MATH 314 Assignment #5

1. Find the following limits.

   (a) $\lim_{x\to 1} \dfrac{x^2+2x-3}{x^2-1}$.

   Solution. We have

$$\lim_{x\to 1} \frac{x^2+2x-3}{x^2-1} = \lim_{x\to 1} \frac{(x-1)(x+3)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{x+3}{x+1} = \frac{4}{2} = 2.$$

   (b) $\lim_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$.

   Solution. We have

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}.$$

2. Find the following limits.

   (a) $\lim_{x\to\infty} \dfrac{2-5x-4x^2}{3x^2+1}$.

   Solution. We have

$$\lim_{x\to\infty} \frac{2-5x-4x^2}{3x^2+1} = \lim_{x\to\infty} \frac{x^2(-4-\frac{5}{x}+\frac{2}{x^2})}{x^2(3+\frac{1}{x^2})} = \lim_{x\to\infty} \frac{-4-\frac{5}{x}+\frac{2}{x^2}}{3+\frac{1}{x^2}} = -\frac{4}{3}.$$

   (b) $\lim_{x\to\infty} \big(\sqrt{x^2+2x}-x\big)$.

   Solution. We have

$$\sqrt{x^2+2x}-x = \frac{(\sqrt{x^2+2x}-x)(\sqrt{x^2+2x}+x)}{\sqrt{x^2+2x}+x} = \frac{2x}{\sqrt{x^2+2x}+x}.$$

   Note that $2x \le \sqrt{x^2+2x}+x \le 2x+1$. Hence,

$$\frac{2x}{2x+1} \le \sqrt{x^2+2x}-x \le \frac{2x}{2x} = 1.$$

   But $\lim_{x\to\infty}(2x)/(2x+1) = 1$. By the squeeze theorem, we obtain

$$\lim_{x\to\infty} \big(\sqrt{x^2+2x}-x\big) = 1.$$

3. Let $f(x) = \sqrt{4-x}$ for $x \leq 4$ and $g(x) = x^2$ for all $x \in \mathbb{R}$.

(a) Give the domains of the functions $f+g$, fg, $f \circ g$ and $g \circ f$.

Solution. The domains of $f+g$, fg and $g \circ f$ are $(-\infty, 4]$. The domain of $f \circ g$ is $[-2, 2]$.

(b) Find the values $f \circ g(0)$, $g \circ f(0)$, $f \circ g(1)$, $g \circ f(1)$, $f \circ g(2)$ and $g \circ f(2)$.

Solution. We have $f \circ g(0) = 2$, $g \circ f(0) = 4$, $f \circ g(1) = \sqrt{3}$, $g \circ f(1) = 3$, $f \circ g(2) = 0$, $g \circ f(2) = 2$.

(c) Are the functions $f \circ g$ and $g \circ f$ equal?

Solution. Since $f \circ g(0) \neq g \circ f(0)$, the functions $f \circ g$ and $g \circ f$ are not equal.

(d) Are $f \circ g(3)$ and $g \circ f(3)$ meaningful?

Solution. $f \circ g(3)$ is not meaningful, while $g \circ f(3) = 1$.

4. Let f and g be two functions from $\mathbb{R}$ to $\mathbb{R}$. Prove the following statements.

(a) If f is continuous, then the function $|f|$ is continuous.

Proof. We wish to prove that $|f|$ is continuous at every point $c \in \mathbb{R}$. Since f is continuous, for given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - c| < \delta \quad \text{implies} \quad |f(x) - f(c)| < \varepsilon.$$

By the triangle inequality we have $\big||f(x)| - |f(c)|\big| \leq |f(x) - f(c)|$. Hence

$$|x - c| < \delta \quad \text{implies} \quad \big||f(x)| - |f(c)|\big| < \varepsilon.$$

This shows that the function $|f|$ is continuous.

(b) If f and g are continuous, then the function $\max\{f, g\}$ is continuous.

Proof. We have

$$\max\{f, g\} = \frac{f+g}{2} + \frac{|f-g|}{2}.$$

Since f and g are continuous, $f+g$ and $f-g$ are continuous, by Theorem 2.2. Moreover, part (a) tells us that $|f-g|$ is continuous. Invoking Theorem 2.2 again, we conclude that the function $\max\{f, g\}$ is continuous.

5. Let $f(x) := 1 + x^2$ and $g(x) := x(1 - x^2)$, $x \in \mathbb{R}$. Moreover, let h be the function defined by

$$h(x) := \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

(a) Prove that $f \circ h$ and $h \circ f$ are continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Proof. For $x \geq 0$ we have $h(x) = 1$ and hence

$$(f \circ h)(x) = f(h(x)) = f(1) = 1 + 1^2 = 2.$$

For $x < 0$ we have $h(x) = -1$ and hence

$$(f \circ h)(x) = f(h(x)) = f(-1) = 1 + (-1)^2 = 2.$$

Thus, $(f \circ h)(x) = 2$ for all $x \in \mathbb{R}$. As a constant function, $f \circ h$ is continuous on $\mathbb{R}$. We have $f(x) > 0$ for all $x \in \mathbb{R}$. Hence, $(h \circ f)(x) = h(f(x)) = 1$ for all $x \in \mathbb{R}$. As a constant function, $h \circ f$ is continuous on $\mathbb{R}$.

(b) Find the set of discontinuity points of $g \circ h$ and $h \circ g$, respectively.

Solution. Since $h(x) = 1$ or $h(x) = -1$. We have $(g \circ h)(x) = g(h(x)) = 0$ for all $x \in \mathbb{R}$. This, $g \circ h$ is continuous and hence the set of its discontinuous points is the empty set.

For $x \in (-\infty, -1)$ or $x \in (0, 1)$, we have

$$g(x) > 0 \quad \text{and} \quad (h \circ g)(x) = h(g(x)) = 1.$$

For $x \in (-1, 0)$ or $x \in (1, \infty)$, we have

$$g(x) < 0 \quad \text{and} \quad (h \circ g)(x) = h(g(x)) = -1.$$

Therefore, the set of discontinuity points of $h \circ g$ is $\{-1, 0, 1\}$.