The Final Exam (Math 314 A1)

December 14, 2015

Name: ______________________ I.D.#: ______________

1. (10 points) (a) Prove that $a + b\sqrt{2}$ is an irrational number for all rational numbers a and b with $b \neq 0$. (It is known that $\sqrt{2}$ is an irrational number.)

Proof. Let $c := a + b\sqrt{2}$. Since $b \neq 0$, it follows that

$$\sqrt{2} = \frac{c-a}{b}.$$

If c is a rational number, then $\frac{c-a}{b}$ is a rational number, because a and b are rational numbers. But $\sqrt{2}$ is an irrational number. Thus c must be an irrational number.

(b) Find three irrational numbers in the interval $[1, 1.1]$. Justify your answer.

For $k \in \mathbb{N}$, let $c_k := 1 + \frac{\sqrt{2}}{k}$. By part (a), each c_k is an irrational number. Since $0 < \sqrt{2} < 2$, we have

$$0 < \frac{\sqrt{2}}{k} < \frac{2}{20} \quad \text{for } k \geq 20.$$

Thus c_{20}, c_{21}, c_{22} are irrational numbers in $[1, 1.1]$.

2. (15 points) Let $f(x) := x^2 - 2\ln(1+x^2)$, $-\infty < x < \infty$.

(a) Determine the intervals where the function f is increasing or decreasing.

Solution. We have $f'(x) = 2x - 2 \cdot \frac{2x}{1+x^2} = \frac{2x(x+1)(x-1)}{1+x^2}$.

Consequently, $f'(x) < 0$ for $x \in (-\infty, -1)$, $f'(x) > 0$ for $x \in (-1, 0)$, $f'(x) < 0$ for $x \in (0,1)$, and $f'(x) > 0$ for $x \in (1, \infty)$.

By the mean value theorem,

f is strictly decreasing on $(-\infty, -1]$,
f is stricty increasing on $[-1, 0]$,
f is strictly decreasing on $[0, 1]$,
f is strictly increasing on $[1, \infty)$.

(b) Prove that there are two real numbers a_1 and a_2 such that $a_1 < a_2$ and that $f(x) \geq f(a_1) = f(a_2)$ for all $x \in (-\infty, \infty)$.

By part (a), $f(-1) \leq f(x)$ for all $x \in (-\infty, 0]$ and $f(1) \leq f(x)$ for all $x \in [0, \infty)$.

But $f(-1) = f(1)$. Therefore,

$f(-1) = f(1) \leq f(x)$ for all $x \in (-\infty, \infty)$.

(c) Prove that there exist three real numbers b_1, b_2, b_3 such that $b_1 < b_2 < b_3$ and that $f(b_1) = f(b_2) = f(b_3) = 0$.

We have $f(-1) = f(1) = 1 - 2\ln 2 < 0$, since $2\ln 2 = \ln 4 > 1$.

Moreover, $f(-3) = f(3) = 9 - 2\ln 10 = 9 - \ln 100 > 0$, since $\ln 100 < \ln 2^9 < \ln e^9 = 9$.

By the intermediate value theorem, there exist $b_1 \in (-3, -1)$ and $b_3 \in (1, 3)$ such that

$f(b_1) = 0$ and $f(b_3) = 0$.

Choosing $b_2 = 0$ we have $f(b_2) = 0$.

3. (10 points) Let f be defined by

$$f(x) := \begin{cases} -x^2 & \text{for } x < 0, \\ \frac{x \arctan x}{1+x} & \text{for } x \geq 0. \end{cases}$$

(a) Prove that f is strictly increasing on $(-\infty, \infty)$. Moreover, find the range of f. Justify your answer.

f is continuous on $(-\infty, \infty)$.

For $x < 0$ we have $f'(x) = -2x > 0$. By the mean value theorem, f is strictly increasing on $(-\infty, 0]$.

For $x > 0$ we have

$$f'(x) = \frac{(\arctan x + x \cdot \frac{1}{1+x^2})(1+x) - x \arctan x}{(1+x)^2}$$

$$= \frac{\arctan x + \frac{x(1+x)}{1+x^2}}{(1+x)^2} > 0.$$

By the mean value theorem, f is strictly increasing on $[0, \infty)$. Consequently, f is strictly increasing on $(-\infty, \infty)$. Since $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \frac{\pi}{2}$, the range of f is $(-\infty, \frac{\pi}{2})$.

(b) By part (a), f is a one-to-one function from $(-\infty, \infty)$ onto its range J. Let g be its inverse function. Find the value $b := f(1)$ explicitly. Moreover, find $g'(b)$.

We have $b = f(1) = \dfrac{\arctan 1}{2} = \dfrac{\pi}{8}$. Moreover,

$$f'(1) = \frac{\arctan 1 + 1}{4} = \frac{\frac{\pi}{4} + 1}{4} \neq 0.$$

Since $g(f(x)) = x$ for $x \in (-\infty, \infty)$, we have

$$g'(f(x))\, f'(x) = 1.$$

In particular, for $x = 1$ we obtain

$$g'(b) = \frac{1}{f'(1)} = \frac{4}{\frac{\pi}{4}+1} = \frac{16}{\pi + 4}.$$

4. (15 points) Let f be the function on $\mathbb{R}$ defined by

$$f(x) := \begin{cases} x^3 \sin(1/x^2) & \text{if } x \in \mathbb{R} \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Find $f'(x)$ for $x \in \mathbb{R} \setminus \{0\}$.

For $x \in \mathbb{R} \setminus \{0\}$ we have

$$f'(x) = 3x^2 \sin\frac{1}{x^2} + x^3 \cos\left(\frac{1}{x^2}\right)\left(-\frac{2}{x^3}\right)$$

$$= 3x^2 \sin\frac{1}{x^2} - 2\cos\frac{1}{x^2}.$$

(b) Prove that f is differentiable at 0 and find $f'(0)$.

For $h \neq 0$ we have $\dfrac{f(h)-f(0)}{h-0} = h^2 \sin\dfrac{1}{h^2}$.

Since $\left|\sin\frac{1}{h^2}\right| \leq 1$, we have $\left|h^2 \sin\frac{1}{h^2}\right| \leq h^2$.

By the squeeze theorem for limits, $\lim_{h \to 0} \dfrac{f(h)-f(0)}{h} = 0$.

Therefore f is differentiable at 0 and $f'(0) = 0$.

(c) Show that f' is not continuous at 0.

Let $x_n := \dfrac{1}{\sqrt{2n\pi}}$ for $n = 1, 2, \cdots$. Then

$$f'(x_n) = \frac{3}{2n\pi} \sin(2n\pi) - 2\cos(2n\pi) = -2.$$

We have $\lim_{n \to \infty} x_n = 0$, but $f'(x_n) = -2 \neq f'(0)$ for all $n \in \mathbb{N}$. Therefore f' is not continuous at 0.

5. (10 points) Let $g(t) := e^t$, $-\infty < t < \infty$.

(a) Find the third Taylor polynomial of g at 0 and the corresponding Lagrange form of the remainder.

We have $g^{(k)}(t) = e^t$ for $k = 0, 1, 2, \cdots$. It follows that $g^{(k)}(0) = 1$ for all $k \in \mathbb{N}_0$. Thus the third Taylor polynomial of g at 0 is

$$T_3(g,0)(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6}.$$

The corresponding remainder is

$$R_3(g,0)(t) = \frac{g^{(4)}(\xi)}{4!} t^4 = \frac{e^{\xi}}{24} t^4,$$

where ξ is between 0 and t.

(b) Let $f(x) := g(x^2) = e^{x^2}$, $-\infty < x < \infty$. Find a polynomial q of degree six such that $q^{(k)}(0) = f^{(k)}(0)$ for $k = 0, 1, 2, 3, 4, 5, 6$.

The Taylor series of g about 0 is

$$g(t) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n, \quad -\infty < t < \infty.$$

Consequently,

$$f(x) = e^{x^2} = g(x^2) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}, \quad -\infty < x < \infty.$$

Let $q(x) := \sum_{n=0}^{3} \frac{1}{n!} x^{2n} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6}$. Then

$$f(x) - q(x) = \sum_{n=4}^{\infty} \frac{1}{n!} x^{2n} = \frac{x^8}{4!} + \frac{x^{10}}{5!} + \cdots, \quad -\infty < x < \infty.$$

It follows that $(f-q)^{(k)}(0) = 0$ for $k = 0, 1, 2, 3, 4, 5, 6$.

Therefore $q^{(k)}(0) = f^{(k)}(0)$ for $k = 0, 1, 2, 3, 4, 5, 6$.

6. (10 points) The Fundamental Theorem of Calculus will be used in this problem.

(a) Let $F(x) := \int_1^{\ln x} \sqrt{t}e^{-t}\,dt$, $x \in (1, \infty)$. Find $F'(x)$ for $x > 1$. Simplify your solution.

For $x > 1$ we have

$$F'(x) = \sqrt{\ln x}\; e^{-\ln x} \frac{d}{dx}(\ln x)$$

$$= \sqrt{\ln x} \cdot \frac{1}{x}\left(\frac{1}{x}\right)$$

$$= \frac{\sqrt{\ln x}}{x^2}.$$

(b) Let $G(x) := \int_x^0 x\sin(t^2)\,dt$, $x \in \mathbb{R}$. Find $G''(x)$ for $x \in \mathbb{R}$.

We have $G(x) = -x\int_0^x \sin(t^2)\,dt.$

Hence $G'(x) = -\int_0^x \sin(t^2)\,dt - x\sin(x^2).$

Further,

$$G''(x) = -\sin(x^2) - \sin(x^2) - x\cos(x^2)(2x)$$

$$= -2\sin(x^2) - 2x^2\cos(x^2).$$

7. (15 points) Calculate the following integrals.

(a) $\int_0^1 (1-x^2)^8 x^3\,dx$

Let $u = 1-x^2$. Then $du = -2x\,dx$. When $x=0$, $u=1$.
When $x=1$, $u=0$. Hence

$$\int_0^1 (1-x^2)^8 x^3\,dx = \int_1^0 u^8(1-u)\left(-\tfrac{1}{2}\right)du$$

$$= \frac{1}{2}\int_0^1 (u^8-u^9)\,du = \frac{1}{2}\left[\frac{u^9}{9} - \frac{u^{10}}{10}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{9} - \frac{1}{10}\right) = \frac{1}{180}.$$

(b) $\int_1^e (\ln x)^2\,dx$.

Integration by parts gives

$$\int_1^e (\ln x)^2\,dx = x(\ln x)^2\Big|_1^e - \int_1^e x\cdot 2\ln x\cdot \frac{1}{x}\,dx$$

$$= e - 2\int_1^e \ln x\,dx = e - 2x\ln x\Big|_1^e + 2\int_1^e dx$$

$$= e - 2e + 2(e-1) = e - 2.$$

(c) $\int_{\pi/2}^{\pi} \sin^2 x\cos^2 x\,dx$.

We have $\sin^2 x\cos^2 x = (\sin x\cos x)^2 = \left[\frac{\sin(2x)}{2}\right]^2$

$$= \frac{\sin^2(2x)}{4} = \frac{1}{4}\,\frac{1-\cos(4x)}{2} = \frac{1-\cos(4x)}{8}.$$

Hence $\int_{\pi/2}^{\pi} \sin^2 x\cos^2 x\,dx = \int_{\pi/2}^{\pi} \frac{1-\cos(4x)}{8}\,dx$

$$= \left[\frac{x}{8} - \frac{\sin(4x)}{32}\right]_{\pi/2}^{\pi} = \frac{\pi}{16}.$$

8. (15 points) Consider the power series

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{n}(x-1)^n.$$

(a) Let I be the interval of convergence of this power series. Find I.

Let $u_n := \frac{2^{n-1}}{n}(x-1)^n$ for $n = 1, 2, \cdots$. For $x \neq 1$ we have

$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{2^n|x-1|^{n+1}}{n+1} \frac{n}{2^{n-1}|x-1|^n} = \lim_{n\to\infty} 2|x-1|\frac{n}{n+1} = 2|x-1|.$

By the ratio test, the series converges if $2|x-1| < 1$ and diverges if $2|x-1| > 1$.

When $x = \frac{1}{2}$, $\sum_{n=1}^{\infty} \frac{2^{n-1}}{n}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$ converges, by the alternating series test. When $x = \frac{3}{2}$, $\sum_{n=1}^{\infty} \frac{2^{n-1}}{n}(x-1)^n = \sum_{n=1}^{\infty} \frac{1}{2n}$ diverges, because the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Thus $I = [\frac{1}{2}, \frac{3}{2})$.

(b) We use I° to denote the interior of I. For $x \in I^\circ$, let $f(x)$ be the sum of the above power series. Let $g = f'$. Find the power series expansion of g about 1.

For $x \in I^\circ = (\frac{1}{2}, \frac{3}{2})$ we have

$$g(x) = f'(x) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{n} \cdot n(x-1)^{n-1} = \sum_{n=1}^{\infty} 2^{n-1}(x-1)^{n-1}.$$

(c) For $x \in I^\circ$, find $g(x)$ explicitly. Moreover, find an explicit expression of f on I°.

We observe that $\sum_{n=1}^{\infty} 2^{n-1}(x-1)^{n-1}$ is a geometric series. Its initial term is $2^{1-1}(x-1)^{1-1} = 1$, and its ratio is $2(x-1)$. Hence

$$g(x) = \sum_{n=1}^{\infty} 2^{n-1}(x-1)^{n-1} = \frac{1}{1-2(x-1)} = \frac{1}{3-2x}, \quad \frac{1}{2} < x < \frac{3}{2}.$$

But $f'(x) = g(x)$. Consequently, $f(x) = \int \frac{1}{3-2x}\,dx = -\frac{1}{2}\ln(3-2x) + C$. We have $f(1) = 0$. So $-\frac{1}{2}\ln(3-2) + C = 0$. It follows that $C = 0$. Therefore

$$f(x) = -\frac{1}{2}\ln(3-2x), \quad \frac{1}{2} < x < \frac{3}{2}.$$