MATH 314 Assignment #10

1. Calculate the following integrals.

(a) $\displaystyle\int_0^1 e^{-\sqrt{x}}\,dx$.

Solution. Use change of variable: $x = t^2$ for $0 \le t \le 1$. When $t = 0$, $x = 0$, and when $t = 1$, $x = 1$. The substitution together with integration by parts gives

$$\int_0^1 e^{-\sqrt{x}}\,dx = \int_0^1 e^{-t}(2t)\,dt = -2te^{-t}\Big|_0^1 + \int_0^1 2e^{-t}\,dt = -2e^{-1} - 2e^{-1} + 2 = 2 - \frac{4}{e}.$$

(b) $\displaystyle\int_{1/e}^{e} \big|\ln x\big|\,dx$.

Solution. If $1/e \le x \le 1$, then $\ln x \le \ln 1 = 0$ and hence $|\ln x| = -\ln x$. If $1 \le x \le e$, then $\ln x \ge \ln 1 = 0$ and hence $|\ln x| = \ln x$. Thus

$$\int_{1/e}^{e} \big|\ln x\big|\,dx = \int_{1/e}^{1} -\ln x\,dx + \int_1^e \ln x\,dx.$$

Integration by parts gives

$$\int \ln x\,dx = x\ln x - \int x \cdot \frac{1}{x}\,dx = x\ln x - x + C.$$

Consequently we have

$$\int_{1/e}^{1} -\ln x\,dx = \big[-x\ln x + x\big]_{1/e}^{1} = 1 - \Big[-\frac{1}{e}\ln\frac{1}{e} + \frac{1}{e}\Big] = 1 - \frac{2}{e}$$

and

$$\int_1^e \ln x\,dx = \big[x\ln x - x\big]_1^e = [e\ln e - e] - [1\ln 1 - 1] = 1.$$

Finally, we obtain

$$\int_{1/e}^{e} \big|\ln x\big|\,dx = \int_{1/e}^{1} -\ln x\,dx + \int_1^e \ln x\,dx = 1 - \frac{2}{e} + 1 = 2 - \frac{2}{e}.$$

(c) $\displaystyle\int_0^{\pi} x^2\cos x\,dx$.

Solution. Integration by parts gives

$$\int_0^{\pi} x^2\cos x\,dx = x^2\sin x\Big|_0^{\pi} - \int_0^{\pi} (\sin x)2x\,dx = -\int_0^{\pi} 2x\sin x\,dx.$$

Using integration by parts again, we obtain

$$-\int_0^\pi 2x\sin x\,dx = 2x\cos x\Big|_0^\pi - 2\int_0^\pi \cos x\,dx = -2\pi - 2\sin x\Big|_0^\pi = -2\pi.$$

Thus

$$\int_0^\pi x^2\cos x\,dx = -2\pi.$$

(d) $\displaystyle\int_1^{\sqrt{3}} \arctan x\,dx.$

Solution. Integration by parts gives

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

It follows that

$$\int_1^{\sqrt{3}} \arctan x\,dx = \left[x\arctan x - \frac{1}{2}\ln(1+x^2)\right]_1^{\sqrt{3}} = \frac{\sqrt{3}\pi}{3} - \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

2. Find the following integrals.

(a) $\displaystyle\int_0^2 x^3\sqrt{4-x^2}\,dx.$

Solution. Let $u = 4-x^2$. Then $du = -2xdx$ and $x^3dx = x^2xdx = (u-4)/2du$. When $x = 0$, $u = 4$. When $x = 2$, $u = 0$. Hence

$$\int_0^2 x^3\sqrt{4-x^2}\,dx = \int_4^0 \frac{u-4}{2}\sqrt{u}\,du = \int_0^4 \left(2u^{1/2} - u^{3/2}/2\right)du$$
$$= \left[\frac{4}{3}u^{3/2} - \frac{1}{5}u^{5/2}\right]_0^4 = \frac{32}{3} - \frac{32}{5} = \frac{64}{15}.$$

(b) $\displaystyle\int_0^1 x^2(1-x)^{10}\,dx.$

Solution. Let $u = 1-x$. Then $dx = -du$. When $x = 0$, $u = 1$. When $x = 1$, $u = 0$. Hence

$$\int_0^1 x^2(1-x)^{10}\,dx = \int_1^0 -(1-u)^2u^{10}\,du = \int_0^1 u^{10}(1-2u+u^2)\,du$$
$$= \int_0^1 \left(u^{10} - 2u^{11} + u^{12}\right)du = \left[\frac{u^{11}}{11} - \frac{2u^{12}}{12} + \frac{u^{13}}{13}\right]_0^1 = \frac{1}{858}.$$

(c) $\displaystyle\int_0^{\pi} \sin^3 x\, dx.$

Solution. We have $\sin^3 x = \sin x \sin^2 x = \sin x(1-\cos^2 x)$. Hence

$$\int_0^{\pi} \sin^3 x\, dx = \int_0^{\pi} (\sin x - \sin x\cos^2 x)\, dx = \Big[-\cos x + \frac{\cos^3 x}{3}\Big]_0^{\pi} = \frac{4}{3}.$$

(d) $\displaystyle\int_{-\pi/6}^{\pi/3} \sec^2 x \tan x\, dx.$

Solution. Put $u = \tan x$. Then $du = \sec^2 x dx$. When $x = \pi/3$, $u = \tan(\pi/3) = \sqrt{3}$. When $x = -\pi/6$, $u = \tan(-\pi/6) = -1/\sqrt{3}$. Hence

$$\int_{-\pi/6}^{\pi/3} \sec^2 x \tan x\, dx = \int_{-1/\sqrt{3}}^{\sqrt{3}} u\, du = \frac{u^2}{2}\Big|_{-1/\sqrt{3}}^{\sqrt{3}} = \frac{4}{3}.$$

3. Calculate the following integrals.

(a) $\displaystyle\int_{-1}^{0} \frac{x}{x^2+2x+4}\, dx.$

Solution. We have $x^2+2x+4 = (x+1)^2+3$. Hence

$$\int_{-1}^{0} \frac{x}{x^2+2x+4}\, dx = \int_{-1}^{0} \frac{x+1-1}{(x+1)^2+3}\, dx$$
$$= \Big[\frac{1}{2}\ln(x^2+2x+4) - \frac{1}{\sqrt{3}}\arctan\frac{x+1}{\sqrt{3}}\Big]_{-1}^{0} = \frac{1}{2}(\ln 4 - \ln 3) - \frac{\pi}{6\sqrt{3}}.$$

(b) $\displaystyle\int_{-\pi}^{0} \sin^2 x\cos^2 x\, dx.$

Solution. We have $\sin^2 x \cos^2 x = [\sin(2x)/2]^2 = \sin^2(2x)/4 = [1-\cos(4x)]/8$. Hence

$$\int_{-\pi}^{0} \sin^2 x\cos^2 x\, dx = \frac{1}{8}\int_{-\pi}^{0} [1-\cos(4x)]\, dx = \frac{1}{8}\Big[x - \frac{\sin(4x)}{4}\Big]_{-\pi}^{0} = \frac{\pi}{8}.$$

(c) $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\, dx.$

Solution. Let $x = 3\sin t$ for $-\pi/2 \le t \le \pi/2$. Then $dx = 3\cos t dt$. When $t = -\pi/2$, $x = -3$. When $t = \pi/2$, $x = 3$. Moreover, since $\cos t \ge 0$ for $-\pi/2 \le t \le \pi/2$, we have $\sqrt{9-x^2} = \sqrt{9\cos^2 t} = 3\cos t$. Thus

$$\int_{-3}^{3} \sqrt{9-x^2}\, dx = \int_{-\pi/2}^{\pi/2} (3\cos t)(3\cos t)\, dt = 9\int_{-\pi/2}^{\pi/2} \cos^2 t\, dt$$
$$= 9\int_{-\pi/2}^{\pi/2} \frac{1+\cos(2t)}{2}\, dt = 9\Big[\frac{t}{2} + \frac{\sin(2t)}{4}\Big]_{-\pi/2}^{\pi/2} = \frac{9\pi}{2}.$$

(d) $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx.$

Solution. We have

$$\int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx = \ln(x+\sqrt{x^2+1})\Big|_0^1 = \ln(1+\sqrt{2}).$$

4. (a) Find the length of the curve given by the equation $y = e^x$, $0 \le x \le 1$.

Solution. Let L denote the length of the curve. Since $y' = e^x$, we have

$$L = \int_0^1 \sqrt{1+e^{2x}}\,dx.$$

Make change of variable $u = \sqrt{1+e^{2x}}$ in the above integral. For $x = 0$ we have $u = \sqrt{2}$; for $x = 1$ we have $u = \sqrt{1+e^2}$. Moreover, $1 + e^{2x} = u^2$. It follows that

$$2e^{2x}\,dx = 2u\,du \quad \text{and} \quad dx = \frac{udu}{e^{2x}} = \frac{udu}{u^2-1}.$$

Consequently,

$$L = \int_0^1 \sqrt{1+e^{2x}}\,dx = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{u^2}{u^2-1}\,du.$$

We observe that

$$\frac{u^2}{u^2-1} = \frac{u^2-1+1}{u^2-1} = 1 + \frac{1}{u^2-1} = 1 + \frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1}\right).$$

Therefore,

$$\begin{aligned} L &= \left[u + \frac{1}{2}\ln\frac{u-1}{u+1}\right]_{\sqrt{2}}^{\sqrt{1+e^2}} \\ &= \sqrt{1+e^2} + \frac{1}{2}\ln\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1} - \sqrt{2} - \frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1}. \end{aligned}$$

(b) Find the length of the curve given by the parametric equations

$$x = \cos^3 t, \quad y = \sin^3 t \quad \text{for } 0 \le t \le \pi.$$

Solution. Let L denote the length of the curve. Since $x'(t) = 3\cos^2 t(-\sin t)$ and $y'(t) = 3\sin^2 t\cos t$, we have

$$L = \int_0^\pi \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt = \int_0^\pi \sqrt{(3\cos t\sin t)^2}\,dt = 3\int_0^\pi |\cos t\sin t|\,dt.$$

For $0 \le t \le \pi/2$, $\cos t \sin t \ge 0$, and so $|\cos t \sin t| = \cos t \sin t$. For $\pi/2 \le t \le \pi$, $\cos t \sin t \le 0$, and so $|\cos t \sin t| = -\cos t \sin t$. Therefore,

$$L = 3\int_0^{\pi/2} \cos t \sin t \, dt - 3\int_{\pi/2}^{\pi} \cos t \sin t \, dt = \left[\frac{3}{2}\sin^2 t\right]_0^{\pi/2} - \left[\frac{3}{2}\sin^2 t\right]_{\pi/2}^{\pi} = 3.$$

5. Let f be a continuous function on $[0,1]$. Prove the following identities:

$$\int_0^{\pi/2} f(\cos x)\, dx = \int_0^{\pi/2} f(\sin x)\, dx = \frac{1}{2}\int_0^{\pi} f(\sin x)\, dx.$$

Proof. Making change of variable $x = \pi/2 - u$ in the first integral, we obtain

$$\int_0^{\pi/2} f(\cos x)\, dx = \int_{\pi/2}^{0} f\big(\cos(\pi/2 - u)\big)(-du) = \int_0^{\pi/2} f(\sin x)\, dx.$$

Making change of variable $x = \pi - u$ in the second integral, we obtain

$$\int_0^{\pi/2} f(\sin x)\, dx = \int_{\pi}^{\pi/2} f\big(\sin(\pi - u)\big)(-du) = \int_{\pi/2}^{\pi} f(\sin x)\, dx.$$

It follows that

$$\int_0^{\pi} f(\sin x)\, dx = \int_0^{\pi/2} f(\sin x)\, dx + \int_{\pi/2}^{\pi} f(\sin x)\, dx = 2\int_0^{\pi/2} f(\sin x)\, dx.$$