MATH 314 Assignment #3

due on Friday, September 30, 2016

1. (a) Prove that $\lim_{n\to\infty} \sqrt{n} = \infty$ and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

   (b) Prove that if $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} |a_n| = |a|$. Is the converse true? Justify your answer.

2. For each sequence below find its limit and determine whether it converges.

   (a) $a_n = \dfrac{3n^2 - 2n^3}{5 + n^3 + 2n}$ (b) $b_n = \dfrac{1 - n^9}{100n^8 + 9n^2}$

   (c) $c_n = \dfrac{3^n + 2^n}{3^n - 4^n}$ (d) $d_n = \sqrt{n+2} - \sqrt{n}$

3. Let $x_n := \sqrt{n^2 + n} - n$ for $n \in \mathbb{N}$.

   (a) Prove that
   $$x_n = \frac{n}{\sqrt{n^2 + n} + n}.$$

   (b) Show that $2n \leq \sqrt{n^2 + n} + n \leq 2n + 1$.

   (c) Deduce from (a) and (b) that
   $$\frac{n}{2n+1} \leq x_n \leq \frac{1}{2}.$$

   (d) Find $\lim_{n\to\infty} x_n$.

4. Let $a_1 := 1$ and set $a_{n+1} := (2a_n + 5)/6$ for $n = 1, 2, \ldots$.

   (a) Find the first five terms of the sequence $(a_n)_{n=1,2,\ldots}$.

   (b) Use mathematical induction to prove that $a_n \leq 2$ for all $n \in \mathbb{N}$.

   (c) Use mathematical induction to show that the sequence $(a_n)_{n=1,2,\ldots}$ is increasing.

   (d) Prove that the sequence $(a_n)_{n=1,2,\ldots}$ is convergent and find $\lim_{n\to\infty} a_n$.

5. Let $b_1 := 1$ and set $b_{n+1} := \sqrt{2b_n}$ for $n = 1, 2, \ldots$.

   (a) Prove that the sequence $(b_n)_{n=1,2,\ldots}$ is increasing and bounded above by 2.

   (b) Show that the sequence $(b_n)_{n=1,2,\ldots}$ is convergent and find $\lim_{n\to\infty} b_n$.