MATH 314 Assignment #4

1. (a) Let $a_n := 2(-1)^{n+1} + (-1)^{n(n+1)/2}$ for $n \in \mathbb{N}$. Find four subsequences of $(a_n)_{n=1,2,\dots}$ such that they converge to different limits.

Solution. We have

$$a_{4k+1} = 1, \quad a_{4k+2} = -3, \quad a_{4k+3} = 3, \quad a_{4k+4} = -1, \quad k \in \mathbb{N}.$$

Thus $(a_{4k+1})_{k=1,2,\dots}$, $(a_{4k+2})_{k=1,2,\dots}$, $(a_{4k+3})_{k=1,2,\dots}$, and $(a_{4k+4})_{k=1,2,\dots}$ are four subsequences that converge to different limits.

(b) Let $b_n := [1 + (-1)^n]n + 100/n$ for $n \in \mathbb{N}$. Find an increasing subsequence of $(b_n)_{n=1,2,\dots}$. Also, find a convergent subsequence of $(b_n)_{n=1,2,\dots}$.

Solution. We have $b_{2k} = 4k + 50/k$, and $b_{2k+1} = 100/(2k+1)$, $k \in \mathbb{N}$. It is easily seen that $(b_{2k+1})_{k=1,2,\dots}$ is a convergent subsequence of $(b_n)_{n=1,2,\dots}$ and $\lim_{k\to\infty} b_{2k+1} = 0$. Moreover, $(b_{2k+6})_{k=1,2,\dots}$ is an increasing subsequence of $(b_n)_{n=1,2,\dots}$. Indeed, we have

$$\begin{aligned} b_{2(k+1)+6} - b_{2k+6} &= \left[4(k+4) + \frac{50}{k+4}\right] - \left[4(k+3) + \frac{50}{k+3}\right] \\ &= 4 - \frac{50}{(k+3)(k+4)} \ge 4 - \frac{50}{(1+3)(1+4)} > 0. \end{aligned}$$

2. Let $(x_n)_{n=1,2,\dots}$ be the sequence recursively defined by $x_1 := 1$ and

$$x_{n+1} := \frac{1}{4}(x_n^2 + 2), \quad n \in \mathbb{N}.$$

(a) Show that $0 < x_n \le 1$ for all $n \in \mathbb{N}$.

Proof. We have $x_1 = 1$. Suppose $0 < x_n \le 1$. Then $x_{n+1} > 0$ and

$$x_{n+1} = \frac{1}{4}(x_n^2 + 2) \le \frac{1}{4}(1^2 + 2) < 1.$$

By the principle of mathematical induction we conclude that $0 < x_n \le 1$ for all $n \in \mathbb{N}$.

(b) Prove that the sequence $(x_n)_{n=1,2,\dots}$ is contractive.

Proof. For $n \ge 2$ we have $x_{n+1} = (x_n^2 + 2)/4$ and $x_n = (x_{n-1}^2 + 2)/4$. It follows that

$$x_{n+1} - x_n = \frac{1}{4}(x_n^2 + 2) - \frac{1}{4}(x_{n-1}^2 + 2) = \frac{1}{4}(x_n^2 - x_{n-1}^2) = \frac{1}{4}(x_n + x_{n-1})(x_n - x_{n-1}).$$

Since $0 < x_n \le 1$ for all $n \in \mathbb{N}$, we have $|x_n + x_{n-1}| \le 2$. Consequently,

$$|x_{n+1} - x_n| = \frac{1}{4}|x_n + x_{n-1}||x_n - x_{n-1}| \le \frac{1}{2}|x_n - x_{n-1}|.$$

This shows that the sequence $(x_n)_{n=1,2,\ldots}$ is contractive.

(c) Show that the sequence $(x_n)_{n=1,2,\ldots}$ converges and find its limit.

Proof. Since the sequence $(x_n)_{n=1,2,\ldots}$ is contractive, it is convergent. Suppose that $\lim_{n\to\infty} x_n = c$. Since $x_n \le 1$ for all $n \in \mathbb{N}$, we have $c \le 1$. Taking limits of both sides of the equation $x_{n+1} = (x_n^2+2)/4$ as $n \to \infty$, we obtain $c = (c^2+2)/4$. It follows that $c^2 - 4c + 2 = 0$. This together with $c \le 1$ gives the only solution $c = 2 - \sqrt{2}$.

3. Find the sum of the following series.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{10 + (-3)^n}{5^{n-1}}$.

Solution. The series can be expressed as the sum of two geometric series:

$$\sum_{n=2}^{\infty} \frac{10 + (-3)^n}{5^{n-1}} = \sum_{n=2}^{\infty} \frac{10}{5^{n-1}} + \sum_{n=2}^{\infty} \frac{(-3)^n}{5^{n-1}}.$$

For the first geometric series, the initial term is $10/5^{2-1} = 2$ and the ratio is $1/5$. Hence

$$\sum_{n=2}^{\infty} \frac{10}{5^{n-1}} = \frac{2}{1-1/5} = \frac{2}{4/5} = \frac{5}{2}.$$

For the second geometric series, the initial term is $(-3)^2/5^{2-1} = 9/5$ and the ratio is $-3/5$. Hence

$$\sum_{n=2}^{\infty} \frac{(-3)^n}{5^{n-1}} = \frac{9/5}{1-(-3/5)} = \frac{9/5}{8/5} = \frac{9}{8}.$$

Therefore

$$\sum_{n=2}^{\infty} \frac{10 + (-3)^n}{5^{n-1}} = \frac{5}{2} + \frac{9}{8} = \frac{29}{8}.$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 3^n - 3 \cdot 2^n}{6^n}$.

Solution. We have

$$\sum_{n=1}^{\infty} \frac{2\cdot 3^n - 3\cdot 2^n}{6^n} = \sum_{n=1}^{\infty} \frac{2\cdot 3^n}{6^n} - \sum_{n=1}^{\infty} \frac{3\cdot 2^n}{6^n} = \sum_{n=1}^{\infty} \frac{2}{2^n} - \sum_{n=1}^{\infty} \frac{3}{3^n}$$
$$= \frac{1}{1-1/2} - \frac{1}{1-1/3} = 2 - \frac{3}{2} = \frac{1}{2}.$$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(2n-1)(2n+1)}$

Solution. Let $a_n := 1/((2n-1)(2n+1))$ for $n = 1, 2, \ldots$. We have

$$a_n = \frac{1}{(2n-1)(2n+1)} = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}.$$

It follows that

$$s_n = a_2 + \cdots + a_n = \Big(\frac{1}{6} - \frac{1}{10}\Big) + \Big(\frac{1}{10} - \frac{1}{14}\Big) + \cdots + \Big(\frac{1}{4n-2} - \frac{1}{4n+2}\Big) = \frac{1}{6} - \frac{1}{4n+2}.$$

Hence $\lim_{n\to\infty} s_n = 1/6$. Thus, the series converges and its sum is $1/6$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+2)}$.

Solution. We have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+2)} = -\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} - \frac{1}{3\cdot 5} + \frac{1}{4\cdot 6} - \cdots$$
$$= -\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} + \sum_{k=1}^{\infty} \frac{1}{2k(2k+2)}.$$

By using the same method as in (c) we get

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}.$$

In order to find the sum $\sum_{k=1}^{\infty} 1/(2k(2k+2))$, let $b_k := 1/(2k(2k+2))$ for $k = 1, 2, \cdots$ and $t_n := b_1 + b_2 + \cdots + b_n$ for $n = 1, 2, \cdots$. Then

$$b_k = \frac{1}{2k(2k+2)} = \frac{1}{2(2k)} - \frac{1}{2(2k+2)}$$

and

$$t_n = \Big(\frac{1}{4} - \frac{1}{8}\Big) + \Big(\frac{1}{8} - \frac{1}{12}\Big) + \cdots + \Big(\frac{1}{4n} - \frac{1}{4n+4}\Big) = \frac{1}{4} - \frac{1}{4n+4}.$$

It follows that $\lim_{n\to\infty} t_n = 1/4$. Finally, we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+2)} = -\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} + \sum_{k=1}^{\infty} \frac{1}{2k(2k+2)} = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}.$$

4. Test each of the following series for convergence or divergence. If the series converges, determine whether it converges absolutely or conditionally. Justify your conclusions.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{1/n}}$.

Solution. We have $\lim_{n\to\infty} 1/2^{1/n} = 1$. By the divergence test, the series $\sum_{n=1}^{\infty} 1/2^{1/n}$ diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{10}{n} \right)$.

Solution. We assert that

$$\frac{1}{\sqrt{n}} - \frac{10}{n} > \frac{1}{2\sqrt{n}} \quad \text{for } \ n > 400.$$

Since the series $\sum_{n=1}^{\infty} 1/(2\sqrt{n})$ diverges, the series in question also diverges, by the comparison test. To verify our assertion we observe that

$$\frac{1}{\sqrt{n}} - \frac{10}{n} > \frac{1}{2\sqrt{n}} \;\Leftrightarrow\; \frac{1}{2\sqrt{n}} > \frac{10}{n} \;\Leftrightarrow\; \sqrt{n} > 20.$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$.

Solution. Let $u_n := 2^n/n!$ for $n = 1, 2, \ldots$. We have

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} \right) = \lim_{n\to\infty} \frac{2}{n+1} = 0.$$

By the ratio test, the series $\sum_{n=1}^{\infty} 2^n/n!$ converges. But $2^n/n! > 0$. So the series $\sum_{n=1}^{\infty} 2^n/n!$ converges absolutely.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n+1}$.

Solution. Let $b_n := (-1)^n\sqrt{n}/(n+1)$ for $n = 1, 2, \ldots$. We use the alternating series test to show that the series $\sum_{n=1}^{\infty}(-1)^n b_n$ converges. Clearly, $\lim_{n\to\infty} b_n = 0$. To prove $b_n > b_{n+1}$ for all $n \in \mathbb{N}$ we observe that

$$\frac{\sqrt{n}}{n+1} > \frac{\sqrt{n+1}}{n+2} \;\Leftrightarrow\; \frac{n}{(n+1)^2} > \frac{n+1}{(n+2)^2} \;\Leftrightarrow\; n(n+2)^2 > (n+1)^2(n+1).$$

The last inequality is true because

$$n(n+2)^2 - (n+1)^2(n+1) = (n^3+4n^2+4n) - (n^3+3n^2+3n+1) = n^2+n-1 > 0$$

for all $n \in \mathbb{N}$. Thus the series $\sum_{n=1}^{\infty}(-1)^n\sqrt{n}/(n+1)$ converges. But the series

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n\sqrt{n}}{n+1}\right| = \sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+1}$$

diverges. Therefore, the series $\sum_{n=1}^{\infty}(-1)^n\sqrt{n}/(n+1)$ converges conditionally.

5. Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two convergent series.

   (a) Show that the sequence $(b_n)_{n=1,2,\ldots}$ is bounded.

Proof. Since the series $\sum_{n=1}^{\infty} b_n$ converges, we have $\lim_{n\to\infty} b_n = 0$. Consequently, the sequence $(b_n)_{n=1,2,\ldots}$ is bounded, by Theorem 1.2.

   (b) If, in addition, $\sum_{n=1}^{\infty} a_n$ converges absolutely, prove that the series $\sum_{n=1}^{\infty} a_n b_n$ also converges absolutely.

Proof. Since the sequence $(b_n)_{n=1,2,\ldots}$ is bounded, there exists a positive number M such that $|b_n| \le M$ for all $n \in \mathbb{N}$. It follows that $|a_n b_n| \le M|a_n|$ for all $n \in \mathbb{N}$. But the series $\sum_{n=1}^{\infty} a_n$ converges absolutely; hence the series $\sum_{n=1}^{\infty} M|a_n|$ converges. By the comparison test (Theorem 5.3), the series $\sum_{n=0}^{\infty} |a_n b_n|$ converges. This shows that the series $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely.

   (c) Give an example of two conditionally convergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that the series $\sum_{n=1}^{\infty} a_n b_n$ diverges.

Solution. Choose $a_n := (-1)^n/\sqrt{n}$ and $b_n := (-1)^n/\sqrt{n}$ for all $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge. But the series $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} 1/n$ diverges.