MATH 314 Assignment #3

1. (a) Prove that $\lim_{n\to\infty}\sqrt{n}=\infty$ and $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$.

   Proof. For $M>0$, let $N=\lfloor M^2\rfloor+1$. Then $n>N$ implies $\sqrt{n}>\sqrt{M^2}=M$. Hence, $\lim_{n\to\infty}\sqrt{n}=\infty$. By Theorem 1.3, it follows that $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$.

   (b) Prove that if $\lim_{n\to\infty}a_n=a$, then $\lim_{n\to\infty}|a_n|=|a|$. Is the converse true? Justify your answer.

   Proof. If $\lim_{n\to\infty}a_n=a$, then for given $\varepsilon>0$, there exists a positive integer N such that $|a_n-a|<\varepsilon$ whenever $n>N$. Since $\big||a_n|-|a|\big|\le|a_n-a|$, it follows that $\big||a_n|-|a|\big|<\varepsilon$ whenever $n>N$. This shows that $\lim_{n\to\infty}|a_n|=|a|$. The converse is not true. For example, $\lim_{n\to\infty}|(-1)^n|=1$, but $\lim_{n\to\infty}(-1)^n$ does not exist.

2. For each sequence below find its limit and determine whether it converges.

   (a) $a_n=\dfrac{3n^2-2n^3}{5+n^3+2n}$ (b) $b_n=\dfrac{1-n^9}{100n^8+9n^2}$

   (c) $c_n=\dfrac{3^n+2^n}{3^n-4^n}$ (d) $d_n=\sqrt{n+2}-\sqrt{n}$

   Solution. (a) We have

$$a_n=\frac{3n^2-2n^3}{5+n^3+2n}=\frac{-2n^3+3n^2}{n^3+2n+5}=\frac{n^3(-2+\frac{3}{n})}{n^3(1+\frac{2}{n^2}+\frac{5}{n^3})}=\frac{-2+\frac{3}{n}}{1+\frac{2}{n^2}+\frac{5}{n^3}}.$$

   It follows that $\lim_{n\to\infty}a_n=-2$. The sequence converges.

   (b) We have

$$b_n=\frac{1-n^9}{100n^8+9n^2}=\frac{-n^9+1}{100n^8+9n^2}=\frac{n^9(-1+\frac{1}{n^9})}{n^8(100+\frac{9}{n^6})}=n\cdot\frac{-1+\frac{1}{n^9}}{100+\frac{9}{n^6}}.$$

   Consequently, $\lim_{n\to\infty}b_n=-\infty$. The sequence diverges.

   (c) We have

$$c_n=\frac{3^n+2^n}{3^n-4^n}=\frac{3^n(1+\frac{2^n}{3^n})}{-4^n(1-\frac{3^n}{4^n})}=-\Big(\frac{3}{4}\Big)^n\frac{1+(\frac{2}{3})^n}{1-(\frac{3}{4})^n}.$$

   Since $\lim_{n\to\infty}(\frac{3}{4})^n=0$, we obtain $\lim_{n\to\infty}c_n=0$. The sequence converges.

   (d) We have $(\sqrt{n+2}-\sqrt{n})(\sqrt{n+2}+\sqrt{n})=\sqrt{n+2}^2-\sqrt{n}^2=(n+2)-n=2$. It follows that $d_n=2/(\sqrt{n+2}+\sqrt{n})$. Since $\sqrt{n+2}+\sqrt{n}\ge2\sqrt{n}$, we obtain $0<d_n\le1/\sqrt{n}$. Since $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$, by the squeeze theorem we conclude that $\lim_{n\to\infty}d_n=0$.

3. Let $x_n := \sqrt{n^2+n} - n$ for $n \in \mathbb{N}$.

(a) Prove that

$$x_n = \frac{n}{\sqrt{n^2+n}+n}.$$

Proof. We have $(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n) = (n^2+n) - n^2 = n$. It follows that

$$x_n = \sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n}.$$

(b) Show that $2n \le \sqrt{n^2+n} + n \le 2n+1$.

Proof. We have $n^2 \le n^2 + n \le (n+1)^2$. It follows that $n \le \sqrt{n^2+n} \le n+1$. Hence, $2n \le \sqrt{n^2+n} + n \le 2n+1$.

(c) Deduce from (a) and (b) that

$$\frac{n}{2n+1} \le x_n \le \frac{1}{2}.$$

Proof. It follows from (a) and (b) that

$$\frac{n}{2n+1} \le x_n \le \frac{n}{2n} = \frac{1}{2}.$$

(d) Find $\lim_{n\to\infty} x_n$.

Solution. Since $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$. By the squeeze theorem for limits we obtain

$$\lim_{n\to\infty} x_n = \frac{1}{2}.$$

4. Let $a_1 := 1$ and set $a_{n+1} := (2a_n + 5)/6$ for $n = 1, 2, \ldots$.

(a) Find the first five terms of the sequence $(a_n)_{n=1,2,\ldots}$.

Solution. $a_1 = 1$, $a_2 = 7/6$, $a_3 = 11/9$, $a_4 = 67/54$, $a_5 = 101/81$.

(b) Use mathematical induction to prove that $a_n \le 2$ for all $n \in \mathbb{N}$.

Proof. We have $a_1 = 1 \le 2$. Suppose $a_n \le 2$. Then

$$a_{n+1} = \frac{2a_n+5}{6} \le \frac{2\cdot 2+5}{6} < 2.$$

By the principle of mathematical induction we conclude that $a_n \le 2$ for all $n \in \mathbb{N}$.

(c) Use mathematical induction to show that the sequence $(a_n)_{n=1,2,\ldots}$ is increasing.

Proof. We have $a_2 > a_1$. Suppose $a_{n+1} > a_n$. Then

$$a_{n+2} = \frac{2a_{n+1} + 5}{6} > \frac{2a_n + 5}{6} = a_{n+1}.$$

By the principle of mathematical induction we conclude that the sequence $(a_n)_{n=1,2,...}$ is increasing.

(d) Prove that the sequence $(a_n)_{n=1,2,...}$ is convergent and find $\lim_{n\to\infty} a_n$.

Proof. Since the sequence $(a_n)_{n=1,2,...}$ is increasing and bounded above, it converges by Theorem 3.1. Suppose $\lim_{n\to\infty} a_n = a$. Taking limits of both sides of the equation $a_{n+1} = (2a_n + 5)/6$, we obtain

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{2a_n + 5}{6} = \frac{2a + 5}{6}.$$

It follows that $6a = 2a + 5$. Consequently, $\lim_{n\to\infty} a_n = a = 5/4$.

5. Let $b_1 := 1$ and set $b_{n+1} := \sqrt{2b_n}$ for $n = 1, 2, \ldots$.

(a) Prove that the sequence $(b_n)_{n=1,2,...}$ is increasing and bounded above by 2.

Proof. We have $b_1 \leq 2$. Suppose $b_n \leq 2$. Then

$$b_{n+1} = \sqrt{2b_n} \leq \sqrt{2\cdot 2} \leq 2.$$

By the principle of mathematical induction we conclude that $b_n \leq 2$ for all $n \in \mathbb{N}$. We have $b_2 = \sqrt{2} > 1 = b_1$. Suppose $b_{n+1} > b_n$. Then

$$b_{n+2} = \sqrt{2b_{n+1}} > \sqrt{2b_n} = b_{n+1}.$$

By the principle of mathematical induction we conclude that the sequence $(b_n)_{n=1,2,...}$ is increasing.

(b) Show that the sequence $(b_n)_{n=1,2,...}$ is convergent and find $\lim_{n\to\infty} b_n$.

Proof. Since the sequence $(b_n)_{n=1,2,...}$ is increasing and bounded above, it converges, by Theorem 3.1. Suppose $\lim_{n\to\infty} b_n = b$. Taking limits of both sides of the equation $b_{n+1} = \sqrt{2b_n}$, we obtain $b = \sqrt{2b}$. It follows that $b^2 = 2b$. But $b > 0$. Hence, we conclude that $\lim_{n\to\infty} b_n = b = 2$.