The Midterm Exam (Math 314 A1)

October 27, 2016

Name: ______________________ I.D.#: ______________

1. (12 points) (a) Use the interval notation to express the set $\{x \in \mathbb{R} : |2 - 3x| \geq 8\}$.

Note that $|2-3x| \geq 8$ if and only if $2-3x \leq -8$ or $2-3x \geq 8$.
Clearly, $2-3x \leq -8 \Leftrightarrow -3x \leq -8-2 \Leftrightarrow -3x \leq -10 \Leftrightarrow x \geq \frac{10}{3}$.
Moreover, $2-3x \geq 8 \Leftrightarrow -3x \geq 8-2 \Leftrightarrow -3x \geq 6 \Leftrightarrow x \leq -2$.
Thus
$$\{x \in \mathbb{R} : |2-3x| \geq 8\} = (-\infty, -2] \cup [\tfrac{10}{3}, \infty).$$

(b) Let a, b, c, and d be real numbers such that $bd > 0$. Prove

$$\frac{a}{b} < \frac{c}{d} \quad \Longrightarrow \quad \frac{a}{b} < \frac{a+2c}{b+2d}.$$

Proof. Multiplying both sides of the inequality $\frac{a}{b} < \frac{c}{d}$ by $bd > 0$ we obtain
$$bd\left(\frac{a}{b}\right) < bd\left(\frac{c}{d}\right), \quad \text{i.e.,} \quad ad < bc.$$
Since $b(b+2d) = b^2 + 2bd > 0$, we have
$$\frac{a}{b} < \frac{a+2c}{b+2d} \Leftrightarrow a(b+2d) < b(a+2c) \Leftrightarrow ab+2ad < ab+2bc.$$
But
$$ad < bc \Rightarrow 2ad < 2bc \Rightarrow ab+2ad < ab+2bc.$$
Therefore,
$$\frac{a}{b} < \frac{c}{d} \Rightarrow \frac{a}{b} < \frac{a+2c}{b+2d}.$$

2. (18 points) (a) Find two irrational numbers in the interval $[1, 1.1]$. Justify your answer.

Solution. It is known that $\sqrt{2}$ is an irrational number. Moreover, if a and b are rational numbers with $b \neq 0$, then $a + b\sqrt{2}$ is an irrational number. Note that $1 < \sqrt{2} < 2$. So, for $k = 20, 21, \cdots$, $1 + \frac{\sqrt{2}}{k}$ is an irrational number in $[1, 1.1]$.

(b) Let $(a_n)_{n=1,2,\ldots}$ be a strictly increasing sequence of real numbers. Prove that the set $A := \{a_n : n \in \mathbb{N}\}$ has a minimum but does not have a maximum. Justify your answer.

We have $a_1 \leqslant a_n$ for all $n \in \mathbb{N}$. Hence a_1 is the minimum of A. If A has a maximum M, then $M = a_{n_0}$ for some $n_0 \in \mathbb{N}$. But the sequence $(a_n)_{n=1,2,\cdots}$ is strictly increasing. So $a_{n_0+1} > a_{n_0} = M$. Thus M would not be a maximum. This contradiction shows that A does not have a maximum.

(c) Let $b_n := [1 - (-1)^n]n - 3/n$, $n = 1, 2, \ldots$, and let B be the set $\{b_n : n = 1, 2, \ldots\}$. Find $\sup B$ and $\inf B$. Justify your answer.

For $k = 0, 1, 2, \cdots$ we have

$$b_{2k+1} = [1 - (-1)^{2k+1}](2k+1) - \frac{3}{2k+1} = 2(2k+1) - \frac{3}{2k+1}.$$

It follows that $\lim_{k \to \infty} b_{2k+1} = \infty$. Hence $\sup B = +\infty$.

We claim that $\inf B = b_2 = -\frac{3}{2}$.

Indeed, if $n = 2k$ ($k = 1, 2, \cdots$) is even, then

$$b_{2k} = [1 - (-1)^{2k}](2k) - \frac{3}{2k} = -\frac{3}{2k} \geqslant -\frac{3}{2},$$

since $2k \geqslant 2$. If $n = 2k+1$ ($k = 0, 1, \cdots$) is odd, then

$$b_{2k+1} = 2(2k+1) - \frac{3}{2k+1} \geqslant 2 - 3 = -1 > b_2.$$

This justifies our claim.

3. (18 points) Let $a_1 := 1$ and set $a_{n+1} := \sqrt{6 + a_n}$ for $n = 1, 2, \ldots$.

(a) Use mathematical induction to prove that $1 \le a_n < 4$ for all $n \in \mathbb{N}$.

For $n=1$ we have $a_1 = 1$, so $1 \leq a_n < 4$ is true for $n=1$. Suppose that $1 \leq a_n < 4$ is true. Then $a_{n+1} = \sqrt{6+a_n}$ satisfies $\sqrt{6+1} \leq a_{n+1} < \sqrt{6+4}$. But $\sqrt{6+1} \geqslant 1$ and $\sqrt{6+1} < 4$. Hence $1 \leq a_{n+1} < 4$. By the Principle of Mathematical Induction, $1 \leq a_n < 4$ for all $n \in \mathbb{N}$.

(b) Use mathematical induction to show that $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.

For $n=1$ we have $a_2 = \sqrt{6+a_1} = \sqrt{6+1} > 1 = a_1$.
Suppose that $a_{n+1} > a_n$ is true. Then

$$a_{n+2} = \sqrt{6+a_{n+1}} > \sqrt{6+a_n} = a_{n+1}.$$

This completes the induction procedure. Therefore, $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.

(c) As a bounded increasing sequence, $(a_n)_{n=1,2,\ldots}$ converges. Find $\lim_{n\to\infty} a_n$.

By (a) and (b), $(a_n)_{n=1,2,\ldots}$ is a bounded increasing sequence. Hence $(a_n)_{n=1,2,\cdots}$ converges, say $\lim\limits_{n\to\infty} a_n = s$. It follows from $a_{n+1} = \sqrt{6+a_n}$ that $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{6+a_n}$. Thus $s = \sqrt{6+s}$. Consequently, $s^2 - s - 6 = 0$. Either $s=3$ or $s=-2$. But $s = \lim\limits_{n\to\infty} a_n \geqslant 1$. We conclude that $\lim\limits_{n\to\infty} a_n = 3$.

4. (18 points) (a) Find the following limit. Justify your answer.

$$\lim_{n\to\infty}\left(3^{-n}-\frac{(-1)^n}{\sqrt{n}}\right).$$

We have $\lim\limits_{n\to\infty} 3^{-n} = \lim\limits_{n\to\infty} \left(\frac{1}{3}\right)^n = 0.$

Moreover, $-\frac{1}{\sqrt{n}} \leq \frac{(-1)^n}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$ and $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$

By the squeeze theorem for sequences, $\lim\limits_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} = 0.$

We conclude that $\lim\limits_{n\to\infty} \left(3^{-n} - \frac{(-1)^n}{\sqrt{n}}\right) = 0.$

(b) Find the following limit. Justify your answer.

$$\lim_{x\to\infty}\log_2\left(\frac{x^2+3x-4}{4x^2-x+7}\right).$$

We have

$$\lim_{x\to\infty} \frac{x^2+3x-4}{4x^2-x+7} = \lim_{x\to\infty} \frac{x^2\left(1+\frac{3}{x}-\frac{4}{x^2}\right)}{x^2\left(4-\frac{1}{x}+\frac{7}{x^2}\right)} = \frac{1}{4}.$$

Since the function $y \mapsto \log_2 y$ is continuous on $(0,\infty)$, it follows that

$$\lim_{x\to\infty} \log_2\left(\frac{x^2+3x-4}{4x^2-x+7}\right) = \log_2 \frac{1}{4} = 2.$$

(c) Find the sum of the following series:

$$\sum_{n=1}^{\infty}\frac{(-3)^{n-1}-2}{4^n}.$$

We have $\sum\limits_{n=1}^{\infty} \frac{(-3)^{n-1}-2}{4^n} = \sum\limits_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} - \sum\limits_{n=1}^{\infty} \frac{2}{4^n}.$

$\sum\limits_{n=1}^{\infty} \frac{(-3)^{n-1}}{4}$ is a geometric series. Its first term is $\frac{1}{4}$ and its ratio is $\frac{-3}{4}$. So $\sum\limits_{n=1}^{\infty} \frac{(-3)^{n-1}}{4} = \frac{\frac{1}{4}}{1-\left(\frac{-3}{4}\right)} = \frac{1}{7}.$

Similarly, $\sum\limits_{n=1}^{\infty} \frac{2}{4^n} = \frac{\frac{2}{4}}{1-\frac{1}{4}} = \frac{2}{3}.$ Finally

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}-2}{4^n} = \frac{1}{7} - \frac{2}{3} = -\frac{11}{21}.$$

5. (18 points) (a) Let f be the function defined on $\mathbb{R}$ by $f(x) := 3^x - x^3 - 10$, $x \in \mathbb{R}$. Prove that there exists some $c \in \mathbb{R}$ such that $f(c) = 0$.

We have $f(0) = 3^0 - 0^3 - 10 < 0$ and

$$f(4) = 3^4 - 4^3 - 10 = 81 - 64 - 10 > 0.$$

By the intermediate value theorem, there exists some $c \in [0,4]$ such that $f(c) = 0$.

(b) Let g be the function defined on $\mathbb{R}$ by $g(x) := x^4 - x^2$, $x \in \mathbb{R}$. Show that $g(x) \geq 0$ whenever $|x| \geq 1$. Moreover, prove that g attains its minimum on $(-\infty, \infty)$. In other words, there exists a real number t such that $g(x) \geq g(t)$ for all $x \in (-\infty, \infty)$.

We have $g(x) = x^4 - x^2 = x^2(x^2-1)$. For $|x| \geqslant 1$, $x^2 - 1 = |x|^2 - 1 \geqslant 0$. Hence $g(x) = x^2(x^2-1) \geqslant 0$ for $|x| \geqslant 1$.

As a continuous function, g attains its minimum on the closed interval $[-1, 1]$. In other words, there exists some $t \in [-1, 1]$ such that $g(t) \leq g(x)$ for all $x \in [-1, 1]$. In particular, $g(t) \leq g(0) = 0$. But $0 \leq g(x)$ for all $|x| \geqslant 1$. Therefore,

$$g(t) \leq g(x) \text{ for all } x \in (-\infty, \infty).$$

(c) Let h be the function given by $h(x) := \sqrt{x}$ for $x \geq 1/2$. Prove that h is uniformly continuous on $[1/2, \infty)$.

We have $|h(x) - h(y)| = |\sqrt{x} - \sqrt{y}| = \left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| \leqslant |x-y|$, since $\sqrt{x} + \sqrt{y} \geqslant \sqrt{1/2} + \sqrt{1/2} \geqslant 1$ for $x, y \in [\frac{1}{2}, \infty)$.

For given $\varepsilon > 0$, choose $\delta = \varepsilon$. Then $|h(x) - h(y)| < \varepsilon$ whenever $x, y \in [\frac{1}{2}, \infty)$ and $|x-y| < \delta$. Therefore, h is uniformly continuous on $[\frac{1}{2}, \infty)$.

6. (16 points) (a) Let

$$f(x) := \begin{cases} \frac{|x|}{x} & \text{for } x \in \mathbb{R} \setminus \{0\}, \\ 1/2 & \text{for } x = 0. \end{cases}$$

Let $g(x) := xf(x)$ for $x \in \mathbb{R}$. Is f continuous? Is g continuous? Justify your answers.

We have $f(x) = 1$ for $x > 0$ and $f(x) = -1$ for $x < 0$.
It follows that $\lim_{x \to 0^+} f(x) = 1 \neq f(0)$. Hence f is not continuous.

The function g is continuous on $(-\infty, 0)$ and $(0, \infty)$.
At the point 0 we have $g(0) = 0$, $\lim_{x \to 0^-} g(x) = 0$, and $\lim_{x \to 0^+} g(x) = 0$.
Therefore, g is continuous on $\mathbb{R}$.

(b) Let

$$u(x) := \frac{x}{1-x}, \quad x \in [0, 1).$$

Prove that u is strictly increasing on $[0, 1)$. Find the range of the function u. Further, find an explicit expression for the inverse function u^{-1} including its domain.

We wish to show $u(a) < u(b)$ whenever $0 \leq a < b < 1$.
Note that

$$u(a) < u(b) \Leftrightarrow \frac{a}{1-a} < \frac{b}{1-b} \Leftrightarrow a(1-b) < b(1-a)$$

since $1-a > 0$ and $1-b > 0$. But $a < b \Rightarrow a - ab < b - ba \Rightarrow a(1-b) < b(1-a)$.
This shows $u(a) < u(b)$ whenever $0 \leq a < b < 1$.

Since u is strictly increasing and $\lim_{x \to 1^-} \frac{x}{1-x} = \infty$, the range of u is $[u(0), \infty) = [0, \infty)$.

Since u is one-to-one and onto $[0, \infty)$, u^{-1} exists and its domain is $[0, \infty)$. Suppose $u^{-1}(y) = x$. Then $y = u(x) = \frac{x}{1-x}$. It follows that $y(1-x) = x$, i.e., $x = \frac{y}{1+y}$. We conclude that

$$u(y) = \frac{y}{1+y}, \quad y \in [0, \infty).$$