, y) \\
(H(x) - E)G(x, y) &= \delta(x, y) & & (4.107)
\end{aligned}$$

Then we assert that the following is also true:

$$G(x, y) = G_0(x, y) - \nu \int dx' G_0(x, x') \delta H(x') G(x', y), \quad (4.108)$$

which can be seen by multiplying both sides by $(H_y - E)$. The LHS just gives a delta function. For the first term on the RHS, we split H_y as in (4.107), which gives a delta function from the unperturbed piece plus a contribution from the perturbation. Then in the second term, since the integral is over x' , it gives a delta function when operating on $G(x', y)$.

$$\delta(x - y) = \delta(x - y) + \nu\delta H(y)G_0(x, y) - \nu \int dx' G_0(x, x') \delta H(x') \delta(x' - y)$$

$$\delta H(y)G_0(x, y) = \int dx' G_0(x, x') \delta H(x') \delta(x' - y),$$

which is obviously true. Next we expand the green's function in a series in ν :

$$G(x, y) = \sum_{m=0} \nu^m G^{(m)}(x, y) \quad (4.109)$$

Thus $G^{(0)} = G_0$, and G_1 through G_n can be expanded iteratively in the parameter ν , by inserting (4.109) into (4.108) and matching powers of ν :

$$\begin{aligned}
G^{(1)}(x, y) &= - \int dx' G_0(x, x') \delta H(x') G_0(x', y) \\
G^{(2)}(x, y) &= - \int dx' G_0(x, x') \delta H(x') G^{(1)}(x', y) \\
&= + \int dx' G_0(x, x') \delta H(x') \left(\int dx'' G_0(x', x'') \delta H(x'') G_0(x'', y) \right) \\
G^{(3)}(x, y) &= \dots
\end{aligned} \tag{4.110}$$

Now, let us recall the spectral representation

$$G(x, y) = \sum_n \frac{\psi_n^*(y) \psi_n(x)}{E_n - E}, \tag{4.111}$$

and insert a series solution into the above

$$\begin{aligned}
G(x, y) &= \sum_n \frac{(\psi_n^{*0} + \nu \psi_n^{*1} + \dots)_y (\psi_n^0 + \nu \psi_n^1 + \dots)_x}{(E_n^0 + \nu E_n^1 + \dots) - E} \\
&= \sum_n \frac{\psi_n^{*0} \psi_n^0}{E_n^0 - E} + \nu \sum_n \left[\frac{\psi_n^{*0} \psi_n^1 + \psi_n^{*1} \psi_n^0}{E_n^0 - E} - \frac{\psi_n^{*0} \psi_n^0}{(E_n^0 - E)^2} E_n^1 \right] + \dots \\
&= \sum_n G^{(n)}(x, y) \nu^n.
\end{aligned} \tag{4.112}$$

The above is evident by expanding $(E_n^0 + \nu E_n^1 - E)^{-1}$ in a geometric series.

We now set the first order term equal to the first order term in the expansion

(4.110), and get

$$- \int dx' G_0(x, x') \delta H(x') G_0(x', y) = \sum_n \left[\frac{\psi_n^{*0} \psi_n^1 + \psi_n^{*1} \psi_n^0}{E_n^0 - E} - \frac{\psi_n^{*0} \psi_n^0}{(E_n^0 - E)^2} E_n^1 \right]. \tag{4.113}$$

But we can rewrite the LHS in a more interesting form by also inserting the definition of the free Green's function there too

$$G^{(1)} = \int dx' \left(\sum_l \frac{\psi_l^{*0}(x')\psi_l^0(x)}{E_l^0 - E} \right) \delta H \left(\sum_{l'} \frac{\psi_{l'}^{*0}(y)\psi_{l'}^0(x')}{E_{l'}^0 - E} \right). \quad (4.114)$$

If we examine the term with E_n^1 in (4.113), we see that that term will correspond to the residue of the second order pole in (4.114). This occurs when $l = l'$:

$$- \sum_n \frac{\psi_n^{*0}(y)\psi_n^0(x)}{(E_n^0 - E)^2} E_n^1 = \sum_n \int dx' \left(\frac{\psi_n^{*0}(x')\psi_n^0(x)}{E_n^0 - E} \right) \delta H \left(\frac{\psi_n^{*0}(y)\psi_n^0(x')}{E_n^0 - E} \right),$$

from which the familiar result from perturbation theory follows

$$E_n^{(1)} = \int dx' \psi_n^{*0}(x') \delta H(x') \psi_n^0(x'). \quad (4.115)$$

The correction to the wave function from perturbation theory occurs at the first order pole of the integral. This occurs when $n = l$ but $l \neq l'$. Upon adding the two different terms and canceling

$$\psi_n^1(x) = \sum_{l \neq n} \frac{\int dx' \psi_l^0(x') \delta H(x') \psi_n^0(x')}{E_n^0 - E_l^0} \psi_l^0(x) = \sum_{l \neq n} \frac{\langle \psi_l^0 | \delta H | \psi_n^0 \rangle}{E_n^0 - E_l^0} \psi_l^0(x). \quad (4.116)$$

Let us now apply this method to positronium. Consider the time independent Schrodinger equation:

$$\left(\frac{1}{m_e} \nabla^2 + E \right) \psi = -V(\mathbf{r})\psi, \quad (4.117)$$

where we have written it in center of mass coordinates so that $2m_e \rightarrow m_e$, and \mathbf{r} is the separation between the electron and positron. We start by writing the free equation ($V \rightarrow 0$) for the Green's function. Insert both the definition of $G(x, y)$ in momentum space (i.e. its fourier transform) and the definition of

the delta function into (4.104)

$$\begin{aligned}
 \left(\frac{1}{m_e}\nabla_r^2 + E\right)G(r, r') &= \delta(r - r') \\
 \left(\frac{1}{m_e}\nabla_r^2 + E\right)\int\frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{r}}G(k) &= \int\frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{r}} \\
 \int\frac{d^3\mathbf{k}}{(2\pi)^3}\left(E - \frac{1}{m_e}\mathbf{k}^2\right)e^{i\mathbf{k}\cdot\mathbf{r}}G(k) &= \int\frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{r}} \\
 G(k) &= \frac{1}{E - \frac{\mathbf{k}^2}{m_e}}, \tag{4.118}
 \end{aligned}$$

That is $G(r, r') = \int\frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{E - \mathbf{k}^2/m_e}$. Note here that we can also derive the same Green's function from the definitions of the NR propagators by picking up the appropriate pole:

$$\begin{aligned}
 &\int\frac{dk^0 d^3\mathbf{k}}{(2\pi)^4}\frac{1}{k^0 - (E/2 - \mathbf{k}^2/2m_e)}\frac{1}{k^0 + (E/2 - \mathbf{k}^2/2m_e)} \\
 &= \int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{E - \frac{\mathbf{k}^2}{m_e}}. \tag{4.119}
 \end{aligned}$$

Let's see how the perturbation series for the Coulomb Green's function can be derived from the free Green's function solution. If we consider the solution generated by the series shown to 3 terms in (3.9), and we suppress the function variables which quickly become cumbersome to write, we get:

$$G_C = G_0 - \int G_0 V_C G_0 + \int\int G_0 V_C G_0 V_C G_0 + \dots \tag{4.120}$$

In terms of diagrams, we can write G_C as the sum:

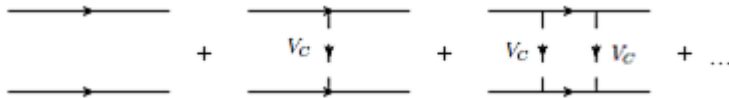


Figure 4.6: Coulomb Green's function as a sum of ladder diagrams

In section 4.1 we showed that all ladder diagrams contribute to the same order in the potential region. At the end of section 4.2, we showed that the second order ladder diagram contained another iteration of the coulomb potential (c.f. (4.31)), and stated without proof that the ladder diagrams produce every iteration of the Coulomb potential. We have now just verified this, and shown how all these diagrams can be formally summed just by using the Coulomb solution as our unperturbed result.

4.8 Positronium Wavefunction

Since we are studying hyperfine splitting in this thesis, we have to make use of the wave function at the origin. Thus, for starters we set ($\mathbf{r} = \mathbf{r}' = 0$) in (4.120), corresponding to the diagrams in figure 4.6 becoming

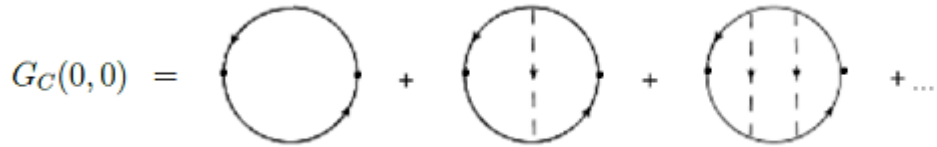


Figure 4.7: Diagrams for Coulomb Green's function at the origin

The Coulomb Green's function is known to have singularities at the origin, which we can regulate within the DR framework. The only two divergent graphs are the first two in the expansion (see figure 4.7). The first diagram gives [42]

$$G_C^{(0)}(0,0) = \int \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}} \frac{m_e}{\mathbf{p}^2 - m_e E} = -\frac{m_e^2}{4\pi} \sqrt{\frac{-E}{m_e}} \quad (4.121)$$

which, by the analytic continuation in the DR scheme, remains finite.

The one photon exchange diagram gives a contribution

$$G_C^{(1)}(0,0) = -\frac{\alpha m_e^2}{8\pi} \left[-\frac{1}{2\epsilon} + \ln\left(\frac{-m_e E}{\mu^2}\right) - 1 + 2 \ln 2 \right]. \quad (4.122)$$

Now, following the discussion at the end of last section, let us consider the situation where the unperturbed Hamiltonian actually contains the Coulomb term. We are interested in treating $V_\delta = C_\delta \delta(\mathbf{r})$ as the perturbation¹

$$G_\delta = G_C - \int G_C V_\delta G_C + \dots, \quad (4.123)$$

from which it immediately follows that:

$$G_\delta(r, r') = G_C(r, r') - C_\delta G_C(r, 0) G_C(0, r') + \dots. \quad (4.124)$$

Therefore we can write $G_\delta(0, 0)$ as

$$G_\delta(0, 0) = G_C(0, 0) - C_\delta G_C(0, 0)^2 + \dots \quad (4.125)$$

If we use the spectral representation for the Green's function (4.105), we can write the above expression as

$$G_\delta = \sum_i \frac{|\psi_{i,C}(0)|^2}{E_{i,C} - E} \left[1 - C_\delta \sum_j \frac{|\psi_{j,C}(0)|^2}{E_{j,C} - E} \right]. \quad (4.126)$$

The sum is over all quantum numbers $\{n, l, m\}$; however, only states with $l = 0$ have a non-vanishing wave function at the origin. Recall that the correction to the wavefunction is found from the residue of the first order pole in the Green function. Thus we find the following expression for the correction to the ground-state Coulomb wavefunction at the origin

$$\delta|\psi_{100}(0)|^2 = -2C_\delta |\psi_{100}(0)|^2 \sum_{n=2} \frac{|\psi_{n00}(0)|^2}{E_{n,C} - E}, \quad (4.127)$$

$$= -2C_\delta |\psi_{100}(0)|^2 \left[G_C(0, 0) - \frac{|\psi_{100}(0)|^2}{E_{1,C} - E} \right], \quad (4.128)$$

and recall that $|\psi_{100}(0)|^2 = \alpha^3 m_e^3 / (8\pi)$. The importance of calculating the

¹ C_δ is just some generic coefficient for the delta function for now. We will relate it to C_{1/m^2} soon.

potential to order ϵ now becomes clear: We are looking to find the finite terms that arise from the interference of the $\mathcal{O}(\epsilon)$ terms with the $\frac{1}{\epsilon}$ poles in the Coulomb Green's function. By examining (4.128), we see that we just take the $\mathcal{O}(\epsilon^{-1})$ term from $G_C^{(1)}$, and multiply by the $\mathcal{O}(\epsilon)$ term arising from (4.35)

$$\begin{aligned} \frac{\delta|\psi_{100}(0)|^2}{|\psi_{100}(0)|^2} &= (-2) \left(\frac{\alpha}{4\pi} \right) \epsilon \left(c_{1/m^2}^{(1,\epsilon)} + \frac{2}{3} c_q^{(1,\epsilon)} + \frac{2}{3} c_p^{(1,\epsilon)} \right) \left(-\frac{4\pi\alpha}{m_e^2} \right) \left(\frac{\alpha m_e^2}{8\pi} \right) \left(\frac{1}{2\epsilon} \right) \\ &= \left(\frac{\alpha^3}{\pi} \right) \left[\frac{c_{1/m^2}^{(1,\epsilon)}}{8} + \frac{c_q^{(1,\epsilon)}}{12} + \frac{c_p^{(1,\epsilon)}}{12} \right]. \end{aligned} \quad (4.129)$$

Note that the derivation of the different numerical coefficients appearing in front of the $c_i^{(1,\epsilon)}$, which are determined by making use of the equations of motion, can be found in [38]. These are of course precisely the coefficients calculated in section 4.6 and Appendix-A, which we repeat here for convenience

$$c_{1/m^2}^{(1,\epsilon)} = \left(-10 + \frac{\pi^2}{36} \right), \quad c_q^{(1,\epsilon)} = \left(2 - \frac{7}{36} \pi^2 \right), \quad c_p^{(1,\epsilon)} = 0. \quad (4.130)$$

4.9 Summary of Results

The results appearing in (4.129) and (4.130) constitute our final results for this chapter, and represent an $\mathcal{O}(\alpha^3)$ correction to the wavefunction at the origin. This was used in the subsequent analysis [1] to calculate the contribution of the one-photon annihilation channel to HFS in positronium at $\mathcal{O}(\alpha^7 m_e)$.

To see the effect of the contribution calculated in this thesis, we recall that at leading order (see section 3.3), the one photon annihilation effect alters the value of the HFS in positronium by

$$\Delta\nu_{ann.}^{LO} = \frac{\alpha^4 m_e}{4}. \quad (4.131)$$

If we now insert (4.130) into (4.129) and multiply by the LO value, we get the

correction

$$\begin{aligned}\Delta\nu^{\delta\psi^{(3)}} &= \Delta\nu_{ann.}^{LO} \left(\frac{\alpha^3}{\pi} \right) \left(-\frac{13}{12} - \frac{11}{864}\pi^2 \right) \\ &= -\frac{\alpha^7 m_e}{48\pi} \left(13 + \frac{11}{72}\pi^2 \right).\end{aligned}\tag{4.132}$$

The perturbative corrections to HFS split into a non-annihilation contribution, a one-photon annihilation contribution, and many photon-annihilation contributions. The second order corrections are heavily dominated by non and one-photon annihilation contributions, which are of the same sign and approximately same magnitude.

We are interested in the one-photon annihilation contribution to the coefficient D (i.e. the $\mathcal{O}(\alpha^3)$ correction to the LO result, see eq. (1.4)) which presumably gives a significant fraction of the third-order correction. In [1], we add the our contribution to the overall $D_{ann}^{1-\gamma}$, which from (4.132) reads

$$D^{\delta\psi^{(3)}} = -\frac{13}{12} - \frac{11}{864}\pi^2.\tag{4.133}$$

The final result for the one-photon annihilation contribution then reads

$$D_{ann}^{1-\gamma} = 84.8 \pm 0.5.\tag{4.134}$$

To summarize, we have calculated the $\mathcal{O}(\alpha^7 m_e)$ one photon annihilation contribution to the positronium HFS, which is the first nontrivial third-order QED result in positronium spectroscopy. This opens the prospect of advancing the theoretical analysis of positronium to a completely different level of precision.

Our final prediction for the positronium HFS including the $\mathcal{O}(\alpha^7 m_e)$ one photon annihilation term reads

$$\Delta\nu^{th} = 203.39191(22)GHZ.\tag{4.135}$$

The error due to the missing part of the $\mathcal{O}(\alpha^7 m_e)$ corrections is given by the size of the evaluated one photon annihilation contribution. The above estimates give only a rough idea of the scale of the missing terms and the calculation of the remaining part of the third-order corrections is mandatory for reducing the theoretical uncertainty significantly below the experimental one.

Chapter 5

Radiative Correction to HFS in Quarkonium

5.1 Matching the perturbative and lattice HFS

We now address the matching of the lattice and perturbative results. The fully perturbative NLO result quoted in [31], is approximately two standard deviations away from the experimentally measured values quoted in the introduction. It is quite reasonable to assume that perturbation theory will give the correct result for the hard modes of the Wilson coefficient. However, we know that due to asymptotic freedom the running coupling is too large in the ultrasoft region where $q \sim \alpha_s^2 m_q$ for perturbation theory to obtain consistent results with any finite order calculation.

The effective pNRQCD Hamiltonian is

$$\begin{aligned} H = & (2\pi)^3 \delta(\mathbf{q}) \left(\frac{\mathbf{p}^2}{m_q} - \frac{\mathbf{p}^4}{4m_q^3} \right) + C_c(\alpha_s) V_C(\mathbf{q}) + C_{1/m}(\alpha_s) V_{1/m}(\mathbf{q}) + \frac{\pi C_F \alpha_s}{m_q^2} \\ & \times \left[C_\delta(\alpha_s) + C_p(\alpha_s) \frac{\mathbf{p}^2 + \mathbf{p}'^2}{2\mathbf{q}^2} + C_{S^2}(\alpha_s) \mathbf{S}^2 + C_\lambda(\alpha_s) \Lambda(\mathbf{p}, \mathbf{q}) + C_c(\alpha_s) T(\mathbf{q}) \right]. \end{aligned} \tag{5.1}$$

The coefficient C_{S^2} will then contain all contributions to the spin-spin coupling (i.e. from both c_F and the four quark operator d_σ in the NRQCD Lagrangian). We define it in such a way that

$$C_{S^2} = \left[\frac{4}{3} + \frac{\alpha_s}{\pi} C_{S^2}^{(1)} + \dots \right]. \quad (5.2)$$

The main idea of our approach is to use continuum QCD only for the hard contribution to the Wilson coefficient C_{S^2} , which is suppressed by powers of the reasonably small expansion parameter $\alpha_s(m_q)$ there. The contribution of momentum $q \sim m_q \alpha_s$ and below, is calculated within the effective NRQCD lattice perturbation theory, by limiting the functional integral to be defined on a finite lattice of spacing $a \gg 1/m_q$. Calculations of this type are quite technically demanding, and as a result they necessitate the use unphysical infrared regulators. Specifically, we use a gluon-mass regulator λ in all gluon propagators.

Now in the end, we want only the contribution of the hard region of momentum $q \sim m_q$ to come from the continuum QCD. However, as pointed out in the previous chapter, we need the infrared behavior to be identical in both calculations in order to perform the matching, which means we must adopt the same infrared regulator λ in the continuum case as used in the effective NRQCD lattice perturbation calculation. This complicates the calculation (see Appendix B) substantially, because we no longer get automatic matching¹ and we must extend our calculation through two different regions of momentum in order to match the theory in a region of overlap.

The dimensionful parameters that remain are the hard scale $\nu_h \sim m_Q$, the soft cutoff $\nu_s \sim 1/a$, where a is the finite lattice spacing that is used in the calculation, and of course the ad hoc gluon mass regulator λ . It is generally true that the condition $\Lambda_{QCD} \ll 1/a \ll m_Q$ is satisfied to ensure that the

¹not to mention we now have an extra scale parameter to deal with in our calculations.

calculations are both valid non-relativistically ($q \ll m_Q$), and capable of being treated in perturbation theory ($q \gg \Lambda_{QCD}$) for the matching region. In order to ensure that we can obtain the Wilson coefficient from the region of virtual momentum of interest $1/a < q < m_q$, we apply a subsidiary condition on the NRQCD lattice cutoff that says $a < a_0^{q\bar{q}}$. In other words, we arrange it so that the lattice cutoff is larger than the reciprocal of the Bohr radius a_0 of quarkonium.

In order to obtain the first-order QCD correction to the Wilson coefficient we must find the contributions of the planar box, the crossed box, and the vertex diagram (both abelian and non-abelian). The details of the calculations are presented in the appendices, and both the planar-box and crossed box have been checked by applying expansion by regions. The results for individual contributions read

$$C_{S^2}^{(1)}|_{p.b.} = C_F \left[-2 + 2 \ln(\lambda/m_Q) \right], \quad (5.3)$$

$$C_{S^2}^{(1)}|_{c.b.} = -2 \left(C_F - \frac{1}{2} C_A \right) \ln(\lambda/m_Q), \quad (5.4)$$

$$C_{S^2}^{(1)}|_{F_2} = \frac{4}{3} \left[C_F + C_A \left(1 + \ln(\lambda/m_Q) \right) \right]. \quad (5.5)$$

In addition, we must also include the effects of two separate two-gluon annihilation box-diagrams, which contribute [43] $C_{S^2}^{(1)}|_{ann.} = 2T_f(1 - \ln(2))$. Summing all these coefficients together we get

$$C_{S^2}^{(1)} = -\frac{2}{3} C_F + \frac{1}{3} C_A \left(4 + 7 \ln(\lambda/m) \right) + 2T_f(1 - \ln(2)). \quad (5.6)$$

The lattice NRQCD result for the first order soft contribution to the Wilson

coefficient can be parameterized as follows

$$C_{S^2}^{(1s)} = A_{lat}^s + \frac{7}{3}C_A \ln(a\lambda), \quad (5.7)$$

where A_{lat}^s is a numerical constant. Since the singular infrared behaviour must be the same in both NRQCD and full QCD, the coefficient of the logarithm of λ is the same as in full QCD. We leave the numerically determined A_{lat}^s as a free parameter in this paper.

To get the hard contribution to the Wilson coefficient consistent with the lattice result, the first-order soft corrections to the spin flip potential computed in perturbative lattice NRQCD, are subtracted from the first-order continuum QCD result for the Wilson coefficient C_{S^2} . Thus the hard contribution to the Wilson coefficient, consistent with the lattice evaluation of the soft contribution, is given by the difference $C_{S^2}^{(1h)} - C_{S^2}^{(1s)}$ and reads

$$C_{S^2}^{(1h)} = \left[-\frac{2}{3}C_F + \frac{4}{3}C_A + 2T_f(1 - \ln(2)) - A_{lat}^s \right] - \frac{7}{3}C_A \ln(m_Q a). \quad (5.8)$$

The above is the main result for this chapter, and it represents a Wilson coefficient whose lower cutoff is now precisely the UV cutoff of the lattice simulations.

5.2 Renormalization Group

QCD is a non-abelian gauge theory. As was mentioned briefly in the previous section, these gauge theories are known to exhibit a property called asymptotic freedom. In order to appreciate the consequences of this, let us first start by going back to our scalar field theory. Let's say that we get an amplitude that looks something like

$$M = -i\lambda + i\lambda^2 B \ln(\Lambda/\mu), \quad (5.9)$$

where Λ is a Pauli-Villars regulator which indicates the relative momentum at

which our knowledge of the theory breaks down. μ is some kinematical invariant for the diagram whose specifics are unimportant, and λ is our coupling. We are then faced with the question of what value to give to Λ and λ in this equation. In other words, we have to give some defining properties to these quantities. One useful way is to define the renormalized coupling λ_R , as the amplitude at some particular value $\mu = \mu_0$,

$$-i\lambda_R(\mu_0) = -i\lambda + i\lambda^2 B \ln(\Lambda/\mu_0). \quad (5.10)$$

In our QED calculation, we used the values at zero momentum transfer and reconciled it with what we observe about free particles. In theories like QCD this is not possible, since a free quark doesn't exist, and we are forced to use some other intermediate value. Now, the above relation is easily inverted to give $-i\lambda = -i\lambda_R - i\lambda_R^2 B \ln(\Lambda/\mu_0)$. Then putting this into our amplitude (5.9), we now have an equation that expresses the amplitude for the process of interest at any energy we want based on measurable quantities only

$$M = -i\lambda_R(\mu_0) + i\lambda_R^2(\mu_0) \ln(\mu_0/\mu). \quad (5.11)$$

Now the issue arises when we wish to measure the process at some momentum μ that is vastly different from our experimental reference point μ_0 ¹. The renormalized coupling constant may still be small, but we can't claim the same thing for the logarithm. This is where the introduction of the renormalization group becomes invaluable. We want to choose the coupling constant that is appropriate for physics at the scale μ so that the logarithm becomes small again. In other words we want it in terms of a new coupling

$$M = -i\lambda_R(\mu'_0) + i\lambda_R^2(\mu'_0) \ln(\mu'_0/\mu). \quad (5.12)$$

¹In what follows, we reason along the same lines as outlined in [44]

We thus want to derive an equation that can tell us how the value of the coupling constant changes as we move to a new scale. A proper treatment of the subject as a whole requires an explanation of Wilsonian renormalization theory and the Callan-Symanzik equation, which is beyond the scope of this discussion. For our simple example however it will be sufficient to use a little differential calculus, and some careful interpretation of the result. Take $\mu'_0 \sim \mu_0 + \delta\mu_0$ for the moment and subtract eq. (5.11) from (5.12) to get:

$$\mu_0 \frac{d\lambda_R}{d\mu_0} = B\lambda_R^2. \quad (5.13)$$

This is known as a renormalization group equation and its solution, which must be supplemented with initial conditions, tells us how the renormalized coupling λ_R changes with the scale of the problem. For the case of QCD, the solution of the renormalization group equation reads [28]

$$\alpha_s(q) = \frac{\alpha_s(\mu_0)}{1 + [\beta_0\alpha_s(\mu_0)/2\pi] \ln(q/\mu_0)}, \quad (5.14)$$

where μ_0 is the reference scale that provided the initial condition and $\beta_0 \equiv \frac{11}{3}C_A - \frac{4}{3}T_F n_l$, is the one loop coefficient of the QCD β -function. Notice that, contrary to QED, this equation predicts that the coupling becomes weaker as we go to higher energies. This is a consequence of the non-abelian nature the SU(3) gauge group and its overall quark content, and is known as asymptotic freedom. It also follows that they are strongly interacting at low energies, where the coupling becomes large, and it is for this reason that perturbation theory becomes less and less effective. At a certain point, usually denoted Λ_{QCD} , perturbation theory breaks down entirely. At this point it is traditional to introduce the point Λ_{QCD} by means of the equation

$$\left(\frac{\beta_0}{2\pi}\right)\alpha_s(\mu_0) \ln(\mu_0/\Lambda_{QCD}) = 1, \quad (5.15)$$

which has been found, from various experiments measuring $\sigma(e^+e^- \rightarrow \text{hadrons})$ [28], to be approximately $\Lambda_{QCD} \sim 200 \text{ MeV}$. This allows the rewriting of (5.14) as:

$$\alpha_s(q) = \frac{2\pi}{\beta_0 \ln(q/\Lambda_{QCD})}. \quad (5.16)$$

Following this discussion, we can now get a more accurate estimation of our previous result (5.8), by including the effects of the leading logs. Leading log (LL) corrections come from inclusion of additional virtual gluon exchanges. We can use the same arguments as used in the above discussion for the effective theory couplings (Wilson coefficients) and, by solving the corresponding effective theory renormalization group equations, sum up the large logarithms in the scale ratio $(\frac{1/a}{m_q})$. In other words, we improve our first order result by resumming, to all-orders, the large logarithms of the lattice spacing a . The leading log result is actually already known from [45]

$$C_{S^2}^{LL} = \alpha_s(\nu_h) \left[1 + \tilde{\beta} \left(z^{-2C_A + \beta_0} - 1 \right) \right], \quad (5.17)$$

where $\tilde{\beta} = \frac{2\beta_0 - 7C_A}{2\beta_0 - 4C_A}$, and

$$z \equiv \left(\frac{\alpha_s(\nu_s)}{\alpha_s(\nu_h)} \right)^{1/\beta_0} = \left(\frac{\ln(1/a)}{\ln(m_q)} \right)^{1/\beta_0}. \quad (5.18)$$

5.3 Summary of Results

We can add the contribution from (5.17) into our previous result, *provided* we remove the one loop log from our original result (5.8) to avoid double counting

$$\begin{aligned} C_{S^2}^{(1h)}(\alpha_s, m_Q, a) = & \left[-\frac{2}{3} C_F + \frac{4}{3} C_A + 2T_f(1 - \ln(2)) - A_{lat}^s \right] \\ & + \frac{4\pi}{3} \left[1 + \tilde{\beta} \left\{ \left(\frac{\alpha_s(1/a)}{\alpha_s(m_Q)} \right)^{1 - \frac{2C_A}{\beta_0}} - 1 \right\} \right]. \end{aligned} \quad (5.19)$$

The above relation constitutes our final (LL improved) result for the Hard coefficient. We can find the correction that it makes to the HFS in Quarkonium as follows

$$\Delta E_{HFS} = \left(\frac{3\alpha_s}{4\pi} \right) C_{S^2}^{(1h)} E_{hfs}^{LO} = C_{S^2}^{(1h)} \left(\frac{C_F^4 \alpha_s^5 m_q}{4\pi} \right). \quad (5.20)$$

This result is used in combination with the contribution from the numerically computed NRQCD lattice result to determine the total hard contribution in [2]. By including the hard contribution and the full lattice result, we find a new theoretical result for $M(\Upsilon) - M(\eta_b)$:

$$E_{hfs}^{th} = 52.9 \pm 5.5 MeV. \quad (5.21)$$

This result strongly favours the value obtained by the Belle collaboration [30]

$$E_{hfs}^{exp} = 57.9 \pm 2.3 MeV. \quad (5.22)$$

The value reported by the Babar collaboration is significantly above Belle's result. The discrepancy could be related to a large systematic uncertainty coming from the subtraction of the resonance background in Babar's analysis (see for instance [46]). Thus, we use the more accurate result from the Belle collaboration, rather than the average of the experimental values, for comparison with our theoretical prediction. The result (5.21) also corrects a significant error in the continuum QCD calculation reported in [3], that was then used in the subsequent analyses [4] and [5], which reported the values

$$E_{hfs}^{th} = 70 \pm 9 MeV, \quad (5.23)$$

$$E_{hfs}^{th} = 62.8 \pm 6.7 MeV. \quad (5.24)$$

Since our publication of [2], there has been an erratum published in [47], wherein the authors report that they now corroborate our correction. With this, we have reconciled the theoretical predictions with the most accurate experimental data, and effectively solved the longstanding η_b -mass puzzle.

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Appendix A

Evaluation of Diagrams for Positronium

A.1 Born Result

In chapter 3, we skimmed over many of the details of the derivation of the Born-result for the potential. In this section we shall elucidate many of these details, with the caveat that we are interested in the d -dimensional potential and thus, as previously stated, will leave the commutator terms as irreducible. We start with expression (3.12),

$$M_{Born} = [\bar{u}_1 \gamma^0 u_1][\bar{u}_2 \gamma^0 u_2] D_{00} + [\bar{u}_1 \gamma^i u_1][\bar{v}_2 \gamma^j v_2] D_{ij} \quad (\text{A.1})$$

Now, in order to evaluate the above, we need an expression for the NR spinors that appear there. This is given by the following expression

$$u = \sqrt{2m} \begin{pmatrix} \left(1 - \frac{\mathbf{p}^2}{8m^2 c^2}\right) w \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2mc} w \end{pmatrix}. \quad (\text{A.2})$$

With this definition, we get that

$$[\bar{u}_{1'}\gamma^0 u_1] = 2m w_1^\dagger \left(1 - \frac{\mathbf{p}_1^2 + \mathbf{p}'_1{}^2}{8m^2 c^2}\right) w'_1 + \frac{1}{2mc^2} w_1^\dagger (\boldsymbol{\sigma} \cdot \mathbf{p}'_1) (\boldsymbol{\sigma} \cdot \mathbf{p}_1) w'_1 \quad (\text{A.3})$$

We write out the second term with indices explicit, for reasons that will be apparent shortly.

$$\begin{aligned} (\boldsymbol{\sigma} \cdot \mathbf{p}'_1) (\boldsymbol{\sigma} \cdot \mathbf{p}_1) &= \sigma^i \sigma^j p_1^i p_1^{j'} \\ &= \frac{1}{2} \left[[\sigma^i, \sigma^j] + \{\sigma^i, \sigma^j\} \right] p_1^i p_1^{j'} \\ &= \frac{1}{2} \left[[\sigma^i, \sigma^j] + 2\delta^{ij} \right] p_1^i p_1^{j'} \\ &= \frac{1}{2} [\sigma^i, \sigma^j] p_1^i p_1^{j'} + \mathbf{p}_1 \cdot \mathbf{p}'_1. \end{aligned} \quad (\text{A.4})$$

If we were working in the standard four dimensional spacetime, we could simply write $[\sigma^i, \sigma^j] = i\epsilon^{ijk}\sigma^k$, which gives us the crossed product that appears in equation (3.21). Since we are working in d -dimensional spacetime, this is not possible, and the commutator is treated as irreducible. We will deal with this commutator term a little later, but for now we notice that when the above equation is inserted into (A.3), we get

$$[\bar{u}_{1'}\gamma^0 u_1] = 2m w_1^\dagger \left(1 - \frac{\mathbf{q}^2}{8m^2 c^2}\right) w'_1 + \frac{1}{4mc^2} w_1^\dagger [\sigma^i, \sigma^j] p_1^i p_1^{j'} w'_1 \quad (\text{A.5})$$

The second term in (A.1) is similar in its treatment.

$$\begin{aligned} [\bar{u}_{1'}\gamma^i u_1] &= (1/c) w_1^\dagger \left[\sigma^i (\boldsymbol{\sigma} \cdot \mathbf{p}_1) + (\boldsymbol{\sigma} \cdot \mathbf{p}'_1) \sigma^i \right] w'_1 \\ &= (1/2c) w_1^\dagger \left(\{\sigma^i, \sigma^j\} (p_1^j + p_1^{j'}) + [\sigma^j, \sigma^i] (p_1^{i'} - p_1^i) \right) w'_1 \\ &= (1/2c) w_1^\dagger \left([\sigma^j, \sigma^i] q^j + 4p_1^i + 2q^i \right) w'_1 \end{aligned} \quad (\text{A.6})$$

Now we will be applying the spin-triplet projection operator to the derived expressions which is accomplished by

$$\frac{\text{Tr}(\sigma^a M_{scatt} \sigma^a)}{\text{Tr}(\sigma^a \sigma^a)} \quad (\text{A.7})$$

Obviously there is no effect on any terms not involving sigma matrices (Pauli matrices), and these terms will be identical to the four dimensional result. The corresponding spinor combination to the right of the ones examined can be easily obtained by changing the labels from $1 \rightarrow 2$ and $\mathbf{q} \rightarrow -\mathbf{q}$. If we look for example at (A.5), we see that when multiplied by the second spinor combination, it gives

$$[\bar{u}_1 \gamma^0 u_1][\bar{u}_2 \gamma^0 u_2] = (2m)^2 w_1^\dagger w_2^\dagger \left(1 - \frac{\mathbf{q}^2}{4m^2 c^2}\right) w' + \frac{p_1^i p_1^{j'} + p_2^i p_2^{j'}}{4mc^2} w^\dagger [\sigma^i, \sigma^j] w'_2 w'_1 \quad (\text{A.8})$$

The second term however, will vanish after performing the projection because of the cyclic property of traces that says $\text{Tr}(ABCD) = \text{Tr}(BCDA) = \text{Tr}(CDBA)$, i.e.

$$\begin{aligned} \text{Tr}(\sigma^a [\sigma^i, \sigma^j] \sigma^a) &= \text{Tr}((\sigma^i \sigma^j - \sigma^j \sigma^i) (\sigma^a)^2) \\ &= (d-1) \text{Tr}(\sigma^i \sigma^j - \sigma^j \sigma^i) \\ &= 0 \end{aligned} \quad (\text{A.9})$$

We now look at the transverse part (A.6), which when multiplied together with its corresponding spinor combination (and ignoring all terms containing only a single commutator) becomes

$$[\bar{u}_1 \gamma^i u_1][\bar{u}_2 \gamma^i u_2] = (1/2c)^2 w_1^\dagger w_2^\dagger \left(4(2\mathbf{p}_1 + \mathbf{q}) \cdot (2\mathbf{p}_2 - \mathbf{q}) + [\sigma^j, \sigma^i][\sigma^i, \sigma^k] q^j q^k\right) w'_2 w'_1 \quad (\text{A.10})$$

We'll start by rewriting the first term, then move on to the second term which will require some massaging to simplify down the projection. Since we're in the center of mass coordinate system $\mathbf{p}_1 = -\mathbf{p}_2$, and thus

$$\begin{aligned}
(2\mathbf{p}_1 + \mathbf{q}) \cdot (2\mathbf{p}_2 - \mathbf{q}) &= -(2\mathbf{p}_1 + \mathbf{q})^2 \\
&= -(\mathbf{p}_1^2 + \mathbf{p}_1'^2 + 2\mathbf{p}_1 \cdot \mathbf{p}_1') \\
&= -(2\mathbf{p}_1^2 + 2\mathbf{p}_1'^2 - \mathbf{q}^2)
\end{aligned} \tag{A.11}$$

Now for the next term, we remind the reader that $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$, is independent of the dimensionality of the spacetime. The full contraction however does change to $\sigma^i \sigma^i = d - 1$. With these points we can easily prove a couple of identities that will be very useful in the evaluation of the second term. For instance, we have that

$$\begin{aligned}
[\sigma^i, \sigma^j] &= \sigma^i \sigma^j - \sigma^j \sigma^i \\
&= 2\sigma^i \sigma^j - 2\delta^{ij}
\end{aligned} \tag{A.12}$$

$$\begin{aligned}
\sigma^i \sigma^a \sigma^i &= -\sigma^i \sigma^i \sigma^a + 2\delta^{ia} \sigma^a \\
&= -(d - 3)\sigma^a
\end{aligned} \tag{A.13}$$

We will also find it convenient to introduce the notation $\sigma^{ij\dots l} \equiv \sigma^i \sigma^j \dots \sigma^l$ for what follows. Now, when we apply the triplet projection operator, we will need to evaluate the trace of

$$\frac{1}{4} \sigma^a [\sigma^j, \sigma^i] \sigma^a [\sigma^i, \sigma^k]. \tag{A.14}$$

We begin by applying the identities proved in (A.12) and (A.13), to the equation (A.14).

$$\begin{aligned}
\frac{1}{4}\sigma^a [\sigma^i, \sigma^j] \sigma^a [\sigma^i, \sigma^k] &= -\sigma^a (\sigma^{ji} - \delta^{ij}) \sigma^a (\sigma^{ik} - \delta^{ik}) \\
&= -\sigma^{ajiaik} + \delta^{ik} \sigma^{ajia} + \sigma^{a^2} \delta^{ij} (\sigma^{ik} - \delta^{ik}) \\
&= (d-3)\sigma^{ajak} + \sigma^{akja} + (d-1)(\sigma^{ik} - \delta^{ik}) \\
&= -(d-3)^2 \sigma^{jk} + (d-1)\sigma^{kj} + (d-1)(\sigma^{ik} - \delta^{ik})
\end{aligned} \tag{A.15}$$

The final line of the above was obtained by using the fact that this expression is contained inside of a trace, which allowed the cyclic property to be used.¹ Finally we need to know that $\text{Tr}[\sigma^i \sigma^j] = 2\delta^{ij}$, from which we see that

$$\begin{aligned}
\text{Tr}\left[\frac{1}{4}\sigma^a [\sigma^i, \sigma^j] \sigma^a [\sigma^i, \sigma^k]\right] &= 2\left[(d-1) - (d-3)^2\right]\delta^{jk} \\
&= 2\left[(3-2\epsilon) - (1-4\epsilon+4\epsilon^2)\right]\delta^{jk} \\
&= 4(1+\epsilon-2\epsilon^2)\delta^{jk}
\end{aligned} \tag{A.16}$$

Putting all this together with the other term gives the results

$$\frac{1}{4m^2} [\bar{u}_{1'} \gamma^0 u_1] [\bar{u}_{2'} \gamma^0 u_2] = w_1^\dagger w_2^\dagger \left(1 - \frac{\mathbf{q}^2}{4m^2 c^2}\right) w_2' w_1' \tag{A.17}$$

$$\frac{1}{4m^2} [\bar{u}_{1'} \gamma^i u_1] [\bar{u}_{2'} \gamma^j u_2] g_{ij} = \frac{\mathbf{p}_{\mathbf{1}}^2}{2m^2 c^2} + \frac{\mathbf{p}_{\mathbf{1}'}^2}{2m^2 c^2} - \frac{\mathbf{q}^2}{m^2 c^2} \left(\frac{1}{4} + \frac{1+\epsilon-2\epsilon^2}{6-4\epsilon}\right) \tag{A.18}$$

This can be written in the form of a potential, where in what follows we adopt the usual convention of setting $c = 1$.

$$V_{Born} = -\frac{4\pi\alpha}{\mathbf{q}^2} \left[1 - \left(\frac{4-\epsilon-2\epsilon^2}{6-4\epsilon}\right) \frac{\mathbf{q}^2}{m^2} + \frac{\mathbf{p}_{\mathbf{1}}^2 + \mathbf{p}_{\mathbf{1}'}^2}{2m^2}\right] \tag{A.19}$$

A.2 Photon Propagator

In this thesis, we are only considering one loop order, and as such there is only one diagram which corrects the photon propagator in this order.

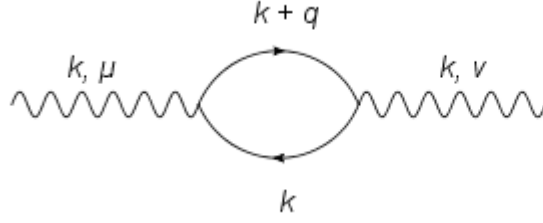


Figure A.1: Photon propagator correction diagram

$$\begin{aligned}
 i\Pi_2^{\mu\nu} &= (-ie)^2 \int \frac{d^d k}{(2\pi)^d} (-1) \text{Tr} \left\{ \frac{i\gamma^\mu (\not{k} + \not{q} + m) i\gamma^\nu (\not{k} + m)}{[(k+q)^2 - m^2][k^2 - m^2]} \right\} \\
 &= -e^2 \int \frac{d^d k}{(2\pi)^d} \text{Tr} \left\{ \frac{\gamma^\mu (\not{k} + \not{q} + m) \gamma^\nu (\not{k} + m)}{[(k+q)^2 - m^2][k^2 - m^2]} \right\}. \tag{A.20}
 \end{aligned}$$

In the above, we use the standard slashed notation that is defined for any four-vector A^μ by $\not{A} = A^\mu \gamma_\mu$. Let's tackle the numerator algebra first.

$$N^{\mu\nu} = \gamma^\mu (\not{k} + \not{q} + m) \gamma^\nu (\not{k} + m) \tag{A.21}$$

Now we can simplify by first noticing that the trace of any odd number of gamma matrices is always zero, i.e.

$$\text{Tr}\{\gamma^\mu \gamma^\alpha \gamma^\nu (k+q)_\alpha m + \gamma^\mu \gamma^\nu \gamma^\beta m k_\beta\} = 0. \tag{A.22}$$

¹note this is generally true as well, but requires the application of the identities (A.12) and (A.13) a few times to prove.

Thus we are left with $N^{\mu\nu} = \gamma^\mu(\not{k} + \not{q})\gamma^\nu\not{k} + \gamma^\mu\gamma^\nu m^2$. We can then use the following identities to simplify the remaining two terms.

$$\begin{aligned}\text{Tr}\{\gamma^\mu\gamma^\nu\} &= 4g^{\mu\nu} \\ \text{Tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}).\end{aligned}\quad (\text{A.23})$$

We note that although we are performing these calculations systematically in d -dimensions, the identities above contain no γ^5 matrix and are thus independent of dimensionality. We are then left with

$$\begin{aligned}N^{\mu\nu} &= 4(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\alpha\nu})(k+q)_\alpha k_\beta + 4g^{\mu\nu}m^2 \\ &= 4[(k+q)^\mu k^\nu + (k+q)^\nu k^\mu - (k \cdot (k+q) - m^2)g^{\mu\nu}].\end{aligned}\quad (\text{A.24})$$

Now let us return to the evaluation of the integral

$$i\Pi_2^{\mu\nu} = -e^2 \int \frac{d^d k}{(2\pi)^d} \frac{N^{\mu\nu}(k)}{[(k+q)^2 - m^2][k^2 - m^2]}.\quad (\text{A.25})$$

We can combine the denominator in a standard Feynman parametrization according to

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[B + (A-B)x]^2}\quad (\text{A.26})$$

Which gives

$$i\Pi_2^{\mu\nu} = -e^2 \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{N^{\mu\nu}}{[k^2 - m^2 + (q^2 + 2k \cdot q)x]^2}.\quad (\text{A.27})$$

We can then perform the change of variables $k \rightarrow l \equiv k + qx$, so that

$l^2 = k^2 + 2k \cdot qx + q^2 x^2$. This of course changes the numerator to

$$\frac{1}{4}N^{\mu\nu}(l) = (l - qx)^\mu (l - qx + q)^\nu + \mu \leftrightarrow \nu - [(l - qx) \cdot (l - qx + q) - m^2]g^{\mu\nu},$$

but terms linear in the loop momentum can be dropped because they do not

contribute to the integral owing to symmetric integration. This means that essentially $\left\{ [(1-x)l^\mu q^\nu - xl^\nu q^\mu] + [\mu \leftrightarrow \nu] - [(1-x)l \cdot q - xl \cdot q]g^{\mu\nu} \right\} \rightarrow 0$. We can also replace all instances of $l^\mu l^\nu \rightarrow \frac{1}{d}l^2 g^{\mu\nu}$, also by symmetry, so that

$$\begin{aligned} \frac{1}{4}N^{\mu\nu}(l) &= 2(l^\mu l^\nu + x(x-1)q^\mu q^\nu) - (l^2 + x(x-1)q^2 - m^2)g^{\mu\nu} \\ &= \frac{1}{d}(2-d)l^2 g^{\mu\nu} - 2x(1-x)q^\mu q^\nu + (m^2 - x(1-x)q^2)g^{\mu\nu} \\ &= \frac{1}{d}(2-d)l^2 g^{\mu\nu} + B^{\mu\nu}(x) \end{aligned} \quad (\text{A.28})$$

where we have defined $B^{\mu\nu}(x)$ as a piece independent of the loop momentum for convenience sake. We also do a similar thing in the denominator, and define $\Delta = x(x-1)q^2 + m^2$, so that

$$\begin{aligned} i\Pi_2^{\mu\nu} &= -4e^2 \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{\frac{1}{d}(2-d)l^2 g^{\mu\nu} + B^{\mu\nu}(x)}{[l^2 - \Delta]^2} \\ &= -4e^2 \int_0^1 dx \left[\frac{-i(1-d/2)g^{\mu\nu} \Gamma(1-d/2)}{(4\pi)^{d/2} \Gamma(2)} \left(\frac{1}{\Delta}\right)^{1-d/2} \right. \\ &\quad \left. + \frac{iB^{\mu\nu} \Gamma(2-d/2)}{(4\pi)^{d/2} \Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-d/2} \right]. \end{aligned} \quad (\text{A.29})$$

The second line was obtained in Appendix B of [28]. The expression can be rewritten in a little nicer form by noting that $(1-d/2)\Gamma(1-d/2) = \Gamma(2-d/2)$, so that on regrouping we have

$$i\Pi_2^{\mu\nu} = 4ie^2 \int_0^1 \frac{dx}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left[\Delta g^{\mu\nu} - B^{\mu\nu}(x) \right]. \quad (\text{A.30})$$

Now inserting the definitions for $B^{\mu\nu}(x)$ and Δ , we get

$$\begin{aligned} i\Pi_2^{\mu\nu} &= 4ie^2 \int_0^1 \frac{dx}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left[-2x(1-x)q^2 g^{\mu\nu} + 2x(1-x)q^\mu q^\nu \right] \\ &= q^2 \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) i\Pi_2(q^2), \end{aligned} \quad (\text{A.31})$$

where

$$\Pi_2(q^2) = \frac{-8e^2}{(4\pi)^{d/2}} \int_0^1 dx(1-x)x \frac{\Gamma(2 - \frac{d}{2})}{\Delta^{1-d/2}}. \quad (\text{A.32})$$

Now let us parameterize the dimensionality according to the relation $d \equiv 4 - 2\epsilon$ and perform an expansion in ϵ . We have

$$\Pi_2(q^2) = \frac{-8e^2}{(4\pi)^2} \int_0^1 dx(1-x)x \left(\frac{1}{\epsilon} - \ln \Delta - \gamma + \ln(4\pi) + \mathcal{O}(\epsilon) \right). \quad (\text{A.33})$$

A.3 Electron Propagator

Recall that we have defined the function $-i\Sigma(p)$ to represent the sum of all 1PI diagrams. From the ensuing geometric series, we got that the exact electron propagator was

$$S_F = \frac{i}{\not{p} - m_0 - \Sigma(p) + i\eta}. \quad (\text{A.34})$$

The one-loop correction Σ_2 , is represented by the following diagram.

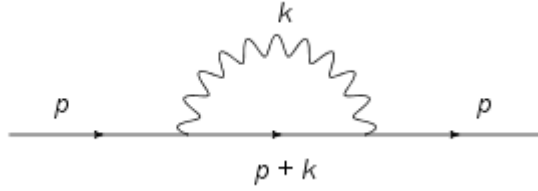


Figure A.2: Electron propagator correction diagram

$$i\Sigma_2(p) = (-ie)^2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{\gamma^\mu i(\not{p} + \not{k} + m) \gamma^\nu}{[(p+k)^2 - m^2]} \right) \left(\frac{ig_{\mu\nu}}{k^2} \right) \quad (\text{A.35})$$

We can then combine the denominator using Feynman parameters by again making use of (A.26)

$$i\Sigma_2(p) = e^2 \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu (\not{p} + \not{k} + m) \gamma_\mu}{[k^2 + 2(p \cdot k)x + (p^2 - m^2)x]^2}. \quad (\text{A.36})$$

Now in d -dimensions, the typical contraction identities for gamma matrices become altered. For our needs, we will make use of the following two identities

$$\begin{aligned} \gamma^\mu \not{A} \gamma_\mu &= (2-d)\not{A} \\ \gamma^\mu \gamma_\mu &= d. \end{aligned} \quad (\text{A.37})$$

With these identities, the amplitude becomes

$$i\Sigma_2(p) = e^2 \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{(2-d)(\not{p} + \not{k}) + dm}{[k^2 + 2(p \cdot k)x + (p^2 - m^2)x]^2}. \quad (\text{A.38})$$

We perform a change of variables $k \rightarrow l \equiv k + px$, which allows us to write $l^2 - p^2 x^2 = k^2 + 2(p \cdot k)x$. If we further define the quantity $\Delta \equiv p^2 x^2 - (p^2 - m^2)x$, we get that

$$i\Sigma_2(p) = e^2 \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{(2-d)(l + (1-x)\not{p}) + dm}{[l^2 - \Delta]^2}. \quad (\text{A.39})$$

By the same reasoning as that used in the photon-propagator correction, we can neglect terms linear in the loop momentum, effectively setting

$$\begin{aligned} i\Sigma_2(p) &\rightarrow e^2 \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{(2-d)(1-x)\not{p} + dm}{[l^2 - \Delta]^2} \\ &= e^2 \int_0^1 dx \left(\frac{(2-d)(1-x)\not{p} + dm}{(4\pi)^{d/2}} \right) \frac{i\Gamma(2-d/2)}{\Delta^{2-d/2}}. \end{aligned} \quad (\text{A.40})$$

It will now prove useful to introduce some new notation. We rewrite $\Sigma_2(p)$ as $i\Sigma_2(p) = \frac{i\alpha}{4\pi} [A(p^2)m + B(p^2)\not{p}]$, with the following definitions

$$\begin{aligned} A(p^2) &\equiv \int_0^1 dx \Gamma(\epsilon)(4-2\epsilon)[xm^2 - x(x-1)p^2]^{-\epsilon} \\ B(p^2) &\equiv - \int_0^1 dx \Gamma(\epsilon)(2-2\epsilon)(1-x)[xm^2 - x(x-1)p^2]^{-\epsilon}. \end{aligned} \quad (\text{A.41})$$

In order to perform the above integrals, we first expand in p^2/m^2

$$[xm^2 - x(x-1)p^2]^{-\epsilon} \sim \left(\frac{1}{x^2 m^2} \right)^\epsilon \left[1 + \frac{\epsilon(1-x)}{x} (p^2/m^2 - 1) \right] + \mathcal{O}(p^4/m^4). \quad (\text{A.42})$$

With this prescription for instance we can carry out the integrals involved in

$A(p^2)$ and $B(p^2)$,

$$\begin{aligned} A(p^2) &= -(2 - \epsilon)\Gamma(\epsilon)\frac{p^2/m^2 - 3}{(m^2/\mu^2)^\epsilon(1 - 2\epsilon)} \\ B(p^2) &= \Gamma(\epsilon)\frac{p^2/m^2 - 2}{(m^2/\mu^2)^\epsilon(1 - 2\epsilon)} \end{aligned} \quad (\text{A.43})$$

We then perform an expansion in ϵ , and keep terms up to $\mathcal{O}(\epsilon)$. In the result below, there should also be terms like $\ln(m^2/\mu^2)$ and $\ln^2(m^2/\mu^2)$, as well as transcendental factors like γ . We omit these in the interest of brevity, as all the terms will cancel in the addition to the vertex, which is shown in section 3.2.2.

$$\begin{aligned} A(p^2) &= \left(3 - \frac{p^2}{m^2}\right)\left(\frac{2}{\epsilon} + 3 + \epsilon\left(6 + \frac{\pi^2}{6}\right)\right) \\ B(p^2) &= -\left(2 - \frac{p^2}{m^2}\right)\left(\frac{1}{\epsilon} + 2 + \epsilon\left(4 + \frac{\pi^2}{12}\right)\right). \end{aligned} \quad (\text{A.44})$$

Finally, the expression for the on shell wave-function renormalization from this diagram is

$$\begin{aligned} \delta Z &= -\frac{\alpha}{4\pi}\left[B(m^2) + 2m^2\left(\frac{\partial A}{\partial p^2} + \frac{\partial B}{\partial p^2}\right)_{p^2=m^2}\right] \\ &= \frac{\alpha}{\pi}\left[\frac{3}{4\epsilon} + 1 + \epsilon\left(2 + \frac{\pi^2}{16}\right)\right], \end{aligned} \quad (\text{A.45})$$

where again the logarithms and transcendental constants are omitted.

A.4 Vertex Function

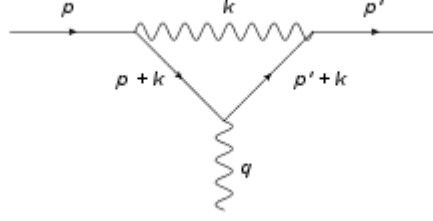


Figure A.3: Vertex correction diagram

In Figure (A.3) above, q is the energy transfer between the electron and positron, and is defined as $q \equiv p' - p$. We define the vertex function by

$$\begin{aligned} iM_V &= -ie \bar{u}(p') \Gamma^\mu u(p) \\ &= -ie \bar{u}(p') [\gamma^\mu + \Gamma_{(2)}^\mu + \dots] u(p) \end{aligned} \quad (\text{A.46})$$

With this definition, we get

$$\begin{aligned} i\Gamma_2^\mu(p', p) &= (-ie)^2 \int \frac{d^d k}{(2\pi)^d} \frac{-ig_{\alpha\beta}}{k^2} \frac{\gamma^\alpha i(\not{p}' + \not{k} + m) \gamma^\beta i(\not{p} + \not{k} + m) \gamma^\sigma}{[(p+k)^2 - m^2][(p'+k)^2 - m^2]} \\ &= e^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\alpha (\not{p}' + \not{k} + m) \gamma^\mu (\not{p} + \not{k} + m) \gamma_\alpha}{k^2 [(p+k)^2 - m^2][(p'+k)^2 - m^2]}. \end{aligned} \quad (\text{A.47})$$

The numerator algebra encountered in the above diagram is considerably more involved than the two seen previously. Indeed it will be a fair bit more complicated than the diagram would be if we set $d = 4$ as is usual. In d -dimensions, we have the following identities which help to simplify the numerator.

1. $\gamma^\alpha \gamma^\mu \gamma_\alpha = -2\gamma^\mu + (4-d)\gamma^\mu$
2. $\gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha = 4g^{\mu\nu} - (4-d)\gamma^\mu \gamma^\nu$
3. $\gamma^\alpha \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\alpha = -2\gamma^\rho \gamma^\nu \gamma^\mu + (4-d)\gamma^\mu \gamma^\nu \gamma^\rho$

Let us make use of them on the numerator

$$N^\mu \equiv \gamma^\alpha (\not{p}' + \not{k} + m) \gamma^\mu (\not{p} + \not{k} + m) \gamma_\alpha \quad (\text{A.48})$$

1. $m^2 \gamma^\alpha \gamma^\mu \gamma_\alpha = -2m^2 \gamma^\mu + (4-d)m^2 \gamma^\mu$
2. $m \gamma^\alpha (\not{p}' + \not{k}) \gamma^\mu \gamma_\alpha + m \gamma^\alpha \gamma^\mu (\not{p} + \not{k}) \gamma_\alpha = 4m(p + p' + 2k)^\mu - (4-d)m [(\not{p}' + \not{k}) \gamma^\mu + \gamma^\mu (\not{p} + \not{k})]$
3. $\gamma^\alpha (\not{p}' + \not{k}) \gamma^\mu (\not{p} + \not{k}) \gamma_\alpha = -2(\not{p} + \not{k}) \gamma^\mu (\not{p}' + \not{k}) + (4-d)(\not{p}' + \not{k}) \gamma^\mu (\not{p} + \not{k})$

Thus in total we have

$$\begin{aligned} N^\mu &= -2m^2 \gamma^\mu + 4m(p + p' + 2k)^\mu - 2(\not{p} + \not{k}) \gamma^\mu (\not{p}' + \not{k}) \\ &\quad + (4-d)(\not{p}' + \not{k} - m) \gamma^\mu (\not{p} + \not{k} - m). \end{aligned} \quad (\text{A.49})$$

We can then apply the equations of motion to the term with coefficient $(4-d)$ since the vertex $i\Gamma^\mu$ will be sandwiched between $\bar{u}(p')$ and $u(p)$. To be explicit, we use $\bar{u}(p')\not{p}' = m\bar{u}(p')$ and $\not{p}u(p) = mu(p)$, to get

$$\begin{aligned} N^\mu &= -2m^2 \gamma^\mu + 4m(p + p' + 2k)^\mu - 2(\not{p} + \not{k}) \gamma^\mu (\not{p}' + \not{k}) \\ &\quad + (4-d)\not{k} \gamma^\mu \not{k}. \end{aligned} \quad (\text{A.50})$$

We can do the same to the 3rd term, but it requires some rearrangement by continuous use of $\gamma^\mu \not{a} = 2a^\mu - \not{a} \gamma^\mu$. That is, if we denote the terms $a = (\not{p} + \not{k})$ and $b = (\not{p}' + \not{k})$ we have that

$$\begin{aligned} \not{a} \gamma^\mu \not{b} &= 2a^\mu \not{b} - \gamma^\mu \not{a} \not{b} \\ &= 2a^\mu \not{b} - 2a \cdot b \gamma^\mu + \gamma^\mu \not{b} \not{a} \\ &= 2a^\mu \not{b} - 2a \cdot b \gamma^\mu + 2b^\mu \not{a} - \not{b} \gamma^\mu \not{a}. \end{aligned} \quad (\text{A.51})$$

When we insert (A.51) into (A.50), we get the following

$$\begin{aligned}
-2(\not{p} + \not{k})\gamma^\mu(\not{p}' + \not{k}) &= -4[(\not{p}' + \not{k})(p + k)^\mu + (\not{p} + \not{k})(p' + k)^\mu - (p + k) \cdot (p' + k)\gamma^\mu] \\
&\quad + 2(\not{p}' + \not{k})\gamma^\mu(\not{p} + \not{k}) \\
&= -4[(\not{k} + m)(p + p' + 2k)^\mu - (p \cdot p' + k^2 + (p + p') \cdot k)\gamma^\mu] \\
&\quad + 2(\not{k} + m)\gamma^\mu(\not{k} + m). \tag{A.52}
\end{aligned}$$

The equations of motion have again been used in the second equality, and we see that the first term proportional to “ m ” in the above, will cancel the second term in (A.50).

$$\begin{aligned}
N^\mu &= -2m^2\gamma^\mu - 4[\not{k}(p + p' + 2k)^\mu - (p \cdot p' + k^2 + (p + p') \cdot k)\gamma^\mu] \\
&\quad 2(\not{k} + m)\gamma^\mu(\not{k} + m) + (4 - d)\not{k} \gamma^\mu \not{k}. \tag{A.53}
\end{aligned}$$

The $-2m^2\gamma^\mu$ term cancels with the first term on the second line and we can combine the last two terms there into $(6 - d)\not{k}\gamma^\mu\not{k} = 2(6 - d)\not{k}k^\mu - (6 - d)k^2\gamma^\mu$, so that finally

$$\begin{aligned}
N^\mu &= [4p \cdot p' + (d - 2)k^2 + 4(p + p') \cdot k]\gamma^\mu \\
&\quad - 4(p + p')^\mu\not{k} - 2(d - 2)\not{k}k^\mu + 4mk^\mu \tag{A.54}
\end{aligned}$$

This completes the simplification of the numerator algebra, however it will be beneficial, for the evaluation of the integrals, to rewrite it in the form

$$N^\mu = A\gamma^\mu + B_\alpha^\mu k^\alpha + C_{\alpha\beta}^\mu k^\alpha k^\beta. \tag{A.55}$$

In (A.55), we must of course make the following identifications

$$\begin{aligned}
A &= 4p \cdot p' = 2(2m^2 - q^2), \\
B_\alpha^\mu &= 4(p + p')_\alpha \gamma^\mu - 4(p + p')^\mu \gamma_\alpha + 4m\delta_\alpha^\mu, \\
C_{\alpha\beta}^\mu &= (d - 2)g_{\alpha\beta} \gamma^\mu - 2(d - 2)\gamma_\alpha \delta_\beta^\mu,
\end{aligned} \tag{A.56}$$

where the second equality in A, follows simply from the definition $q \equiv p' - p$. Next we wish to use a Feynman parameter scheme to simplify the denominator; to this end we will need the identity

$$\frac{1}{A_1 A_2 \dots A_N} = \int_0^1 dx_1 \dots dx_N \delta(\Sigma x_i - 1) \frac{(N - 1)!}{[x_1 A_1 + x_2 A_2 + \dots + x_N A_N]^N}. \tag{A.57}$$

In the denominator we have $k^2 [(p + k)^2 - m^2] [(p' + k)^2 - m^2]$, which upon applying the prescription of (A.57), we get

$$\begin{aligned}
D &= x[(p + k)^2 - m^2] + y[(p' + k)^2 - m^2] + zk^2 \\
&= (x + y + z)k^2 + 2(px + p'y) \cdot k + x(p^2 - m^2) + y(p'^2 - m^2) \\
&= k^2 + 2\Delta \cdot k.
\end{aligned} \tag{A.58}$$

In the last line, we have made use of the delta function that appears in (A.57); we have set the external momenta to be on-shell, which is true up to the order of interest; and we have defined $\Delta \equiv (px + p'y)$.

$$i\Gamma_2^\mu = 2e^2 \int_0^1 dx dy dz \delta(x + y + z - 1) \int \frac{d^d k}{(2\pi)^d} \frac{A\gamma^\mu + B_\alpha^\mu k^\alpha + C_{\alpha\beta}^\mu k^\alpha k^\beta}{[k^2 + 2\Delta \cdot k]^3} \tag{A.59}$$

We can evaluate each of the three different terms in the numerator as separate integrals. Firstly we have that in d -dimensions

$$I_0 = \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2 + 2\Delta \cdot k]^n} = \frac{i(-1)^{d-n} \Gamma(n-d/2)}{(4\pi)^{d/2}} \frac{1}{\Gamma(n)} \frac{1}{(\Delta^2)^{n-d/2}}. \quad (\text{A.60})$$

Then, we have that

$$I_\mu = \int \frac{d^d k}{(2\pi)^d} \frac{k_\mu}{[k^2 + 2\Delta \cdot k]^3} = -\Delta_\mu I_0 \quad (\text{A.61})$$

$$I_{\mu\nu} = \int \frac{d^d k}{(2\pi)^d} \frac{k_\mu k_\nu}{[k^2 + 2\Delta \cdot k]^3} = \left[\Delta_\mu \Delta_\nu + \frac{\frac{1}{2} g_{\mu\nu} (-\Delta^2)}{2-d/2} \right] I_0. \quad (\text{A.62})$$

If we plug in $n = 3$ and $d = 4 - 2\epsilon$ into the above equations, we get

$$I_0 = -\frac{i\Gamma(1+\epsilon)}{2(4\pi)^{2-\epsilon}} \frac{1}{(\Delta^2)^{1+\epsilon}}. \quad (\text{A.63})$$

It therefore follows that

$$i\Gamma_2^\mu = \frac{\alpha}{4\pi} \int_{x,y,z} \frac{-i\Gamma(1+\epsilon)}{(\Delta^2)^{1+\epsilon}} \left[A\gamma^\mu - B_\alpha^\mu \Delta^\alpha + C_{\alpha\beta}^\mu \left(\Delta^\alpha \Delta^\beta - \frac{1}{4-d} g^{\alpha\beta} \Delta^2 \right) \right]. \quad (\text{A.64})$$

Let us first simplify the x -integral using the delta function. We have that

$$\begin{aligned} \Delta^2 &= (px + p'y)^2 = m^2(x^2 + y^2) + 2p \cdot p'(xy) \\ &= m^2(x^2 + y^2) + (2m^2 - q^2)(xy) \\ &= m^2(x+y)^2 - q^2 xy \\ &= m^2(1-z)^2 - q^2(1-y-z)z. \end{aligned} \quad (\text{A.65})$$

We then make a change of variables as defined by $y = \omega\xi$ and $z = 1-\omega$, which of course implies that $x = \omega(1-\xi)$. This effectively sets $\Delta^2 \rightarrow \omega^2 [m^2 - \xi(1-\xi)q^2]$,

and

$$\int_0^1 dx dy dz \delta(x + y + z - 1) \rightarrow \int_0^1 \omega d\omega d\xi. \quad (\text{A.66})$$

It will again be simplest to evaluate each of these integrals as separate terms.

For the $A\gamma^\mu$ term we have

$$\begin{aligned} i\Gamma_A^\mu &= -i\Gamma(1 + \epsilon) \frac{\alpha}{2\pi} \int_0^1 \int_0^1 \frac{\omega d\omega d\xi}{w^{2+2\epsilon}} \frac{(2 - q^2/m^2)\gamma^\mu}{[1 - \xi(1 - \xi)q^2/m^2]^{1+\epsilon}} \\ &= \frac{i\Gamma(1 + \epsilon)}{4\epsilon} \frac{\alpha}{2\pi} \int_0^1 d\xi \frac{(2 - q^2/m^2)\gamma^\mu}{[1 - \xi(1 - \xi)q^2/m^2]^{1+\epsilon}} \\ &= \frac{i\alpha}{2\pi} \left[\frac{1}{\epsilon} \left(1 - \frac{1}{3} \frac{q^2}{m^2} \right) + \frac{1}{6} \frac{q^2}{m^2} + \frac{\pi^2}{12} \left(1 - \frac{1}{3} \frac{q^2}{m^2} \right) \epsilon \right] \gamma^\mu. \end{aligned} \quad (\text{A.67})$$

Evaluating the $B_\alpha^\mu \Delta^\alpha$ will require some simplification first as will the $C_{\alpha\beta}^\mu$. For the former we have

$$\begin{aligned} -B_\alpha^\mu \Delta^\alpha &= \left[-4(p + p')_\alpha \gamma^\mu + 4(p + p')^\mu \gamma_\alpha - 4m\delta_\alpha^\mu \right] [p^\alpha \xi + p'^\alpha (1 - \xi)] \omega \\ &= -4\omega \left[(m^2 + p \cdot p') \gamma^\mu - (p + p')^\mu (\not{p}\xi + \not{p}'(1 - \xi)) + m(p^\mu \xi + p'^\mu (1 - \xi)) \right] \\ &= -4\omega \left[(2m^2 - \frac{1}{2}q^2) \gamma^\mu - m(p + p')^\mu + m(p^\mu \xi + p'^\mu (1 - \xi)) \right], \end{aligned} \quad (\text{A.68})$$

where in the last line we have again used the equations of motion to eliminate the \not{p} and \not{p}' terms in favor of the mass term. We will separate all the contributions into only two distinct terms according to (4.76)

$$\bar{u}(p') \Gamma^\mu u(p) = \bar{u}(p') \left[F_1(q^2) \gamma^\mu + F_2(q^2) \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(p), \quad (\text{A.69})$$

Let us first calculate the correction to the Dirac form factor arising from the second order correction. According to the Gordon identity (4.75), if we are only interested in the contribution to the Dirac form factor, then we can just replace $(p + p')^\mu \rightarrow 2m\gamma^\mu$ in (A.68). At first it seems there is an asymmetry between p and p' in the last term, but in fact by changing $1 - \xi \rightarrow \xi$ in the term

in front of p' , we get that $-B_\alpha^\mu \Delta^\alpha \rightarrow 2\omega(q^2 - 4m^2\xi)\gamma^\mu$. Thus the contribution to the Dirac form factor from the second term is just

$$\begin{aligned} i\Gamma_B^\mu &= -i\Gamma(1+\epsilon)\frac{\alpha}{4\pi}\int_0^1\frac{\omega^2 d\omega}{\omega^{2+2\epsilon}}\int_0^1 d\xi\frac{2(q^2/m^2-4\xi)\gamma^\mu}{[1-\xi(1-\xi)q^2/m^2]^{1+\epsilon}} \\ &= \frac{i\alpha}{2\pi}\left[1-\frac{1}{3}\frac{q^2}{m^2}+\left(4-\frac{q^2}{m^2}\right)\epsilon\right]\gamma^\mu. \end{aligned} \quad (\text{A.70})$$

Finally we pass to the evaluation of the contribution involving $C_{\alpha\beta}^\mu$. The largest term is

$$\begin{aligned} C_{\alpha\beta}^\mu \Delta^\alpha \Delta^\beta &= \omega^2(d-2)[g_{\alpha\beta}\gamma^\mu - 2\gamma_\alpha\delta_\beta^\mu][p^\alpha\xi + p'^\alpha(1-\xi)][p^\beta\xi + p'^\beta(1-\xi)] \\ &= \omega^2(d-2)\left[(m^2 - \xi(1-\xi)q^2)\gamma^\mu - 2(p^\mu\xi + p'^\mu(1-\xi))(\not{p} + \not{p}'(1-\xi))\right] \\ &= (d-2)\Delta^2\gamma^\mu - 2(d-2)m(p^\mu\xi + p'^\mu(1-\xi)). \end{aligned} \quad (\text{A.71})$$

Thus in total we have

$$\begin{aligned} C_{\alpha\beta}^\mu \Delta^\alpha \Delta^\beta - \frac{1}{4-d}C_{\alpha\beta}^\mu g^{\alpha\beta} \Delta^2 &= \left(1 - \frac{d-2}{4-d}\right)(d-2)\Delta^2\gamma^\mu - 2(d-2)m(p^\mu\xi + p'^\mu(1-\xi)) \\ &= (2-1/\epsilon)(2-2\epsilon)\Delta^2\gamma^\mu - 4(2-2\epsilon)m^2\xi\gamma^\mu. \end{aligned} \quad (\text{A.72})$$

If we now insert this into the appropriate integral, we get

$$\begin{aligned} i\Gamma_C^\mu &= -i\Gamma(1+\epsilon)(1-\epsilon)\frac{\alpha}{2\pi}\int_0^1\frac{\omega^3 d\omega}{\omega^{2+2\epsilon}}\int_0^1 d\xi\left\{\frac{(2-1/\epsilon)\gamma^\mu}{[1-\xi(1-\xi)q^2/m^2]^\epsilon} - \frac{4\xi\gamma^\mu}{[1-\xi(1-\xi)q^2/m^2]^{1+\epsilon}}\right\} \\ &= \frac{i\alpha}{2\pi}\left[\frac{1}{2\epsilon} + \frac{1}{4}\frac{q^2}{m^2} + \frac{\pi^2}{24}\epsilon\right]. \end{aligned} \quad (\text{A.73})$$

We can now add all the pieces that contribute to the Dirac form factor and get

that the second order correction reads

$$\delta F_1(q^2) = \frac{\alpha}{\pi} \left[\frac{3}{4\epsilon} + 1 - \frac{q^2}{m^2} \left(\frac{1}{6\epsilon} + \frac{1}{8} \right) \right] + \frac{\alpha}{\pi} \left[2 + \frac{\pi^2}{16} - \frac{q^2}{m^2} \left(\frac{1}{2} - \frac{\pi^2}{72} \right) \right] \epsilon. \quad (\text{A.74})$$

It is a rather simple matter to get the Pauli form factor from the work that has already been presented. For instance, looking again at the Gordon identity (4.75), we see that we just have to replace all instances of $(p+p')^\mu$ with $-i\sigma^{\mu\nu}q_\nu$. If we do this, we obtain the result that

$$F_2(q^2) = \frac{\alpha}{2\pi} \left[1 + \frac{1}{6} \frac{q^2}{m^2} + \left(4 + \frac{5}{6} \frac{q^2}{m^2} \right) \epsilon \right] \quad (\text{A.75})$$

A.5 Pauli Form Factor Contribution

The way in which the Pauli form factor enters is as

$$M_{Pauli} = -e^2 [\bar{u}_{1'} \gamma^\rho u_1] [\bar{u}_2 i\sigma^{\mu\nu} u_{2'}] \left(\frac{q_\nu}{2m} \right) D_{\rho\mu}(q^2) \quad (\text{A.76})$$

We know immediately that $\nu = 0$ contributes nothing because $q_0 = 0$. We also in this case might as well make $q_i \rightarrow -q^i$ to get a standard three vector (i.e. change to contravariant components). It will be simplest to separate this out, as in the Born case, into contributions from scalar and three vector terms. We note for clarity here that we are still using the Feynman gauge

$$\begin{aligned} D_{\mu\rho} &= \frac{4\pi}{q^2} g_{\mu\rho} \\ &= -\frac{4\pi}{\mathbf{q}^2} g_{\mu\rho}. \end{aligned} \quad (\text{A.77})$$

It follows that, in terms of the separation just mentioned, we have

$$M_{Pauli} = -\left(\frac{4\pi\alpha}{\mathbf{q}^2} \right) \left(\frac{q^j}{2m} \right) \left[[\bar{u}_{1'} \gamma^0 u_1] [\bar{u}_2 i\sigma^{0j} u_{2'}] - [\bar{u}_{1'} \gamma^i u_1] [\bar{u}_2 i\sigma^{ij} u_{2'}] \right] \quad (\text{A.78})$$

Now we know that $\sigma^{oi} = i\alpha^i = i\gamma^0\gamma^i$. We can of course use (A.5), but (A.6) will change slightly due to a minus sign, which makes the first term (in natural units)

$$-(1/2m) [\bar{u}_{1'} \gamma^0 u_1] [u_2^\dagger \gamma^j u_{2'}] = \frac{1}{2} w_1^\dagger w_2^\dagger \left(1 - \frac{\mathbf{q}^2}{8m^2} \right) \left([\sigma^k, \sigma^j] (2p_1^k + q^k) + 4p_1^j + 2q^j \right) w_2' w_1' \quad (\text{A.79})$$

This gives only one term of interest to the $1/m^2$ operator, which contributes to the tree-diagram's portion of the potential¹ coming from (A.78) as

$$M_{Pauli}^{(1)} = -\left(\frac{4\pi\alpha}{\mathbf{q}^2}\right)\left(\frac{-\mathbf{q}^2}{4m^2}\right) \quad (\text{A.80})$$

Now for the next term in (A.78). First, just a quick clarification on notation; we note that the reader is likely familiar with the expression $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, from which it is typically written that

$$\sigma^{ij} = \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \quad (\text{A.81})$$

This expression however is actually derived from $[\sigma^i, \sigma^j] = 2i \epsilon^{ijk} \sigma^k$, which of course we can't use. Instead, we must use the unreduced expression

$$\sigma^{ij} = \frac{1}{2i} \begin{pmatrix} [\sigma^i, \sigma^j] & 0 \\ 0 & [\sigma^i, \sigma^j] \end{pmatrix}. \quad (\text{A.82})$$

We can now insert this into (A.78), but we note that this matrix is diagonal and the other set of spinors that multiplies this one will be of order $1/m$ (or $1/c$ in non-natural units); therefore we need only the leading order spinor definition here.

$$[\bar{u}_2 i\sigma^{ij} u_{2'}] = (2m) w_2^\dagger \frac{1}{2}[\sigma^i, \sigma^j] w_2'. \quad (\text{A.83})$$

¹This of course does not include the effect of the actual Pauli form factor yet

It immediately follows that

$$\begin{aligned}
& -\left(\frac{q^j}{2m}\right) [\bar{u}_1 \gamma^i u_1] [\bar{u}_2 i\sigma^{ij} u_2] \\
& = -(4m^2) \left(\frac{q^k q^j}{4m^2}\right) w_1^\dagger w_2^\dagger \frac{1}{4} [\sigma^k, \sigma^i] [\sigma^i, \sigma^j] w_1' w_2' \\
& = (4m^2) \left(\frac{\mathbf{q}^2}{m^2}\right) \frac{1 + \epsilon - 2\epsilon^2}{6 - 4\epsilon}.
\end{aligned} \tag{A.84}$$

More conveniently, for the sake of putting it into the form of a potential later, we write as in (A.80)

$$M_{Pauli}^{(2)} = -\left(\frac{4\pi\alpha}{\mathbf{q}^2}\right) \left(\frac{-\mathbf{q}^2}{m^2}\right) \left(\frac{1 + \epsilon - 2\epsilon^2}{6 - 4\epsilon}\right) \tag{A.85}$$

A.6 Hard Crossed Box

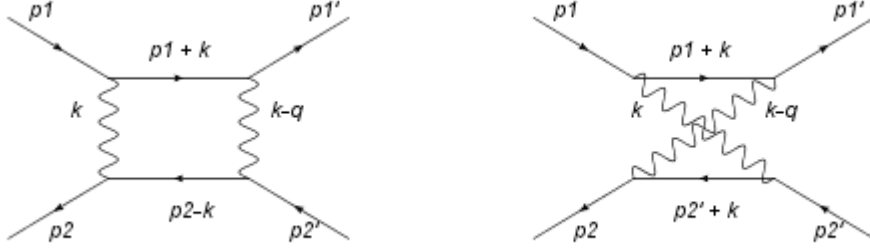


Figure A.4: Crossed Box diagram

$$iM_{cb} = e^4 \int \frac{d^d k}{(2\pi)^d} \frac{[\bar{u}\gamma^\mu(\not{p} - \not{k} + m)\gamma^\nu u][\bar{v}\gamma_\mu(-\not{p} + \not{k} + m)\gamma_\nu v]}{k^4[(p-k)^2 - m^2]^2} \quad (\text{A.86})$$

We begin by sorting out the numerator algebra a little to make the integral easier to handle.

$$N = [\bar{u}\gamma^\mu(\not{p} - \not{k} + m)\gamma^\nu u][\bar{v}\gamma_\mu(-\not{p} + \not{k} + m)\gamma_\nu v]. \quad (\text{A.87})$$

We can first rearrange the terms a little using

$$\begin{aligned} \gamma^\mu(\not{p} + m) &= 2p^\mu - (\not{p} - m)\gamma^\mu, \\ &= 2p^\mu \end{aligned} \quad (\text{A.88})$$

where in the second equality we used the fact that $\bar{u}(\not{p} - m) = 0$. We can perform a similar operation on the piece sandwiched between \bar{v} and v and note that there $\bar{v}(\not{p} + m) = 0$. Upon doing this we get

$$N = -[\bar{u}(2p^\mu - \gamma^\mu \not{k})\gamma^\nu u][\bar{v}(2p_\mu - \gamma_\mu \not{k})\gamma_\nu v]. \quad (\text{A.89})$$

We want to turn this into the form $N = A + B^\mu k_\mu + C^{\mu\nu} k_\mu k_\nu$, from which we

extrapolate

$$\begin{aligned}
A &= -4[\bar{u}\gamma^\nu u][\bar{v}\gamma_\nu v]p^\mu p_\mu \\
&= -4m^2[\bar{u}\gamma^\mu u][\bar{v}\gamma_\mu v] \\
B^\mu k_\mu &= 2[\bar{u}\gamma^\nu u][\bar{v}\not{p}k\gamma_\nu v] + 2[\bar{u}\not{p}k\gamma^\nu u][\bar{v}\gamma_\nu v] \\
&= 2m[\bar{u}\gamma^\nu u][\bar{v}\gamma_\mu\gamma_\nu v]k^\mu + 2m[\bar{u}\gamma^\mu\gamma^\nu u][\bar{v}\gamma_\nu v]k_\mu \\
C^{\mu\nu}k_\mu k_\nu &= -[\bar{u}\gamma^\rho\gamma^\mu\gamma^\sigma u][\bar{v}\gamma_\rho\gamma_\nu\gamma_\sigma v]k_\mu k^\nu.
\end{aligned} \tag{A.90}$$

We will leave it here for now and return to the integral which now looks like

$$iM_{cb} = e^4 \int \frac{d^d k}{(2\pi)^d} \frac{A + B^\mu k_\mu + C^{\mu\nu} k_\mu k_\nu}{k^4 [(p-k)^2 - m^2]^2}. \tag{A.91}$$

We can get a Feynman parametrization from this by using the form already used in (A.26) and just taking the derivative

$$\begin{aligned}
\frac{1}{a^2 b^2} &= \frac{\partial}{\partial a} \frac{\partial}{\partial b} \int_0^1 \frac{dx}{[a + (b-a)x]^2} \\
&= 3! \int_0^1 \frac{x(1-x)dx}{[a + (b-a)x]^4}
\end{aligned} \tag{A.92}$$

Which applied to our integral after defining $\Delta \equiv -xp$, gives

$$iM_{cb} = 6e^4 \int_0^1 x(1-x)dx \int \frac{d^d k}{(2\pi)^d} \frac{A + B^\mu k_\mu + C^{\mu\nu} k_\mu k_\nu}{[k^2 + 2\Delta \cdot k]^4}. \tag{A.93}$$

The integrals can again be evaluated by referring back to the equations from the vertex calculation (A.60) to (A.62). When we do this we find that

$$iM_{cb} = i\alpha^2 \int_0^1 x(1-x)dx \frac{\Gamma(2+\epsilon)}{(\Delta^2)^{2+\epsilon}} \left[A - B^\mu \Delta_\mu + C^{\mu\nu} \Delta_\mu \Delta_\nu - C^{\mu\nu} g_{\mu\nu} \frac{\Delta^2}{1+\epsilon} \right]. \tag{A.94}$$

We return now to finish up with the numerator. We will need only the leading order approximation for the spinors since the integral will already be of order

$1/m^4$. The A -piece is pretty much done therefore, since only the piece with $\mu = 0$ will contribute in leading order, giving

$$A = -4m^2. \quad (\text{A.95})$$

For the B -piece we will need the identity that

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} + \sigma^{\mu\nu}. \quad (\text{A.96})$$

If we insert this into the definition for B in (A.90)

$$\frac{1}{2m} B^\mu k_\mu = [\bar{u} \gamma^\nu u] [\bar{v} (g_{\mu\nu} + \sigma_{\mu\nu}) v] k^\mu + [\bar{u} (g^{\mu\nu} + \sigma^{\mu\nu}) u] [\bar{v} \gamma_\nu v] k_\mu \quad (\text{A.97})$$

We can tell right away that the pieces containing $\sigma^{\mu\nu}$ will not contribute because the gamma matrix sandwiched between the other spinors is forced to have an index of zero, but of course $\sigma^{00} = 0$ and $\sigma^{0i} = \alpha^i$, which does not contribute when only zeroth order spinors are used. thus we are left with

$$\begin{aligned} \frac{1}{2m} B^\mu k_\mu &\sim [\bar{u} \gamma^\nu u] [\bar{v} g_{\mu\nu} v] k^\mu + [\bar{u} g^{\mu\nu} u] [\bar{v} \gamma_\nu v] k_\mu \\ &= [\bar{u} \gamma^\mu u] [\bar{v} v] k_\mu + [\bar{u} u] [\bar{v} \gamma^\mu v] k_\mu. \end{aligned} \quad (\text{A.98})$$

Again we can immediately conclude based on the structure here, that only the term with $\mu = 0$ will contribute, so in equation (A.122) we have $B^\mu \Delta_\mu \rightarrow B^0 \Delta_0 = -4m \Delta_0$.

Finally we pass to the evaluation of the largest piece. It's evaluation will be more useful by separating it into terms that are spin-dependent and spin-independent. First off though, we notice the piece $C^{\mu\nu} \Delta_\mu \Delta_\nu \rightarrow C^{00} \Delta_0^2$, since by definition of Δ the other terms are subleading in α . It will be useful then to see what the 00^{th} term will be. If we neglect explicitly writing the spinors,

we have using direct product notation

$$\begin{aligned}
C^{00} &= [\gamma^\mu \gamma^0 \gamma^\nu] \otimes [\gamma_\mu \gamma^0 \gamma_\nu] \\
&= [(2g^{0\mu} - \gamma^0 \gamma^\mu) \gamma^\nu] \otimes [(2g_{0\mu} - \gamma^0 \gamma_\mu) \gamma_\nu] \\
&= [g^{\mu\nu} + \sigma^{\mu\nu}] \otimes [g_{\mu\nu} + \sigma_{\mu\nu}] \\
&= [\sigma^{\mu\nu}] \otimes [\sigma_{\mu\nu}] + [g^{\mu\nu}] \otimes [g_{\mu\nu}] \\
&= [\sigma^{ij}] \otimes [\sigma_{ij}] + d[1] \otimes [1].
\end{aligned} \tag{A.99}$$

To get the fourth line we used the fact that an antisymmetric tensor fully contracted with a symmetric one will always give zero.

Now let us examine the spin-independent part of $C^{\mu\nu} g_{\mu\nu}$, and again we will neglect explicitly writing the spinors.

$$\begin{aligned}
& - [\gamma^\mu \gamma^\rho \gamma^\nu] \otimes [\gamma_\mu \gamma_\sigma \gamma_\nu] g_\rho^\sigma \\
= & - \frac{1}{4} [2g^{\mu\rho} \gamma^\nu + 2g^{\nu\rho} \gamma^\mu - \gamma^\rho \gamma^\mu \gamma^\nu - \gamma^\mu \gamma^\nu \gamma^\rho] \otimes [2g_{\mu\rho} \gamma_\nu + 2g_{\nu\rho} \gamma_\mu - \gamma_\rho \gamma_\mu \gamma_\nu - \gamma_\mu \gamma_\nu \gamma_\rho].
\end{aligned} \tag{A.100}$$

We might as well just work with one piece at a time. We separate it out as follows

$$\begin{aligned}
& \frac{1}{2} [2g^{\mu\rho} \gamma^\nu + 2g^{\nu\rho} \gamma^\mu - \gamma^\rho \gamma^\mu \gamma^\nu - \gamma^\mu \gamma^\nu \gamma^\rho] \\
= & \frac{1}{2} [2g^{\mu\rho} \gamma^\nu + 2g^{\nu\rho} \gamma^\mu - \gamma^\rho (g^{\mu\nu} + \sigma^{\mu\nu}) - (g^{\mu\nu} + \sigma^{\mu\nu}) \gamma^\rho] \\
= & \frac{1}{2} [2g^{\mu\rho} \gamma^\nu + 2g^{\nu\rho} \gamma^\mu - 2g^{\mu\nu} \gamma^\rho - \gamma^\rho \sigma^{\mu\nu} - \sigma^{\mu\nu} \gamma^\rho].
\end{aligned} \tag{A.101}$$

When all multiplied together again, we see that the first three terms of expression (A.101) form the spin independent part, and the last two terms are

entirely spin dependent. The cross term will vanish since for instance

$$\begin{aligned}
& [g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu - g^{\mu\nu}\gamma^\rho] \otimes [\gamma_\rho\sigma_{\mu\nu} + \sigma_{\mu\nu}\gamma_\rho] \\
&= [\gamma^\nu] \otimes [\gamma^\mu\sigma_{\mu\nu} + \sigma_{\mu\nu}\gamma^\mu] + [\gamma^\mu] \otimes [\gamma^\nu\sigma_{\mu\nu} + \sigma_{\mu\nu}\gamma^\nu] \\
&= [\gamma^\nu] \otimes [\gamma^\mu\sigma_{\mu\nu} + \sigma_{\mu\nu}\gamma^\mu] + [\gamma^\nu] \otimes [\gamma^\mu\sigma_{\nu\mu} + \sigma_{\nu\mu}\gamma^\mu] \\
&= 0
\end{aligned} \tag{A.102}$$

In the third line we just renamed the dummy indices to show that it vanishes since $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$.

We are of course only interested in the spin independent piece right now, which is simply

$$\begin{aligned}
& [g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu - g^{\mu\nu}\gamma^\rho] \otimes [g_{\mu\rho}\gamma_\nu + g_{\nu\rho}\gamma_\mu - g_{\mu\nu}\gamma_\rho] \\
&= 3d[\gamma^\mu] \otimes [\gamma_\mu] + [\gamma^\nu] \otimes [g_{\nu\rho}\gamma_\mu - g_{\mu\nu}\gamma_\rho]g^{\mu\rho} \\
&\quad + [\gamma^\mu] \otimes [g_{\nu\rho}\gamma_\mu - g_{\mu\nu}\gamma_\rho]g^{\nu\rho} - [\gamma^\rho] \otimes [g_{\mu\rho}\gamma_\nu + g_{\nu\rho}\gamma_\mu]g^{\mu\nu} \\
&= (3d - 2)[\gamma^\mu] \otimes [\gamma_\mu].
\end{aligned} \tag{A.103}$$

Now for the spin-dependent piece we have that

$$\begin{aligned}
[\gamma^\mu\gamma^\rho\gamma^\nu] \otimes [\gamma_\mu\gamma_\rho\gamma_\nu] &= [\gamma^\mu\gamma^0\gamma^\nu] \otimes [\gamma_\mu\gamma_0\gamma_\nu] + [\gamma^\mu\gamma^i\gamma^\nu] \otimes [\gamma_\mu\gamma_i\gamma_\nu] \\
&= [\gamma^i\gamma^0\gamma^j] \otimes [\gamma_i\gamma_0\gamma_j] + [\gamma^0\gamma^i\gamma^j] \otimes [\gamma_0\gamma_i\gamma_j] \\
&\quad + [\gamma^i\gamma^j\gamma^0] \otimes [\gamma_i\gamma_j\gamma_0] \\
&= 3[\gamma^i\gamma^j] \otimes [\gamma_i\gamma_j] \\
&= 3[\sigma^{ij}] \otimes [\sigma_{ij}] \\
&= \frac{3}{4}[\sigma^i, \sigma^j] \otimes [\sigma^i, \sigma^j].
\end{aligned} \tag{A.104}$$

In the above equalities, we only retained pieces that contribute to the spin-dependent part, and dropped all others. Note also that in the combination $\gamma^0\gamma^i\gamma^\nu$, the only leading order contribution will be that for which $\nu = j$.

We return to the integral (A.122), which we repeat here for your reading convenience

$$iM_{cb} = i\alpha^2 \int_0^1 x(1-x) dx \frac{\Gamma(2+\epsilon)}{(\Delta^2)^{2+\epsilon}} \left[A - B^\mu \Delta_\mu + C^{\mu\nu} \Delta_\mu \Delta_\nu - C^{\mu\nu} g_{\mu\nu} \frac{\Delta^2}{1+\epsilon} \right]. \quad (\text{A.105})$$

We will have to evaluate the terms one by one again. Starting with the A -integral, we first need that

$$\int_0^1 dx \frac{x(1-x)}{(m^2 x^2)^{2+\epsilon}} = \frac{1}{m^4} \frac{1}{(2+2\epsilon)(1+2\epsilon)}. \quad (\text{A.106})$$

When we plug this into the amplitude with the value of $A = -4m^2$, we get that

$$iM_{cb}^A = \frac{\alpha^2 A}{m^4} \frac{i\Gamma(2+\epsilon)}{(2+2\epsilon)(1+2\epsilon)} = -\frac{i\alpha^2}{m^2} (2-4\epsilon) + O(\epsilon^2). \quad (\text{A.107})$$

Now for the B -integral, we saw that only the $B^0 \Delta_0$ term will contribute, where $\Delta_0 = -mx$. Thus we need that

$$\int_0^1 dx \frac{mx^2(1-x)}{(m^2 x^2)^{2+\epsilon}} = \frac{1}{m^3} \frac{1}{2\epsilon(1+2\epsilon)}, \quad (\text{A.108})$$

from which it follows that

$$iM_{cb}^B = \frac{\alpha^2 B_0}{m^3} \frac{i\Gamma(2+\epsilon)}{2\epsilon(1+2\epsilon)} = -\frac{i\alpha^2}{m^2} \left(\frac{2}{\epsilon} - 2 + 4\epsilon + \frac{\pi^2}{6}\epsilon \right) + O(\epsilon^2). \quad (\text{A.109})$$

Finally for the C -integral, we need that

$$\int_0^1 dx \frac{m^2 x^3(1-x)}{(m^2 x^2)^{2+\epsilon}} = -\frac{1}{m^2} \frac{1}{2\epsilon(1-2\epsilon)}, \quad (\text{A.110})$$

which gives

$$\begin{aligned}
iM_{cb}^C &= -\frac{i\alpha^2}{m^2} \left[\left(\frac{1}{2\epsilon} + \frac{3}{2} + 3\epsilon + \frac{\pi^2}{24}\epsilon \right) C^{00} - \left(\frac{1}{4\epsilon} + \frac{1}{2} + \epsilon + \frac{\pi^2}{48}\epsilon \right) C^{\mu\nu} g_{\mu\nu} \right] \\
&= -\frac{i\alpha^2}{m^2} \left(\frac{1}{2\epsilon} + \frac{3}{2} + 3\epsilon + \frac{\pi^2}{24}\epsilon \right) \left(d + \frac{1}{4} [\sigma^i, \sigma^j] \otimes [\sigma^i, \sigma^j] \right) \\
&\quad + \frac{i\alpha^2}{m^2} \left(\frac{1}{4\epsilon} + \frac{1}{2} + \epsilon + \frac{\pi^2}{48}\epsilon \right) \left((3d - 2) + \frac{3}{4} [\sigma^i, \sigma^j] \otimes [\sigma^i, \sigma^j] \right) + O(\epsilon^2)
\end{aligned} \tag{A.111}$$

The contribution from the commutators is slightly different than it was in (A.16) since now there are no uncontracted indices. We can easily get the required result however by noticing that we can just write $[\sigma^i, \sigma^j] \otimes [\sigma^i, \sigma^j] = [\sigma^i, \sigma^j] \otimes [\sigma^i, \sigma^k] \delta^{jk}$. Upon applying the projection we get an extra factor of $\delta^{jk} \delta^{jk} = 3 - 2\epsilon$. Now adding all the pieces together, and again keeping spin and spin-independent pieces (respectively) separate, we get

$$M_{cb} = -\frac{\alpha^2}{m^2} \left(\frac{1}{4\epsilon} + \frac{\pi^2}{48}\epsilon \right) \left(2 + 2\epsilon - 4\epsilon^2 \right) + \frac{\alpha^2}{m^2} \left(\frac{3}{2\epsilon} - \frac{5}{2} + 10\epsilon + \frac{\pi^2}{8}\epsilon \right). \tag{A.112}$$

A.7 Hard Planar Box

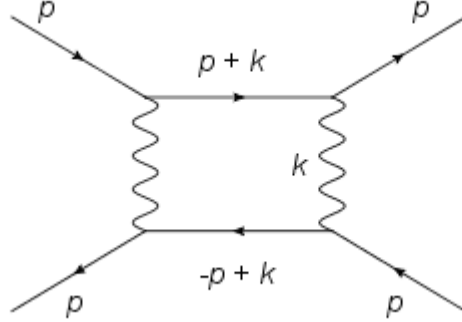


Figure A.5: Planar Box diagram

$$iM_{pb} = e^4 \int \frac{d^d k}{(2\pi)^d} \frac{[\bar{u}\gamma^\mu(\not{p} + \not{k} + m)\gamma^\nu u][\bar{v}\gamma_\nu(-\not{p} + \not{k} + m)\gamma_\mu v]}{k^4[(p+k)^2 - m^2][(p-k)^2 - m^2]} \quad (\text{A.113})$$

The numerator algebra is very similar in the planar box, to the crossed box which has already been discussed. We will go over it schematically with some details, but leave out many intermediate steps whose technique have already been elucidated. To start with we see that the order of the gamma matrices are switched in the second term and that the \not{k} factor in the first term is now with a positive sign. Performing the same initial steps as before, one arrives at the following form

$$N = -[\bar{u}(2\not{p} + \gamma^\mu \not{k})\gamma^\nu u][\bar{v}\gamma_\nu(2\not{p} - \not{k})\gamma_\mu v]. \quad (\text{A.114})$$

We can see right away that the A term will be identical to that of the crossed box. For the B^μ term, we will see that the integral gives zero, and we therefore pursue it no further. Let's take a look therefore at the $C^{\mu\nu}$ term. We have that

$$C^{\mu\nu} k_\mu k_\nu = [\bar{u}\gamma^\mu \not{k}\gamma^\nu u][\bar{v}\gamma_\nu \not{k}\gamma_\mu v] \quad (\text{A.115})$$

It is a simple matter from here to see the difference in this term between the crossed box and the planar box that we are now examining. For instance, we can expand exactly as we did before in (A.101). In the first term we will get exactly the same thing, which we repeat here for your reading convenience

$$\frac{1}{2} [2g^{\mu\rho}\gamma^\nu + 2g^{\nu\rho}\gamma^\mu - 2g^{\mu\nu}\gamma^\rho - \gamma^\rho\sigma^{\mu\nu} - \sigma^{\mu\nu}\gamma^\rho]. \quad (\text{A.116})$$

In the second term, we just have to switch $\mu \leftrightarrow \nu$, giving

$$\frac{1}{2} [2g_{\mu\rho}\gamma_\nu + 2g_{\nu\rho}\gamma_\mu - 2g_{\nu\mu}\gamma_\rho - \gamma_\rho\sigma_{\nu\mu} - \sigma_{\nu\mu}\gamma_\rho]. \quad (\text{A.117})$$

We can therefore immediately conclude that the spin-independent part does not change sign while the spin dependent part does owing to the fact that $\sigma^{\nu\mu} = -\sigma^{\mu\nu}$. Then there was also the minus sign difference in the definition as well so that we have

$$C^{\mu\nu}g_{\mu\nu} = -(3d-2)[\gamma^\mu] \otimes [\gamma_\mu] + \frac{3}{4}[\sigma^i, \sigma^j] \otimes [\sigma^i, \sigma^j]. \quad (\text{A.118})$$

We now pass to the evaluation of the integral which is

$$iM_{pb} = e^4 \int \frac{d^d k}{(2\pi)^d} \frac{A + B^\mu k_\mu + C^{\mu\nu} g_{\mu\nu}}{k^4 [(p+k)^2 - m^2] [(p-k)^2 - m^2]}. \quad (\text{A.119})$$

This can be Feynman parameterized by means of the formula

$$\begin{aligned} \frac{1}{a^2bc} &= -\frac{\partial}{\partial a} \int_0^1 \int_0^x \frac{2dy dx}{[a + (b-a)x + (c-b)y]^3} \\ &= 3! \int_0^1 \int_0^x \frac{(1-x)dy dx}{[a + (b-a)x + (c-b)y]^4}, \end{aligned} \quad (\text{A.120})$$

which leads to

$$iM_{pb} = e^4 \int \frac{d^d k}{(2\pi)^d} \frac{A + B^\mu k_\mu + C^{\mu\nu} g_{\mu\nu}}{[k^2 + 2(x-2y)p \cdot k]}. \quad (\text{A.121})$$

The evaluation of the momentum integral is identical to the one performed in the crossed box case, and gives

$$iM_{pb} = i\alpha^2 \int_0^1 \int_0^x (1-x) dy dx \frac{\Gamma(2+\epsilon)}{(\Delta^2)^{2+\epsilon}} \left[A - B^\mu \Delta_\mu + C^{\mu\nu} \Delta_\mu \Delta_\nu - C^{\mu\nu} g_{\mu\nu} \frac{\Delta^2}{1+\epsilon} \right]. \quad (\text{A.122})$$

With the caveat that Δ is now defined by $\Delta \equiv (x-2y)p$. These can again be evaluated one at a time, starting with the A integrals

$$\begin{aligned} & \int_0^1 \int_0^x \frac{(1-x) dy dx}{[m^2(x-2y)^2]^{2+\epsilon}} \\ &= -\frac{1}{m^4} \int_0^1 \frac{(1-x) dx}{(3+2\epsilon)x^{3+2\epsilon}} \\ &= \frac{1}{m^4(3+2\epsilon)(2+2\epsilon)(1+2\epsilon)}. \end{aligned} \quad (\text{A.123})$$

Thus in total we have

$$\begin{aligned} iM_{pb}^A &= \frac{i\alpha^2}{m^4} \frac{\Gamma(2+\epsilon) A}{(3+2\epsilon)(2+2\epsilon)(1+2\epsilon)} \\ &= -\frac{i\alpha^2}{m^2} \left(-\frac{2}{3} + \frac{16}{9}\epsilon \right) + O(\epsilon^2). \end{aligned} \quad (\text{A.124})$$

For the B -integral, we have simply

$$\begin{aligned} iM_{pb}^B &= -\frac{i\alpha^2}{m^2} \Gamma(2+\epsilon) \int_0^1 \int_0^x \frac{(1-x) dy dx}{(x-2y)^{4+2\epsilon}} B^0(x-2y) \\ &= 0. \end{aligned} \quad (\text{A.125})$$

Finally for the C -integral we have

$$\begin{aligned}
iM_{pb}^C &= \frac{i\alpha^2}{m^2} \Gamma(2 + \epsilon) \int_0^1 \int_0^x \frac{(1-x)dy dx}{(x-2y)^{4+2\epsilon}} \left(C^{00} m^2 (x-2y)^2 - C^{\mu\nu} g_{\mu\nu} \frac{m^2 (x-2y)^2}{1+\epsilon} \right) \\
&= \frac{-i\alpha^2}{m^2} \Gamma(2 + \epsilon) \int_0^1 \int_0^x \frac{x(1-x) dx}{(1+2\epsilon)x^{2+2\epsilon}} \left(C^{00} - \frac{C^{\mu\nu} g_{\mu\nu}}{1+\epsilon} \right) \\
&= \frac{i\alpha^2}{m^2} \frac{\Gamma(2 + \epsilon)}{2\epsilon(1+2\epsilon)(1-2\epsilon)} \left(C^{00} - \frac{C^{\mu\nu} g_{\mu\nu}}{1+\epsilon} \right). \tag{A.126}
\end{aligned}$$

If we now insert the definitions for the constants and expand out to $O(\epsilon)$ as usual, we obtain the result

$$\begin{aligned}
iM_{pb}^C &= \frac{i\alpha^2}{m^2} \left(\frac{1}{2\epsilon} + \frac{1}{2} + 2\epsilon + \frac{\pi^2}{24}\epsilon \right) \left(-d + \frac{1}{4} [\sigma^i, \sigma^j] \otimes [\sigma^i, \sigma^j] \right) \\
&\quad - \frac{i\alpha^2}{m^2} \left(\frac{1}{4\epsilon} + \epsilon + \frac{\pi^2}{48}\epsilon \right) \left(-(3d-2) + \frac{3}{4} [\sigma^i, \sigma^j] \otimes [\sigma^i, \sigma^j] \right) + O(\epsilon^2) \tag{A.127}
\end{aligned}$$

We now just add up all the pieces and arrange them as spin and spin-independent pieces respectively, which reads

$$M_{pb} = \frac{\alpha^2}{m^2} \left(\frac{1}{4\epsilon} - \frac{1}{2} + \epsilon + \frac{\pi^2}{48}\epsilon \right) \left(2 + 2\epsilon - 4\epsilon^2 \right) + \frac{\alpha^2}{m^2} \left(-\frac{1}{2\epsilon} + \frac{19}{6} - \frac{43}{9}\epsilon - \frac{\pi^2}{24}\epsilon \right). \tag{A.128}$$

A.8 Soft Boxes

We will be doing the soft boxes by using the Feynman rules for NRQED as discussed earlier. The rules for full QED are well known, and can be found in any text on QFT. The rules for NRQED are not so well known, so to start with, we must elucidate what these rules are. We choose to work in the Feynman gauge in which the photon propagator is, in line with the notation below, $D_{\mu\nu}(q^2) = \frac{g_{\mu\nu}}{q^2+i\eta}$. But note that it is not uncommon for the Coulomb gauge to be used in NR calculations, in which case longitudinal and transverse photons are respectively:

$$\begin{aligned}\langle A_0 A_0 \rangle &= \frac{1}{\mathbf{q}^2}, \\ \langle A_i A_j \rangle &= \frac{1}{q^2 + i\eta} \left(\delta_{ij} - \frac{q_i q_j}{\mathbf{q}^2} \right).\end{aligned}\tag{A.129}$$

In either case the longitudinal photons are represented by dashed lines, while the transverse photons are represented by curly lines.¹ Pictorially we have:



Figure A.6: Photons in NRQED.

¹not to be confused of course with the gluon lines used in QCD

We now list below only those rules that are necessary for our purposes; a more extensive list can be found in [48].

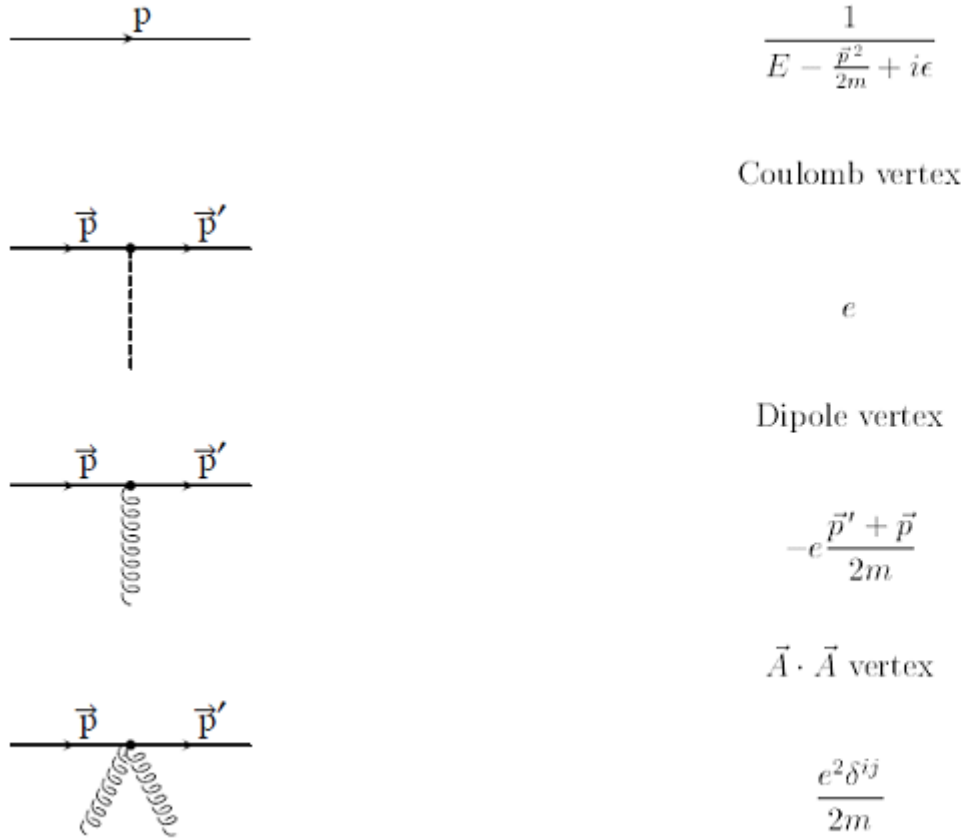


Figure A.7: Feynman Rules For NRQED

The propagators for both the electron and positron fields are identical, and in calculations we must only make the adjustment $e \rightarrow -e$ where necessary.¹ Finally we have the following diagrams, where the wavy lines just mean a generic photon (i.e. transverse or longitudinal is unspecified).

¹Note that in non-abelian theories like NRQCD we must also make the adjustment to the colour matrices $T^a \rightarrow (T^a)^T$

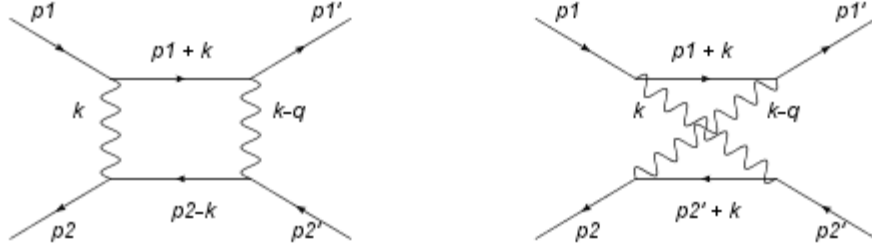


Figure A.8: Boxes in NRQED

Since we are working in the center of mass system, it is useful in the following scenario to elucidate several key properties and identities of this framework.

We have by definition:

$$\begin{aligned} \mathbf{p}_1 &= -\mathbf{p}_2, \quad \mathbf{p}'_1 = -\mathbf{p}'_2; \quad p'_{1,0} = p_{1,0}, \quad p'_{2,0} = p_{2,0} \\ q &= p'_1 - p_1 = p_2 - p'_2 \quad \rightarrow \quad q_0 = 0 \end{aligned} \quad (\text{A.130})$$

Then, some properties that follow from these are for example

$$p_1^2 = p'^2_1 = m^2 \quad \Rightarrow \quad \mathbf{p}'^2_1 = \mathbf{p}_1^2, \quad (\text{A.131})$$

$$\begin{aligned} \mathbf{q}^2 &= \mathbf{p}'^2_1 + \mathbf{p}_1^2 - 2\mathbf{p}'_1 \cdot \mathbf{p}_1 = 2\mathbf{p}_1 \cdot (\mathbf{p}_1 - \mathbf{p}'_1) \\ \Rightarrow \quad \mathbf{p}_1 \cdot \mathbf{q} &= -\frac{1}{2}\mathbf{q}^2. \end{aligned} \quad (\text{A.132})$$

Let us begin by examining the denominators of these diagrams. The planar box has the following denominator

$$\begin{aligned} D_{pb} &= k^2(k-q)^2 \left[(E+k_0) - \frac{(\mathbf{k} + \mathbf{p}_1)^2}{2m} \right] \left[(E-k_0) - \frac{(\mathbf{p}_2 - \mathbf{k})^2}{2m} \right] \\ &= -k^2(k-q)^2 k_0^2 \left[1 - \frac{\mathbf{k} \cdot (\mathbf{k} + 2\mathbf{p}_1)}{2mk_0} \right] \left[1 + \frac{\mathbf{k} \cdot (\mathbf{k} + 2\mathbf{p}_1)}{2mk_0} \right]. \end{aligned} \quad (\text{A.133})$$

The denominator for the crossed box is quite similar

$$\begin{aligned}
D_{cb} &= k^2(k-q)^2 \left[(E+k_0) - \frac{(\mathbf{k} + \mathbf{p}_1)^2}{2m} \right] \left[(E+k_0) - \frac{(\mathbf{p}'_2 + \mathbf{k})^2}{2m} \right] \\
&= k^2(k-q)^2 k_0^2 \left[1 - \frac{\mathbf{k} \cdot (\mathbf{k} + 2\mathbf{p}_1)}{2mk_0} \right] \left[1 - \frac{\mathbf{k} \cdot (\mathbf{k} - 2\mathbf{p}'_1)}{2mk_0} \right]. \quad (\text{A.134})
\end{aligned}$$

We see from (A.133) and (A.134), that expanding the denominators gives us $D = k^2(k-q)^2 k_0^2 (1 + O(1/m))$. Two vertices connected with a transverse photon will contribute to $O(1/m^2)$, while those with longitudinal photons will be $O(1)$. Recall that we are looking for contributions to the potential term that are $O(1/m^2)$. Thus if we are considering the exchange of transverse photons we will need only the leading behaviour from the denominator. It is only when the diagram consists of longitudinal photons that we require a higher order expansion of the denominator in $1/m$. Now let us take a look at the numerators of each diagram.

$$\begin{aligned}
N_{pb} &= e^4 \left[1 - \frac{1}{4m^2} (2\mathbf{p}_1 + \mathbf{k}) \cdot (2\mathbf{p}_2 - \mathbf{k}) \right] \left[1 - \frac{1}{4m^2} (\mathbf{k} + \mathbf{p}'_1 + \mathbf{p}_1) \cdot (\mathbf{p}'_2 + \mathbf{p}_2 - \mathbf{k}) \right] \\
&= \frac{e^4}{4m^2} \left[4m^2 + (2\mathbf{p}_1 + \mathbf{k})^2 + (\mathbf{k} + \mathbf{q} + 2\mathbf{p}_1)^2 \right] + O(1/m^4) \quad (\text{A.135})
\end{aligned}$$

$$\begin{aligned}
N_{cb} &= e^4 \left[1 - \frac{1}{4m^2} (2\mathbf{p}_1 + \mathbf{k}) \cdot (2\mathbf{p}'_2 - \mathbf{k}) \right] \left[1 - \frac{1}{4m^2} (\mathbf{k} + \mathbf{p}'_1 + \mathbf{p}_1) \cdot (\mathbf{k} + \mathbf{p}'_2 + \mathbf{p}_2) \right] \\
&= \frac{e^4}{4m^2} \left[4m^2 - (\mathbf{k} + 2\mathbf{p}_1) \cdot (\mathbf{k} - 2\mathbf{p}'_1) - (\mathbf{k} + \mathbf{q} + \mathbf{p}_1) \cdot (\mathbf{k} - \mathbf{q} - 2\mathbf{p}_1) \right] + O(1/m^4) \\
&\quad (\text{A.136})
\end{aligned}$$

Instead of evaluating each diagram separately, we find that many cancellations will occur when we do them together. For instance, when we add the contributions to the numerators from the transverse photons together we get

$$\begin{aligned}
N_{T.P.} &= -2(2\mathbf{p}_1 + \mathbf{k}) \cdot (\mathbf{k} - \mathbf{q}) - 2\mathbf{k} \cdot (\mathbf{k} + \mathbf{q} + 2\mathbf{p}_1) \\
&= -4\mathbf{k}^2 + 4\mathbf{k} \cdot \mathbf{q} - 2\mathbf{q}^2. \quad (\text{A.137})
\end{aligned}$$

In the above we used the fact that we can effectively set $\mathbf{k} \cdot \mathbf{p}_1 \rightarrow -\frac{1}{2}\mathbf{k} \cdot \mathbf{q}$ as is clear from the symmetry of the problem. Recall also that the denominator of the planar box had a negative sign relative to the crossed box. Thus to required order, the transverse photons in the combined diagrams make a contribution of

$$iM_{T.P.} = -\frac{e^4}{4m^2} \int \frac{d^d k}{(2\pi)^d} \frac{4\mathbf{k}^2 - 4\mathbf{k} \cdot \mathbf{q} + 2\mathbf{q}^2}{k^2(k-q)^2 k_0^2} \quad (\text{A.138})$$

Having 3-vectors in the numerator and 4-vectors in the denominator means we'll need some fancy footwork to get this into a form that is easily solvable. Let's do it term by term starting with the first one. For this one, we note that we can write $\mathbf{k}^2 = k_0^2 - k^2$, giving

$$\begin{aligned} iM_{T.P.}^{(1)} &= -\frac{e^4}{m^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k-q)^2} \left(\frac{1}{k^2} - \frac{1}{k_0^2} \right) \\ &= -\frac{e^4}{m^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k-q)^2} \end{aligned} \quad (\text{A.139})$$

The second term could be set to zero because a simple change of variables $k \rightarrow k - q$ makes the integral scaleless,¹ which are zero by definition in the effective theory. This integral is just a table integral, for example in Appendix A of [40]

$$\int \frac{d^d k}{(-k^2 + m^2)^a (-k^2)^b} = i\pi^{d/2} \frac{\Gamma(a+b-2+\epsilon)\Gamma(-b-\epsilon+2)}{\Gamma(a)\Gamma(b)(m^2)^n}. \quad (\text{A.140})$$

When we apply this to our integral, we get

$$iM_{T.P.}^{(1)} = -\frac{i\alpha^2}{m^2} \left(\frac{1}{\epsilon} + 2 + 4\epsilon - \frac{\pi^2}{12}\epsilon \right) + O(\epsilon^2). \quad (\text{A.141})$$

¹Recall that $q_0 = 0$.

Now for the second term we will need to rewrite it a little

$$\begin{aligned} 2\mathbf{k} \cdot \mathbf{q} &= (k^2 - 2\mathbf{k} \cdot \mathbf{q} + q^2) - k^2 - q^2 \\ &= (k - q)^2 - k^2 + \mathbf{q}^2. \end{aligned} \quad (\text{A.142})$$

When we compare what denominators are left under each of these factors, we see that the first leaves $k^2 k_0^2$, while the second leaves $(k - q)^2 k_0^2$, both of which are scaleless and set to zero. The third factor in (A.142), when multiplied by 2, cancels the third term in the overall integral (A.138). Thus we actually just have

$$iM_{T.P.} = -\frac{i\alpha^2}{m^2} \left(\frac{1}{\epsilon} + 2 + 4\epsilon - \frac{\pi^2}{12}\epsilon \right) + O(\epsilon^2). \quad (\text{A.143})$$

The second part of this calculation is that which has longitudinal photons, and requires a higher order expansion in $1/m$ of the denominators. Referring back to equations (A.133) and (A.134), we see that

$$\begin{aligned} \frac{1}{D_{pb}} + \frac{1}{D_{cb}} &= \frac{1}{k^2(k - q)^2 k_0^2 \left(1 - \frac{\mathbf{k} \cdot (\mathbf{k} + 2\mathbf{p}_1)}{2mk_0}\right)} \left[\frac{1}{1 - \frac{\mathbf{k} \cdot (\mathbf{k} - 2\mathbf{p}'_1)}{2mk_0}} - \frac{1}{1 + \frac{\mathbf{k} \cdot (\mathbf{k} + 2\mathbf{p}_1)}{2mk_0}} \right] \\ &= \frac{1}{k^2(k - q)^2 k_0^2} \left[\frac{\mathbf{k}^2 - \mathbf{k} \cdot \mathbf{q}}{mk_0} + \frac{(\mathbf{k}^2 - \mathbf{k} \cdot \mathbf{q})^2}{2m^2 k_0^2} \right] + O(1/m^3). \end{aligned} \quad (\text{A.144})$$

The first of these terms is of order $1/m$ and is thus not of interest for us; we'll have to expand the second term before it becomes useful.

$$\begin{aligned} \frac{1}{D_{pb}} + \frac{1}{D_{cb}} &= \frac{1}{2m^2} \frac{\mathbf{k}^2(\mathbf{k}^2 - 2\mathbf{k} \cdot \mathbf{q}) + (\mathbf{k} \cdot \mathbf{q})^2}{k^2(k - q)^2 k_0^4} \\ &= \frac{1}{2m^2} \left[\frac{\mathbf{k}^2 - 2\mathbf{k} \cdot \mathbf{q}}{k^2(k - q)^2 k_0^2} + \frac{(\mathbf{k} \cdot \mathbf{q})^2}{k^2(k - q)^2 k_0^4} \right] \\ &= \frac{1}{2m^2} \left[\frac{1}{k^2(k - q)^2} - \frac{\mathbf{q}^2}{k^2(k - q)^2 k_0^2} + \frac{(\mathbf{k} \cdot \mathbf{q})^2}{k^2(k - q)^2 k_0^4} \right] \end{aligned} \quad (\text{A.145})$$

To get the second line, we just applied the trick that we already used in (A.139)

to the first term. To get the third line we again used that same trick on the first term, while the second term follows from equation (A.142) and the argument that followed it. We already know what the value of the first term in the integral will be from the evaluation in equation (A.141):

$$\frac{e^4}{2m^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k-q)^2} = \frac{i\alpha^2}{m^2} \left(\frac{1}{2\epsilon} + 1 + 2\epsilon - \frac{\pi^2}{24}\epsilon \right) + O(\epsilon^2). \quad (\text{A.146})$$

The second term is again just a tabulated integral that, like (A.140), can be found in [40]

$$\begin{aligned} & \int \frac{d^d k}{(2\pi)^d} \frac{1}{(-k^2)^a [-(k-q)^2]^b (-2k \cdot p)^c} \\ &= i\pi^{d/2} \frac{\Gamma(-a-c/2-\epsilon+2)\Gamma(-b-c/2-\epsilon+2)}{\Gamma(-a-b-c-2\epsilon+4)} \\ & \times \frac{\Gamma(a+b+c/2+\epsilon-2)\Gamma(c/2)}{2\Gamma(a)\Gamma(b)\Gamma(c)(-q^2)^{a+b+c/2+\epsilon-2}(p^2)^{c/2}}, \end{aligned} \quad (\text{A.147})$$

provided that $q \cdot p = 0$. It therefore follows that by choosing for instance $p = \langle \frac{1}{2}, 0, 0, 0 \rangle$, we get

$$-\frac{e^4}{2m^2} \int \frac{d^d k}{(2\pi)^d} \frac{\mathbf{q}^2}{k^2(k-q)^2 k_0^2} = \frac{i\alpha^2}{m^2} \left(\frac{2}{\epsilon} - \frac{\pi^2}{6}\epsilon \right) + O(\epsilon^2). \quad (\text{A.148})$$

The last term requires the most amount of work.

$$\begin{aligned} \frac{(\mathbf{k} \cdot \mathbf{q})^2}{k^2(k-q)^2 k_0^4} &= \frac{(k-q)^2}{4k^2 k_0^4} - \frac{(k^2+q^2)}{2k^2 k_0^4} + \frac{(k^2+q^2)^2}{4k^2(k-q)^2 k_0^4} \\ &= \frac{k^4 + 2k^2 q^2 + q^4}{4k^2(k-q)^2 k_0^4} \\ &= \frac{k^2}{4(k-q)^2 k_0^4} + \frac{q^2}{(k-q)^2 k_0^4} + \frac{q^4}{4k^2(k-q)^2 k_0^4} \\ &= \frac{q^4}{4k^2(k-q)^2 k_0^4} \end{aligned} \quad (\text{A.149})$$

As usual, scaleless integrals were set to zero as they showed up. We can now just apply (A.147) to the remaining term to get

$$\frac{e^4}{8m^2} \int \frac{d^d k}{(2\pi)^{d/2}} \frac{\mathbf{q}^4}{k^2(k-q)^2 k_0^4} = -\frac{i\alpha^2}{m^2} \left(\frac{2}{3\epsilon} + \frac{4}{3} - \frac{\pi^2}{18} \epsilon \right). \quad (\text{A.150})$$

Finally, we can add all the contributions of (A.143),(A.146),(A.148) and (A.150) to get

$$iM_{S.B.} = \frac{i\alpha^2}{m^2} \left(\frac{5}{6\epsilon} - \frac{7}{3} - 2\epsilon - \frac{5\pi^2}{72} \epsilon \right) \quad (\text{A.151})$$

A.9 Double Photon Vertex

In the non-relativistic effective theory sketched briefly in the previous section, we see that there is an additional diagram that contributes. The $\mathbf{A} \cdot \mathbf{A}$ vertex on the electron line (i.e. the last diagram in figure A.7), provides us with a $1/m^2$ contribution when coupled to its counterpart diagram on the positron line.

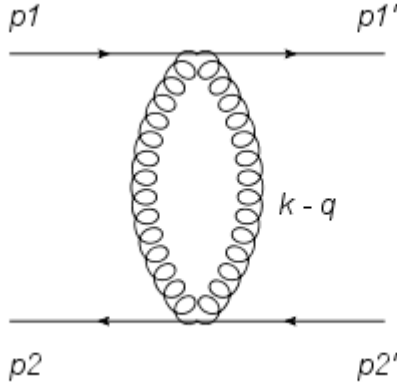


Figure A.9: $\mathbf{A} \cdot \mathbf{A}$ contribution in NRQED.

Using the rules we get

$$iM_{\mathbf{A}^2} = 2 \frac{e^4}{4m^2} \int \frac{d^d k}{(2\pi)^d} \frac{\delta^{ij} \delta^{ij}}{k^2 (k - q)^2}, \quad (\text{A.152})$$

where the factor of two out front is a symmetry factor. We actually already know how to evaluate this integral. By referring back to (A.146), and noting that $\delta^{ij} \delta^{ij} = d - 1$, we get

$$\begin{aligned} iM_{\mathbf{A}^2} &= \frac{i\alpha^2(3 - 2\epsilon)}{m^2} \left(\frac{1}{2\epsilon} + 1 + 2\epsilon - \frac{\pi^2}{24}\epsilon \right) \\ &= \frac{i\alpha^2}{m^2} \left(\frac{3}{2\epsilon} + 2 + 4\epsilon - \frac{\pi^2}{8}\epsilon \right) + O(\epsilon^2). \end{aligned} \quad (\text{A.153})$$

Appendix B

Evaluation of Diagrams for Quarkonium

B.1 Crossed Box

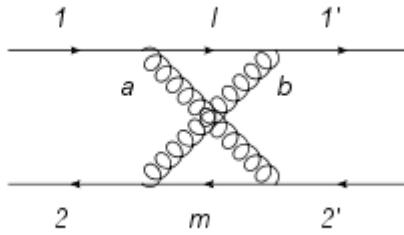


Figure B.1: Crossed Box

B.1.1 Colour Factors

The $q\bar{q}$ is necessarily in the colour singlet configuration, which in terms of standard colour spinor ket notation of red, blue, green can be written simply as

$$\frac{1}{\sqrt{3}}[|r\bar{r}\rangle + |b\bar{b}\rangle + |g\bar{g}\rangle]. \quad (\text{B.1})$$

The reason for writing this expression in such a pedestrian way, is because the colour calculation will be carried out for a general $SU(N)$ gauge group and it gives us the form of a general singlet configuration projection operator in order to carry out the calculation. From the above we see that the general singlet projector is

$$|q\bar{q}\rangle_{colour} = \frac{1}{\sqrt{N}} \sum_{i=1}^N |\mathbf{q}_i\rangle |\bar{\mathbf{q}}_i\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N |\mathbf{e}_q^i\rangle \otimes |\mathbf{e}_{\bar{q}}^i\rangle, \quad (\text{B.2})$$

where the column vectors have N components, and a value of one at the position i and 0 elsewhere. An analogous expression for the post collision particles is also used $|q\bar{q}'\rangle'_{colour}$. We can use the rules for an $SU(N)$ gauge group to get the expression for the colour \tilde{C}

$$\tilde{C} = \frac{1}{N} \langle q_i | T_{il}^a T_{lj}^b | q'_j \rangle \langle \bar{q}'_j | T_{jm}^a T_{mi}^b | \bar{q}_i \rangle = \frac{1}{N} T_{il}^a T_{lj}^b T_{jm}^a T_{mi}^b. \quad (\text{B.3})$$

We will want to commute the matrices in order to simplify, thus matrix notation will now be used. We have that

$$\tilde{C} = \frac{1}{N} (\mathbf{T}^a \cdot \mathbf{T}^b) \cdot (\mathbf{T}^a \cdot \mathbf{T}^b), \quad (\text{B.4})$$

which we can rewrite with the commutator rule: $\mathbf{T}^b \cdot \mathbf{T}^a = \mathbf{T}^a \cdot \mathbf{T}^b - i f^{abc} \mathbf{T}^c$ of the lie-algebra, along with the associativity of matrix multiplication.

Plugging in and using the definition of the quadratic Casimir operator in the first term,

$$\tilde{C} = \frac{1}{N} C_F^2 \delta^{ij} \delta^{ji} - \frac{i f^{abc}}{N} \mathbf{T}^a \cdot \mathbf{T}^c \cdot \mathbf{T}^b = C_F^2 + \frac{i f^{abc}}{N} \mathbf{T}^a \cdot \mathbf{T}^b \cdot \mathbf{T}^c. \quad (\text{B.5})$$

Concentrating on the second term $\tilde{C}^{(2nd)}$, we can use the identity

$$\begin{aligned}
2\mathbf{T}^{\mathbf{a}} \cdot \mathbf{T}^{\mathbf{b}} &= \frac{1}{N} \delta^{ab} \mathbf{1} + (d^{abd} + i f^{abd}) \mathbf{T}^{\mathbf{d}} \\
\Rightarrow \tilde{C}^{(2^{nd})} &= \frac{i f^{abc}}{2N} \left(\frac{1}{N} \delta^{ab} \mathbf{1} + (d^{abd} + i f^{abd}) \mathbf{T}^{\mathbf{d}} \right) \cdot \mathbf{T}^{\mathbf{c}}. \quad (\text{B.6})
\end{aligned}$$

Since contracting an antisymmetric tensor with a symmetric one will obviously give zero, this leaves only

$$\tilde{C}^{(2^{nd})} = -\frac{f^{abc} f^{abd}}{2N} \mathbf{T}^{\mathbf{d}} \cdot \mathbf{T}^{\mathbf{c}}. \quad (\text{B.7})$$

Then using one final identity $f^{acd} f^{bcd} = N \delta^{ab}$, we get in total

$$\begin{aligned}
\tilde{C} &= C_F^2 - \frac{\delta^{cd}}{2} \mathbf{T}^{\mathbf{d}} \cdot \mathbf{T}^{\mathbf{c}} \\
&= C_F^2 - \frac{N}{2} C_F \\
&= C_F^2 - \frac{1}{2} C_A C_F. \quad (\text{B.8})
\end{aligned}$$

We might as well just get the planar box colour factor here too. Examining Figure A.2 (p.47), it is apparent that we only need to switch the colour matrix indices on the two middle matrices of the above. This means there is no need to commute the colour matrix and we just get C_F^2 .

B.1.2 Calculation of Amplitude

The above diagram can be written in the following schematic form (where the colour factor is left out, and $m = m_q$)

$$iM = i g_s^4 \int \frac{d^4 q}{(2\pi)^4} \frac{A + B^\mu q_\mu + C^{\mu\nu} q_\mu q_\nu}{(q^2 - \lambda^2)^2 (q^2 - 2m q_0 + i\epsilon)^2}. \quad (\text{B.9})$$

It will be shown when we discuss the numerator algebra that the only term

important for spin is the term second order in loop momentum $C^{\mu\nu}q_\mu q_\nu$. As a result we consider only

$$iM_{spin} = ig_s^4 \int \frac{d^4q}{(2\pi)^4} \frac{C^{\mu\nu}q_\mu q_\nu}{(q^2 - \lambda^2)^2(q^2 - 2mq_0 + i\epsilon)^2}. \quad (\text{B.10})$$

In order to solve this integral it is simplest to use a simple Feynman parametrization

$$\frac{1}{a^2b^2} = 3! \int_0^1 \frac{x(1-x)dx}{[a + (b-a)x]^4}, \quad (\text{B.11})$$

which can be derived by twice differentiating the standard relation

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[a + (b-a)x]^2}. \quad (\text{B.12})$$

So we now have the form

$$iM_{spin} = 3! ig_s^4 \int_0^1 x(1-x)dx \int \frac{d^4q}{(2\pi)^4} \frac{C^{\mu\nu}q_\mu q_\nu}{[(q^2 - \lambda^2) + (-2mq_0 + \lambda^2 + i\epsilon)x]^4}. \quad (\text{B.13})$$

In order to bring this into the form of a standard loop integral, we make the substitution $q_0 \rightarrow q'_0 = q_0 - mx$ so that $q'^2_0 - m^2x^2 = q^2_0 - 2mxq_0$. This will of course change the numerator but we note that $C^{0i} \sim O(\vec{k})$ and the term $C^{00}(2mxq_0)$ contributes an odd term which drops upon integrating over the loop momentum.

$$iM_{spin} = 3! ig_s^4 \int_0^1 x(1-x)dx \int \frac{d^4q}{(2\pi)^4} \frac{C^{\mu\nu}q_\mu q_\nu + C^{00}m^2x^2}{[q^2 - \{m^2x^2 + \lambda^2(1-x) - i\epsilon x\}]^4}. \quad (\text{B.14})$$

This integral is now a standard loop integral with the term in the curly brackets independent of the loop momentum. This can be evaluated by looking it up in a table (see for instance [28] appendix B), giving:

$$iM_{spin} = -\frac{g_s^4}{(4\pi)^2} \int_0^1 x(1-x) dx \left[\frac{-\frac{1}{2}C^{\mu\nu}g_{\mu\nu}}{\Delta} + \frac{C^{00}m^2x^2}{\Delta^2} \right], \quad (\text{B.15})$$

where $\Delta \equiv \{m^2x^2 + \lambda^2(1-x) - i\epsilon x\}$, and we write $C^{\mu\nu}g_{\mu\nu} = C^{00} - C^{ii}$ for convenience.

$$\frac{-\frac{1}{2}C^{00}\Delta + C^{00}m^2x^2}{\Delta^2} = \frac{\frac{1}{2}C^{00}[m^2x^2 - \lambda^2(1-x) + i\epsilon x]}{\Delta^2}, \quad (\text{B.16})$$

Thus in total the x-integral is:

$$iM_{spin} = -\frac{\alpha_s^2}{2} \int_0^1 x(1-x) dx \left[\frac{C^{00}[m^2x^2 - \lambda^2(1-x) + i\epsilon x]}{\Delta^2} + \frac{C^{ii}}{\Delta} \right]. \quad (\text{B.17})$$

It will be simplest to evaluate each of these separately. Starting with the first term we have

$$iM_{spin}^a = -\frac{\alpha_s^2 C^{00}}{2m^2} \int_0^1 dx \frac{[x^2 - \delta^2(1-x)][1-x]x}{[x^2 + \delta^2(1-x) - i\epsilon x]^2}, \quad (\text{B.18})$$

where the m^2 terms, which lead to an overall $\frac{1}{m^2}$ multiplying the integral, have been factored out. Finally replacements with the dimensionless constant $\delta = \frac{\lambda}{m}$ have been done. the details of this integral are not very interesting and we will simply quote the result

$$iM_{spin}^a = \frac{\alpha_s^2 C^{00}}{2m^2} \left(2 + \ln(\lambda/m) \right). \quad (\text{B.19})$$

The second integral is likewise evaluated

$$iM_{spin}^b = -\frac{\alpha_s^2 C^{ii}}{2m^2} \int_0^1 dx \frac{(1-x)x}{x^2 + \delta^2(1-x) - i\epsilon x} = \frac{\alpha_s^2 C^{ii}}{2m^2} \left(1 + \ln(\lambda/m) \right). \quad (\text{B.20})$$

We now pass to the evaluation of the numerator algebra, which was written out in terms of tensors of different rank, and is equivalent to the following.

$$[\bar{u}_r(p')\gamma^\mu(\not{s} + m)\gamma^\nu u_s(p)][\bar{v}_t(\bar{p})\gamma_\mu(\not{s}' + m)\gamma_\nu v_w(\bar{p}')] \quad (\text{B.21})$$

where $s = p' - q$ and $s' = -\bar{p} + q$. We use the identity $\gamma^\mu \not{k} = 2k^\mu - \not{k}\gamma^\mu$,

$$[\bar{u}_r(p')(2s^\mu + (-\not{s} + m)\gamma^\mu)\gamma^\nu u_s(p)][\bar{v}_t(\bar{p})(2s'_\mu + (-\not{s}' + m)\gamma_\mu)\gamma_\nu v_w(\bar{p}')]. \quad (\text{B.22})$$

Then of course $\bar{u}_s(p')[\not{p}' - m] = 0$ and $\bar{v}_t(\bar{p})[\bar{p} + m] = 0$, leaving us with

$$[\bar{u}_r(2s^\mu + \not{q}\gamma^\mu)\gamma^\nu u_s][\bar{v}_t(2s'_\mu - \not{q}\gamma_\mu)\gamma_\nu v_w]. \quad (\text{B.23})$$

If we enumerate the terms in the first bracket as a,b and the second as c,d; we can take a closer look and save some work. Any term combined with a(c) will necessarily come with a factor s^μ (s'_μ), thus leaving only a single gamma matrix sandwiched between the spinors. So let's look at the form of the gamma matrices to see why this has no interest for us.

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{B.24})$$

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \quad (\text{B.25})$$

The first matrix has no pauli matrices and thus contributes nothing to the spin. The second matrix contains pauli factors but they cross the two top and bottom components of the 4-spinors and in our non-relativistic approximation do not contribute. Thus only the term (b,d) will contribute to the spin structure in our approximation

$$-[\bar{u}_r \not{q}\gamma^\mu \gamma^\nu u_s][\bar{v}_t \not{q}\gamma_\mu \gamma_\nu v_w]. \quad (\text{B.26})$$

We can now use some gamma matrix identities to simplify the expression,

$$\gamma^\mu \gamma^\nu = \frac{1}{2}[\gamma^\mu, \gamma^\nu] + \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu} + \sigma^{\mu\nu}, \quad (\text{B.27})$$

$$g^{\mu\nu} g_{\mu\nu} = 4 \quad , \quad g^{\mu\nu} \sigma_{\mu\nu} = 0. \quad (\text{B.28})$$

As before we will only be interested in terms that generate spin, so the terms where the metric has been fully contracted will leave a \not{q} which is just a summation of a single gamma matrix at a time, and drops as before. Thus we are left with

$$-[\bar{u}_r \not{q} \sigma^{\mu\nu} u_s][\bar{v}_t \not{q} \sigma_{\mu\nu} v_w]. \quad (\text{B.29})$$

Now the form of $\sigma^{\mu\nu}$ is that of a second rank antisymmetric tensor, and as such can be represented as a two-vector $\sigma^{\mu\nu} = \langle \sigma^{0i}, \sigma^{ij} \rangle = \langle \vec{\alpha}, i\vec{\Sigma} \rangle$, where its covariant equivalent is $\sigma_{\mu\nu} = \langle -\vec{\alpha}, i\vec{\Sigma} \rangle$. Here $\vec{\Sigma}$, is just the spin operator double stacked (a 4D irreducible representation), so that it also covers positron states. With a couple lines of algebra, you can convince yourself that,

$$\sigma^{\mu\nu} \sigma_{\mu\nu} = -2[\vec{\alpha}_1 \cdot \vec{\alpha}_2 + \vec{\Sigma}_1 \cdot \vec{\Sigma}_2]. \quad (\text{B.30})$$

This brings our expression into the form:

$$\begin{aligned} & 2[\bar{u}_r \not{q} \vec{\alpha}_1 u_s] \cdot [\bar{v}_t \not{q} \vec{\alpha}_2 v_w] + 2[\bar{u}_r \not{q} \vec{\Sigma}_1 u_s] \cdot [\bar{v}_t \not{q} \vec{\Sigma}_2 v_w] \\ &= 2([\bar{u}_r \gamma^\mu \vec{\alpha}_1 u_s] \cdot [\bar{v}_t \gamma^\nu \vec{\alpha}_2 v_w] + [\bar{u}_r \gamma^\mu \vec{\Sigma}_1 u_s] \cdot [\bar{v}_t \gamma^\nu \vec{\Sigma}_2 v_w]) q_\mu q_\nu. \end{aligned} \quad (\text{B.31})$$

This is good, because it tells us right away that the only important piece for the spin is the tensor term in the integral, as previously stated. Now again we are interested in the spin terms, which means we want the ones that have a Σ matrix sandwiched between the spinors. This means of course that in term 1 both μ and ν must *not* be zero, and in the second term they *must be zero*.

Let's start by simplifying the first term, remembering $\gamma^i = \alpha^i \gamma^0$:

$$C^{ij} q_i q_j = 2 [u_r^\dagger \gamma^i \gamma^k u_s] [v_t^\dagger \gamma^j \gamma^k v_w] q_i q_j. \quad (\text{B.32})$$

We then re-express the gamma product again, and again drop terms like $g^{ij} g_{ij}$, and replace σ^{ij} matrices by their definition

$$\sigma^{ij} = i \epsilon^{ijk} \Sigma^k, \quad (\text{B.33})$$

$$C^{ij} = 2 i^2 [u_r^\dagger \epsilon^{ikl} \Sigma_1^l u_s] [v_t^\dagger \epsilon^{jkm} \Sigma_2^m v_w]. \quad (\text{B.34})$$

We simplify as follows :

$$\begin{aligned} \epsilon^{ikl} \epsilon^{jkm} &= \delta^{ij} \delta^{lm} - \delta^{im} \delta^{lj}, \\ (\delta^{ij} \delta^{lm} - \delta^{im} \delta^{lj}) \Sigma_1^l \Sigma_2^m &= \delta^{ij} \Sigma_1 \cdot \Sigma_2 - \Sigma_1^i \Sigma_2^j, \\ \Sigma &= \sigma \otimes \mathbf{1}_{2 \times 2}. \end{aligned} \quad (\text{B.35})$$

Now we have that

$$C^{ij} = 2 [\xi_r^\dagger \zeta_t^\dagger (\sigma_1^i \sigma_2^j - \delta^{ij} \sigma_1 \cdot \sigma_2) \xi_s \zeta_w]. \quad (\text{B.36})$$

For later convenience:

$$C^{ii} = -4 [\xi_r^\dagger \zeta_t^\dagger (\sigma_1 \cdot \sigma_2) \xi_s \zeta_w], \quad (\text{B.37})$$

where they are now 2-spinors. It should be clear from the above that the second term (where $\mu = \nu = 0$) leads to:

$$C^{00} = 2 [\xi_r^\dagger \zeta_t^\dagger (\sigma_1 \cdot \sigma_2) \xi_s \zeta_w]. \quad (\text{B.38})$$

If we write the amplitude out in all its glory we get

$$iM_{spin}^{c.b.} = \frac{C_F^2 - \frac{1}{2} C_A C_F}{2m^2} \alpha_s^2 \left[C^{00} (2 + \ln(\lambda/m)) + C^{ii} (1 + \ln(\lambda/m)) \right]. \quad (\text{B.39})$$

B.2 Planar Box

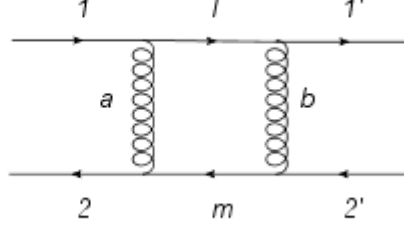


Figure B.2: Planar Box

In the planar box, only the denominator and the colour factor change. For the latter $(C_F^2 - \frac{1}{2}C_A C_F) \rightarrow C_F^2$, while the integral becomes the following

$$iM_{spin} = ig_s^4 \int \frac{d^4 q}{(2\pi)^4} \frac{C^{\mu\nu} q_\mu q_\nu}{(q^2 - \lambda^2)^2 (q^2 - 2mq_0 + i\epsilon)(q^2 + 2mq_0 + i\epsilon)}. \quad (\text{B.40})$$

If we're only considering the spin contribution, the form factors C^{00} and C^{ii} are identical to the ones outlined for the crossed box diagram. We can now immediately use a Feynman parametrization,

$$\begin{aligned} \frac{1}{a^2 bc} &= \frac{\partial}{\partial a} \int_0^1 \int_0^x \frac{2 dy dx}{[a + (b-a)x + (c-b)y]^3} \\ &= 3! \int_0^1 dx \int_0^x dy \frac{(1-x)}{[a + (b-a)x + (c-b)y]^4}, \end{aligned} \quad (\text{B.41})$$

$$\Rightarrow iM_{spin} = 3! ig_s^4 \int_0^1 dx \int_0^x dy \int \frac{d^4 q}{(2\pi)^4} \frac{(1-x)C^{\mu\nu} q_\mu q_\nu}{[(q^2 - \lambda^2) + (2mq_0 + \lambda^2 + i\epsilon)x - 4mq_0 y]^4} \quad (\text{B.42})$$

We will make the following change of variable $q_0 \rightarrow q'_0 = q_0 + m(x - 2y)$

$$\rightarrow q_0'^2 = q_0^2 + 2mq_0(x - 2y) + m^2(x - 2y)^2.$$

$$\begin{aligned}
iM_{spin} &= 3! i g_s^4 \int_0^1 dx \int_0^x dy \int \frac{d^4 q}{(2\pi)^4} \frac{(1-x)[C^{\mu\nu} q_\mu q_\nu + C^{00} m^2(x-2y)^2]}{[q^2 - \{m^2(x-2y)^2 + \lambda^2(1-x) - i\epsilon x\}]^4} \\
&= 3! i g_s^4 \int_0^1 dx \int_0^x dy \int \frac{d^4 q}{(2\pi)^4} \frac{(1-x)[C^{\mu\nu} q_\mu q_\nu + C^{00} m^2(x-2y)^2]}{[q^2 - \Delta]^4} \\
&= -\frac{g_s^4}{4\pi^2} \int_0^1 dx \int_0^x dy (1-x) \left[\frac{-\frac{1}{2} C^{\mu\nu} g_{\mu\nu} \Delta + C^{00} m^2(x-2y)^2}{\Delta^2} \right] \\
&= -\frac{\alpha_s^2}{2m^2} \int_0^1 dx \int_0^x dy (1-x) \left[\frac{C^{00}[(x-2y)^2 - \delta^2(1-x) + i\epsilon x]}{\Delta'^2} + \frac{C^{ii}}{\Delta'} \right],
\end{aligned} \tag{B.43}$$

where $\Delta' \equiv (x - 2y)^2 + \delta^2(1 - x) - i\epsilon x$.

Interestingly enough, evaluation of the y-integral in the first term gives

$$\frac{\alpha_s^2 C^{00}}{2m^2} \int_0^1 dx \frac{(1-x)x}{x^2 + \delta^2(1-x) - i\epsilon x}, \tag{B.44}$$

which we know from (A.12) is just

$$iM_{spin}^a = -\frac{\alpha_s^2 C^{00}}{2m^2} \left(1 + \ln(\lambda/m) \right). \tag{B.45}$$

Evaluation of the second term is a little more interesting, it is this term that will give us a singular pole term in $\frac{1}{\lambda}$, along with another set of logarithmic and constant terms contributing to the spin. It is nearly identical to the second term in eqn. (9), except for the extra integral, and missing x-term in the numerator.

But as we will see these minor changes create large differences in the answers.

$$iM_{spin}^b = -C^{ii} \frac{\alpha_s^2}{2m^2} \int_0^1 dx \int_0^x dy \frac{1-x}{(x-2y)^2 + \delta^2(1-x) - i\epsilon x} \quad (\text{B.46})$$

$$= -C^{ii} \frac{\alpha_s^2}{2m^2} \int_0^1 dx \frac{1-x}{\delta\sqrt{1-x(1+i\epsilon)}} \tan^{-1} \left(\frac{x}{\delta\sqrt{1-x(1+i\epsilon)}} \right), \quad (\text{B.47})$$

the epsilons will be left in until we're sure there's no danger of a divergence, but to save space $b \equiv 1 + i\epsilon$. Evaluation of the x-integral gives

$$I_{spin}^b = -\frac{1}{2} \ln(1-i\epsilon) + \frac{2}{3\delta\sqrt{-b^2}} \tanh^{-1} \left(\frac{\delta^2 - 2}{2\delta\sqrt{-b^2}} \right) + \ln(\delta) - \frac{2}{3\delta\sqrt{-b^2}} \tanh^{-1} \left(\frac{\delta}{2\sqrt{-b^2}} \right) + \frac{1}{3}. \quad (\text{B.48})$$

We can now easily see that the $\sqrt{-b^2}$ terms can safely be set to i .

$$I_{spin}^b = -\frac{2i}{3\delta} \tanh^{-1} \left(\frac{i}{\delta} \right) + \frac{2i}{3\delta} \tanh^{-1} \left(\frac{-i\delta}{2} \right) + \ln(\delta) + \frac{1}{3} \\ = \frac{2}{3\delta} \left[\tan^{-1} \left(\frac{1}{\delta} \right) + \tan^{-1} \left(\frac{\delta}{2} \right) \right] + \ln(\delta) + \frac{1}{3}, \quad (\text{B.49})$$

where the identity $i \tanh^{-1}(z) = \tan^{-1}(iz)$, was used. Now we have that

$$\begin{aligned}\tan^{-1}(1/x) &= \frac{\pi}{2} - \tan^{-1}(x) \quad \forall x : [0, 1) \\ \tan^{-1}(x) &\approx x, \quad x \ll 1.\end{aligned}$$

Thus upon using the expansion in our expression we get that

$$\begin{aligned}I_{spin}^b &= \frac{\pi}{3\delta} + \frac{2}{3\delta} \left(\frac{-\delta}{2} \right) + \ln(\delta) + \frac{1}{3} \\ &= \frac{\pi}{3\delta} + \ln(\delta) \\ &= \frac{\pi m}{3\lambda} + \ln\left(\frac{\lambda}{m}\right)\end{aligned}\tag{B.50}$$

So now, all in all, we have

$$iM_{spin}^{p.b.} = -\frac{C_F^2}{2m^2} \alpha_s^2 \left[C^{00} \left(1 + \ln\left(\frac{\lambda}{m}\right) \right) + C^{ii} \left(\frac{\pi m}{3\lambda} + \ln\left(\frac{\lambda}{m}\right) \right) \right], \tag{B.51}$$

$$iM_{spin}^{c.b.} = \frac{C_F^2 - \frac{1}{2}C_A C_F}{2m^2} \alpha_s^2 \left[C^{00} (2 + \ln(\lambda/m)) + C^{ii} (1 + \ln(\lambda/m)) \right]. \tag{B.52}$$

Plugging in the definitions of the coefficients:

$$C^{ii} = -4 [\xi_r^\dagger \zeta_t^\dagger (\sigma_1 \cdot \sigma_2) \xi_s \zeta_w] = -8\mathbf{S}^2 \quad ; \quad C^{00} = 2 [\xi_r^\dagger \zeta_t^\dagger (\sigma_1 \cdot \sigma_2) \xi_s \zeta_w] = 4\mathbf{S}^2$$

$$V_{spin}^{p.b.} = \frac{C_F^2}{m^2} \alpha_s^2 \left[-2 + 2\ln(\lambda/m) + \frac{4\pi m}{3\lambda} \right] \mathbf{S}^2, \tag{B.53}$$

$$V_{spin}^{c.b.} = -\frac{C_F^2 - \frac{1}{2}C_A C_F}{m^2} \alpha_s^2 [2\ln(\lambda/m)] \mathbf{S}^2. \quad (\text{B.54})$$

The equations (B.53) and (B.54) constitute our final result for these diagrams. This result will be checked by a more modern analytical technique, known as expansion by regions, later.

B.3 Non-Abelian Vertex

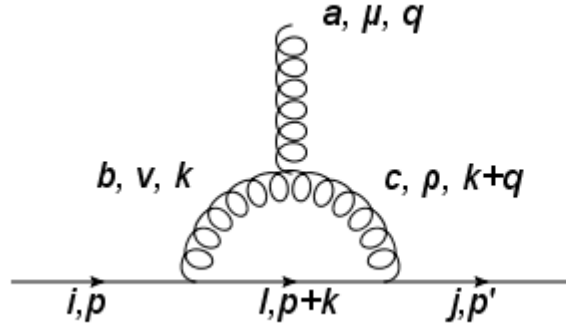


Figure B.3: Non-Abelian Vertex

If we perform this calculation in the background field formulation it will make things considerably easier to carry out. Following the Feynman rules for the background formalism [49]

$$i\Gamma^\mu \sim \int d^D \hat{k} \frac{\gamma^\beta \left[g_{\alpha\beta} (2k - q)^\mu - g_\alpha^\mu \left(k + q + \frac{1}{\xi} (q - k) \right)_\beta + g_\beta^\mu \left(2q - k + \frac{1}{\xi} k \right)_\alpha \right] (\not{p} + \not{k} + m) \gamma^\alpha}{[(p+k)^2 - m^2][(k-q)^2 - \lambda^2][k^2 - \lambda^2]}. \quad (\text{B.55})$$

In the above, constant prefactors have been temporarily omitted in the interest

of space. Then, in the Feynman gauge the numerator simplifies substantially

$$i\Gamma_{NA}^\mu(p, p') = ig_s^3 C_{ji}^a \int d^D \hat{k} \frac{\gamma^\beta [g_{\alpha\beta}(2k - q)^\mu - 2g_\alpha^\mu q_\beta + 2g_\beta^\mu q_\alpha] (\not{p} + \not{k} + m) \gamma^\alpha}{[(p + k)^2 - m^2][(k - q)^2 - \lambda^2][k^2 - \lambda^2]}. \quad (\text{B.56})$$

where $\frac{i}{2} C_A T_{ji}^a \equiv C_{ji}^a$. The work of Manohar [50] will be invaluable in this calculation. The denominator of his calculation of this vertex reads (Eq.'s 31 & 32)

$$m^2(x + y - 1)^2 - q^2 xy. \quad (\text{B.57})$$

We can reproduce precisely this denominator in our work if we use the following Feynman parametrization (working first without the gluon mass)

$$\int_0^1 dx \int_0^{1-x} dy \int d^D \hat{k} \frac{N^\mu}{k^2 + 2p \cdot k + [q^2 - 2(q + p) \cdot k]x - 2p \cdot ky}. \quad (\text{B.58})$$

From the above expression it is clear that the necessary change of variables is $k \rightarrow l = k + [p(1-y) - p'x]$. The quantity in the square brackets then goes back into the denominator squared which gives $m^2(1-y)^2 + m^2 x^2 - 2p \cdot p' x(1-y)$, then upon noting that $-2p \cdot p' = q^2 - 2m^2$ we recover the required denominator. Finally we see that by including a gluon-mass term we incorporate into the denominator a term of the form $\lambda^2(x + y)$. Thus inserting this factor into Manohar's expression for the non-abelian F_2 we get

$$F_2^g = -\frac{\alpha_s}{4\pi} C_A \int_0^1 dx \int_0^{1-x} dy \frac{2(x+y)(1-x-y)m^2}{m^2(x+y-1)^2 - q^2 xy + \lambda^2(x+y)}. \quad (\text{B.59})$$

We are interested only in the limit $q \rightarrow 0$, so for us

$$F_2^g = -\frac{\alpha_s}{4\pi} C_A \int_0^1 dx \int_0^{1-x} dy \frac{2(x+y)(1-x-y)}{(x+y-1)^2 + \frac{\lambda^2}{m^2}(x+y)}. \quad (\text{B.60})$$

Evaluation of this integral gives us

$$F_2^g = \frac{\alpha_s}{4\pi} C_A \left[3 + 2 \ln(\lambda/m) \right] + \mathcal{O}(q^2/m^2). \quad (\text{B.61})$$

We must add to this the contribution of the abelian vertex, which Schwinger tells us (up to colour factors) is

$$F_2^V = \frac{\alpha_s}{2\pi} (C_F - \frac{1}{2}C_A) + \mathcal{O}(q^2/m^2). \quad (\text{B.62})$$

We can thus write out the final result¹

$$\begin{aligned} F_2 &= F_2^V + F_2^g \\ &= \frac{\alpha_s}{2\pi} \left[C_F + C_A \left(1 + \ln(\lambda/m) \right) \right]. \end{aligned} \quad (\text{B.63})$$

¹Note that we will have to multiply this result by two when inserting it into the Born result, to account for the two separate insertions of the vertex corrections