MATH 314 Assignment #4

due on Friday, October 7, 2016

1. (a) Let $a_n := 2(-1)^{n+1} + (-1)^{n(n+1)/2}$ for $n \in \mathbb{N}$. Find four subsequences of $(a_n)_{n=1,2,...}$ such that they converge to different limits.

   (b) Let $b_n := [1 + (-1)^n]n + 100/n$ for $n \in \mathbb{N}$. Find an increasing subsequence of $(b_n)_{n=1,2,...}$. Also, find a convergent subsequence of $(b_n)_{n=1,2,...}$.

2. Let $(x_n)_{n=1,2,...}$ be the sequence recursively defined by $x_1 := 1$ and

$$x_{n+1} := \frac{1}{4}(x_n^2 + 2), \quad n \in \mathbb{N}.$$

   (a) Show that $0 < x_n \leq 1$ for all $n \in \mathbb{N}$.

   (b) Prove that the sequence $(x_n)_{n=1,2,...}$ is contractive.

   (c) Show that the sequence $(x_n)_{n=1,2,...}$ converges and find its limit.

3. Find the sum of the following series.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{10 + (-3)^n}{5^{n-1}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 3^n - 3 \cdot 2^n}{6^n}$

   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(2n-1)(2n+1)}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+2)}$

4. Test each of the following series for convergence or divergence. If the series converges, determine whether it converges absolutely or conditionally. Justify your conclusions.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{1/n}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{10}{n}\right)$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$

5. Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two convergent series.

   (a) Show that the sequence $(b_n)_{n=1,2,...}$ is bounded.

   (b) If, in addition, $\sum_{n=1}^{\infty} a_n$ converges absolutely, prove that the series $\sum_{n=1}^{\infty} a_n b_n$ also converges absolutely.

   (c) Give an example of two conditionally convergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that the series $\sum_{n=1}^{\infty} a_n b_n$ diverges.