MATH 314 Assignment #7

due on Wednesday, November 16, 2016

1. Let $f(x) := x^2$ for $x \geq 0$ and $f(x) := 0$ for $x < 0$.
   (a) Use the definition of derivative to show that f is differentiable at 0.
   (b) Find an explicit expression of $f'(x)$ for $x \in \mathbb{R}$.
   (c) Is f' continuous on $\mathbb{R}$? Is f' differentiable on $\mathbb{R}$?

2. Find the derivative of each of the following functions.
   (a) $g(x) := \sqrt[3]{x^2} - \frac{1}{x}$, $x \neq 0$.
   (b) $h(x) := \frac{1-x^2}{1+x^2}$, $x \in \mathbb{R}$.
   (c) $u(x) := \ln(x + \sqrt{a^2 + x^2})$, $x \in \mathbb{R}$.
   (d) $v(x) := x^x$, $x > 0$.

3. Let f be a real-valued function on an open interval I, and let c be a point in I. Prove the following statements.
   (a) If f is differentiable at c, then $\lim_{n\to\infty} n\big[f\big(c + \frac{1}{n}\big) - f(c)\big] = f'(c)$.
   (b) If f is differentiable at c, then

$$\lim_{h\to 0} \frac{f(c+h) - f(c-h)}{2h} = f'(c).$$

4. Let $f(x) := x \ln\big(1 + \frac{1}{x}\big)$, $0 < x < \infty$.
   (a) Compute $\lim_{x\to\infty} f(x)$.
   (b) Find $f'(x)$ and $f''(x)$ for $0 < x < \infty$.
   (c) Prove that f' is strictly decreasing on $(0, \infty)$ and f is strictly increasing on $(0, \infty)$.

5. Let f be the function on $\mathbb{R}$ given by

$$f(x) := \begin{cases} e^{-1/x} & \text{for } x \in (0, \infty), \\ 0 & \text{for } x \in (-\infty, 0]. \end{cases}$$

   (a) Compute $f'(x)$ and $f''(x)$ for $x > 0$.
   (b) Show that f is differentiable at 0 and find $f'(0)$.
   (c) Show that f' is differentiable at 0 and find $f''(0)$.