# MATH 314 Assignment #6

due on Friday, October 21, 2016

1. Let $f$ be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that $\lim_{|x|\to\infty} f(x) = \infty$, that is, for any real number $M$, there exists a positive real number $K$ such that $f(x) > M$ whenever $|x| \geq K$.
   (a) Fix a point $x_0 \in \mathbb{R}$. Prove that there exists a positive real number $a$ such that $-a \leq x_0 \leq a$ and that $f(x) \geq f(x_0)$ whenever $x \notin [-a, a]$.
   (b) Show that there exists some $c \in \mathbb{R}$ such that $f(c) \leq f(x)$ for all $x \in \mathbb{R}$, that is, $f$ attains its minimum at $c$.
2. The Intermediate Value Theorem will be used in the following problems.
   (a) Show that the equation $2^x = 3x$ has a solution $c \in (1, 4)$.
   (b) Let $p$ be a cubic polynomial, *i.e.*, $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, $x \in \mathbb{R}$, where $a_3 \neq 0$. Prove that $p$ has at least one real root.
3. (a) Suppose that $f$ is a continuous function from $[0, \infty)$ to $\mathbb{R}$. Moreover, there exists some $a > 0$ such that $f$ is uniformly continuous on $[a, \infty)$. Prove that $f$ is uniformly continuous on $[0, \infty)$.
   (b) Let $g$ be the function from $[0, \infty)$ to $\mathbb{R}$ given by $g(x) = \sqrt{x}$, $x \geq 0$. Prove that $g$ is uniformly continuous on $[0, \infty)$.
4. Let $f$ be a real-valued function defined by

$$f(x) = \begin{cases} 2^x & \text{for } 0 \leq x \leq 1, \\ 3 - 1/x^2 & \text{for } 1 < x \leq 2. \end{cases}$$

   (a) Show that $f$ is continuous and strictly increasing on $[0, 2]$.
   (b) Find an explicit expression for the inverse function $f^{-1}$ including its domain and range.
   (c) Is $f^{-1}$ continuous on its domain? Justify your answer.
5. (a) Let $f$ be the function given by

$$f(x) := \begin{cases} 3^{-1/|x|} & \text{for } x \in (-\infty, 0) \cup (0, \infty), \\ 0 & \text{for } x = 0. \end{cases}$$

   Find $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$. Is $f$ continuous on $(-\infty, \infty)$? Justify your answer.
   (b) Let $g(x) := \log_2 x$ for $0 < x < \infty$. Prove that $g$ is *not* uniformly continuous on the interval $(0, 1)$.