MATH 314 Assignment #9

1. Let f be an increasing function on $[a,b]$ with $-\infty < a < b < \infty$.

   (a) Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a,b]$. Prove

   $$U(f,P) - L(f,P) \leq \sum_{i=1}^{n} [f(t_i) - f(t_{i-1})](t_i - t_{i-1}).$$

   (b) Prove that $U(f,P) - L(f,P) \leq [f(b) - f(a)]\delta$ whenever $\|P\| < \delta$.

   (c) Prove that f is integrable on $[a,b]$.

2. Let g be the function on $[0,1]$ defined by $g(0) := 0$ and

   $$g(x) := 2^{-n} \text{ for } 2^{-n-1} < x \leq 2^{-n}, \quad n = 0, 1, 2, \ldots.$$

   (a) Prove that g is integrable on $[0,1]$.

   (b) Find $\int_0^1 g(x)\,dx$.

3. Let f be the function on $\mathbb{R}$ defined as follows: $f(x) := 0$ for $x < 0$; $f(x) := x$ for $0 \leq x \leq 1$; $f(x) := 2$ for $x > 1$.

   (a) Find an explicit expression of the function $F(x) := \int_0^x f(t)\,dt$, $x \in \mathbb{R}$.

   (b) Is F continuous on $\mathbb{R}$? Justify your answer.

   (c) Where is F differentiable? Calculate F' at the points of differentiability.

4. (a) Let $G(x) := \int_{-x}^{x^2} \sqrt{1+t^2}\,dt$, $x \in \mathbb{R}$. Find $G'(x)$ for $x \in \mathbb{R}$.

   (b) Let $H(x) := \int_0^x xe^{t^2}\,dt$ for $x \in \mathbb{R}$. Find $H''(x)$ for $x \in \mathbb{R}$.

5. Let

   $$F(x) := \int_x^{x+\pi} |\cos t|\,dt, \quad x \in \mathbb{R}.$$

   (a) Find $F'(x)$ for $x \in \mathbb{R}$.

   (b) Find an explicit expression for $F(x)$, $x \in \mathbb{R}$.