MATH 314 Assignment #1

1. Let A, B, C, and X be sets. Prove the following statements:

   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

   Proof. Suppose $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$, then x belongs to both $A \cup B$ and $A \cup C$; hence, $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in B$ and $x \in C$; hence, we also have $x \in (A \cup B) \cap (A \cup C)$.
   Conversely, suppose $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then $x \in A \cup (B \cap C)$. If $x \notin A$, then we must have $x \in B$ and $x \in C$. Hence, $x \in B \cap C$ and so $x \in A \cup (B \cap C)$.

   (b) $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

   Proof. Suppose $x \in X \setminus (A \cap B)$. Then $x \in X$ and $x \notin A \cap B$. It follows that $x \notin A$ or $x \notin B$. Hence, $x \in X \setminus A$ or $x \in X \setminus B$, that is, $x \in (X \setminus A) \cup (X \setminus B)$. Conversely, suppose $x \in (X \setminus A) \cup (X \setminus B)$. Then $x \in X \setminus A$ or $x \in X \setminus B$. It follows that $x \in X$, $x \notin A$ or $x \notin B$. Hence, $x \notin A \cap B$, and thereby $x \in X \setminus (A \cap B)$.

2. Use the principle of mathematical induction to prove the following statements:

   (a) $1 + 3 + \cdots + (2n - 1) = n^2$ for all $n \in \mathbf{N}$.

   Proof. Our nth proposition is P_n : "$1 + 3 + \cdots + (2n - 1) = n^2$". Thus P_1 asserts that $1 = 1^2$. This is obviously true. For the induction step, suppose that P_n is true, *i.e.*, $1 + 3 + \cdots + (2n - 1) = n^2$. Since we wish to prove P_{n+1} from this, we add $2n + 1$ to both sides to obtain

   $$1 + 3 + \cdots + (2n - 1) + (2n + 1) = n^2 + 2n + 1 = (n + 1)^2.$$

   Thus, P_{n+1} holds if P_n holds. By the principle of mathematical induction, we conclude that P_n is true for all n.

   (b) $2^n > n^2$ for all $n \geq 5$.

   Proof. For $n = 5$, we have $2^n = 32$ and $n^2 = 25$. So $2^n > n^2$ for $n = 5$. For the induction step, suppose that $2^n > n^2$ and $n \geq 5$. It follows that $2^{n+1} = 2 \cdot 2^n > 2n^2$. For $n \geq 5$ we have

   $$2n^2 - (n+1)^2 = 2n^2 - (n^2 + 2n + 1) = n^2 - 2n - 1 = (n - 1)^2 - 2 \geq (5 - 1)^2 - 2 > 0.$$

   Hence $2^{n+1} > 2n^2 > (n + 1)^2$. This completes the induction procedure.

3. Let A, B, C be sets, and let $f : A \to B$ and $g : B \to C$ be functions. Prove the following statements.

   (a) If f and g are injective, then $g \circ f$ is injective.

   Proof. Suppose that f and g are injective. Let $x, y \in A$. If $(g \circ f)(x) = (g \circ f)(y)$, then $g(f(x)) = g(f(y))$. Since g is injective, we have $f(x) = f(y)$. Further, since f is injective, it follows that $x = y$. Therefore, $g \circ f$ is injective.

   (b) If f and g are surjective, then $g \circ f$ is surjective.

   Proof. Suppose that f and g are surjective. Let c be an arbitrary element of C. Since $g : B \to C$ is surjective, there exists some $b \in B$ such that $g(b) = c$. Further, since $f : A \to B$ is surjective, there exists some $a \in A$ such that $f(a) = b$. Consequently, $(g \circ f)(a) = g(f(a)) = g(b) = c$. This shows that $g \circ f$ is surjective.

   (c) If f and g are bijective, then $g \circ f$ is bijective.

   Proof. Suppose that f and g are bijective. Then they are injective and surjective. By (a) and (b), $g \circ f$ is both injective and surjective. Therefore $g \circ f$ is bijective.

   (d) If f and g are bijective, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

   Proof. Suppose that $f : A \to B$ and $g : B \to C$ are bijective. Then both the functions $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ map C to A. Let $c \in C$, $b = g^{-1}(c)$, and $a = f^{-1}(b)$. Then $g(b) = c$ and $f(a) = b$. It follows that $(g \circ f)(a) = g(f(a)) = g(b) = c$. Hence $(g \circ f)^{-1}(c) = a$. This shows that $(g \circ f)^{-1}(c) = (f^{-1} \circ g^{-1})(c)$ for all $c \in C$. Therefore $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

4. Let a and b be two elements of an ordered commutative ring. Prove the following statements.

   (a) $|a| - |b| \leq |a - b|$.

   Proof. By the triangle inequality we have $|a| = |(a - b) + b| \leq |a - b| + |b|$. It follows that $|a| - |b| \leq |a - b|$.

   (b) $\big||a| - |b|\big| \leq |a - b|$.

   Proof. If $|a| \geq |b|$, then $\big||a| - |b|\big| = |a| - |b| \leq |a - b|$, by part (a). If $|a| < |b|$, then

$$\big||a| - |b|\big| = |b| - |a| \leq |b - a| = |a - b|.$$

   (c) $2 \max\{a, b\} = (a + b) + |a - b|$.

Proof. If $a \geq b$, then $\max\{a, b\} = a$ and $|a - b| = a - b$. Hence

$$(a + b) + |a - b| = (a + b) + (a - b) = 2a = 2\max\{a, b\}.$$

If $a < b$, then $\max\{a, b\} = b$ and $|a - b| = -(a - b)$. Hence

$$(a + b) + |a - b| = (a + b) - (a - b) = 2b = 2\max\{a, b\}.$$

(d) $2\min\{a, b\} = (a + b) - |a - b|$.

Proof. If $a \geq b$, then $\min\{a, b\} = b$ and $|a - b| = a - b$. Hence

$$(a + b) - |a - b| = (a + b) - (a - b) = 2b = 2\min\{a, b\}.$$

If $a < b$, then $\min\{a, b\} = a$ and $|a - b| = -(a - b)$. Hence

$$(a + b) - |a - b| = (a + b) + (a - b) = 2a = 2\min\{a, b\}.$$

5. Let a, b, c, and d be elements of an ordered field. Prove the following statements.

(a) If $bd > 0$, then $a/b < c/d \Leftrightarrow ad - bc < 0$.

Proof. If $bd > 0$, then

$$a/b < c/d \Leftrightarrow (bd)(a/b) < (bd)(c/d) \Leftrightarrow ad < bc \Leftrightarrow ad - bc < 0.$$

(b) If $bd > 0$ and $a/b < c/d$, then

$$\frac{a}{b} < \frac{a + c}{b + d}.$$

Proof. Suppose that $bd > 0$ and $a/b < c/d$. By (a) we have $ad - bc < 0$. It follows that

$$a(b + d) - b(a + c) = ab + ad - ba - bc = ad - bc < 0.$$

Note that $b(b + d) = b^2 + bd > 0$, since $bd > 0$. Applying (a) again, we obtain $a/b < (a + c)/(b + d)$.