MATH 314 Assignment #2

1. (a) Prove that there is no rational number r such that $r^2 = 3$.

Proof. Consider the set S of all positive integers n such that $(m/n)^2 = 3$ for some $m \in \mathbb{Z}$. If the set S is not empty, then we let n_0 be its least element. For this n_0, there exists some $m_0 \in \mathbb{Z}$ such that $(m_0/n_0)^2 = 3$, *i.e.*, $m_0^2 = 3n_0^2$. Hence 3 divides m_0^2. It follows that 3 divides m_0. So $m_0 = 3m_1$ for some $m_1 \in \mathbb{Z}$. Consequently, $(3m_1)^2 = 3n_0^2$, and so $3m_1^2 = n_0^2$. Hence 3 divides n_0, and so $n_0 = 3n_1$ for some $n_1 \in \mathbb{N}$. Now we have $m_1^2 = 3n_1^2$. Thus $n_1 \in S$ and $n_1 < n_0$. This contradicts the fact that n_0 is the least element of S. Therefore, there is no rational number r such that $r^2 = 3$.

(b) Prove that $a + b\sqrt{2}$ is an irrational number for all rational numbers a and b with $b \neq 0$.

Proof. Let $c := a + b\sqrt{2}$. Since $b \neq 0$, it follows that $\sqrt{2} = (c - a)/b$. If c is a rational number, then $(c - b)/a$ is a rational number because a and b are rational numbers. But $\sqrt{2}$ is an irrational number. Thus c must be an irrational number.

2. Let x and y be real numbers. Prove the following statements.

(a) If $x > 0$, then there exists a unique natural number n such that $n - 1 < x \leq n$.

Proof. Let S be the set $\{k \in \mathbb{N} : k \geq x\}$. By the well ordering principle, S has a least element, say n. Since $n \in S$, we have $n \geq x$. If $n = 1$, then $n - 1 = 0 < x$. If $n > 1$, then $n - 1 \in \mathbb{N}$. But n is the least element of S, so $n - 1 \notin S$. Hence $n - 1 < x$. If n_1 is a natural number satisfying $n_1 - 1 < x \leq n_1$. Then $n_1 \geq x > n - 1$ and $n \geq x > n_1 - 1$. So $n_1 \geq n$ and $n \geq n_1$. Thus $n_1 = n$. This proves the uniqueness.

(b) If $0 < y < 1$, then there exists a unique integer $n \geq 2$ such that

$$\frac{1}{n} \leq y < \frac{1}{n-1}.$$

Proof. Let T be the set $\{k \in \mathbb{N} : ky \geq 1\}$. In light of the Archimedean property, T is not empty. By the well ordering principle, T has a least element, say n. Since $n \in T$, we have $ny \geq 1$. It follows that $n \geq 1/y$. But $0 < y < 1$, so $1/y > 1$. Hence $n > 1$. This shows that $n \geq 2$ and $1/n \leq y$. Moreover, since n is the least element of T, we have $n - 1 \notin T$. It follows that $(n-1)y < 1$. Therefore, $1/n \leq y < 1/(n-1)$. If $n_1 \geq 2$ satisfies $1/n_1 \leq y < 1/(n_1 - 1)$, then $1/n_1 < 1/(n-1)$ and $1/n < 1/(n_1 - 1)$. Therefore, we get $n_1 = n$ and complete the proof of the uniqueness.

3. Write the following sets in interval notation:

(a) $\{x \in \mathbb{R} : |x - 2| < 3\}$ (b) $\{x \in \mathbb{R} : |2x + 1| \leq 1\}$

Solution. We see that $|x-2| < 3$ if and only if $-3 < x-2 < 3$, *i.e.*, $2-3 < x < 2+3$. Hence, $\{x \in \mathbb{R} : |x-2| < 3\} = (-1, 5)$. For (b), $|2x+1| \geq 5$ if and only if $2x+1 \leq -5$ or $2x + 1 \geq 5$. We have $2x + 1 \leq -5 \Leftrightarrow x \leq -3$. Moreover, $2x + 1 \geq 5 \Leftrightarrow x \geq 2$. It follows that $\{x \in \mathbb{R} : |2x + 1| \geq 5\} = (-\infty, -3] \cup [2, \infty)$.

(c) $\{x \in \mathbb{R} : x^2 < 8\}$ (d) $\{x \in \mathbb{R} : x^3 \leq 8\}$

Solution. We have $\{x \in \mathbb{R} : x^2 < 8\} = (-2\sqrt{2}, 2\sqrt{2})$ and $\{x \in \mathbb{R} : x^3 \leq 8\} = (-\infty, 2)$.

3. For each set below, find its maximum, supremum, minimum, and infimum if they exist.

(a) $(0, 3]$ (b) $\{1 - 1/n : n \in \mathbb{N}\}$

Solution. Let $A := (0, 3]$ and $B := \{1-1/n : n \in \mathbb{N}\}$. We have $\max A = 3$, $\sup A = 3$, $\min A$ does not exists, and $\inf A = 0$. Moreover, $\max B$ does not exist, $\sup B = 1$, $\min B = 0$, and $\inf B = 0$.

(c) $\mathbb{R} \setminus [1, \infty)$ (d) $\{n - (-1)^n : n \in \mathbb{N}\}$

Solution. Let $C := \mathbb{R} \setminus [1, \infty) = (-\infty, 1)$ and $D := \{n-(-1)^n : n \in \mathbb{N}\}$. Then $\max C$ does not exist, $\sup C = 1$, $\min C$ does not exist, and $\inf C = -\infty$. Moreover, $\max D$ does not exist, $\sup D = \infty$, $\min D = 1$, and $\inf D = 1$.

5. Let A be a nonempty bounded subset of $\mathbb{R}$, and let $s := \sup A$.

(a) Show that $s \in A$ if and only if $s = \max A$.

Proof. If $s = \sup A \in A$, then $s \geq a$ for all $a \in A$. Hence $s = \max A$. Conversely, if $s = \max A$, then $s \in A$.

(b) Let $-A := \{-x : x \in A\}$. Prove that $\inf(-A) = -\sup A$.

Proof. Let $s := \sup A$. Then $a \leq s$ for all $a \in A$. It follows that $-s \leq -a$ for all $a \in A$. Hence, $-s$ is a lower bound of $-A$. Let t be a lower bound of $-A$. Then $t \leq -a$ for all $a \in A$. It follows that $a \leq -t$ for all $a \in A$. Hence, $-t$ is an upper bound of A. Consequently, $-t \geq s$. So $t \leq -s$ whenever t is a lower bound of $-A$. This shows $-s = \inf(-A)$.