MATH 314 Assignment #6

1. Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that $\lim_{|x|\to\infty} f(x) = \infty$, that is, for any real number M, there exists a positive real number K such that $f(x) > M$ whenever $|x| \ge K$.

   (a) Fix a point $x_0 \in \mathbb{R}$. Prove that there exists a positive real number a such that $-a \le x_0 \le a$ and $f(x) \ge f(x_0)$ whenever $x \notin [-a, a]$.

   Proof. Since $\lim_{|x|\to\infty} f(x) = \infty$, there exists some $K > 0$ such that $f(x) \ge f(x_0)$ whenever $|x| \ge K$. Choose $a := \max\{K, |x_0|\}$. Then we have $-a \le x_0 \le a$ and $f(x) \ge f(x_0)$ whenever $x \notin [-a, a]$.

   (b) Show that there exists some $c \in \mathbb{R}$ such that $f(c) \le f(x)$ for all $x \in \mathbb{R}$, that is, f attains its minimum at c.

   Proof. Since f is continuous, f attains its minimum on the closed interval $[-a, a]$, that is, there exists some $c \in [-a, a]$ such that $f(c) \le f(x)$ for all $x \in [-a, a]$. It follows that $f(c) \le f(x_0)$ since $x_0 \in [-a, a]$. Consequently, for $x \in \mathbb{R} \setminus [-a, a]$ we have $f(c) \le f(x_0) \le f(x)$. Therefore, $f(c) \le f(x)$ for all $x \in \mathbb{R}$.

2. The Intermediate Value Theorem will be used in the following problems.

   (a) Show that the equation $2^x = 3x$ has a solution $c \in (1, 4)$.

   Proof. Let $g(x) := 2^x - 3x$, $1 \le x \le 4$. Then we have $g(1) = 2 - 3 < 0$ and $g(4) = 2^4 - 3 \cdot 4 = 4 > 0$. By the Intermediate Value Theorem, there exists some $c \in (1, 4)$ such that $g(c) = 0$. For this c we have $2^c = 3c$.

   (b) Let p be a cubic polynomial, *i.e.*, $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, $x \in \mathbb{R}$, where $a_3 \ne 0$. Prove that p has at least one real root.

   Proof. Suppose that $a_3 > 0$. Then we have

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} x^3(a_3 + a_2/x + a_1/x^2 + a_0/x^3) = \infty.$$

Hence, there exists some $x_1 \in \mathbb{R}$ such that $p(x_1) > 0$. Moreover,

$$\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} x^3(a_3 + a_2/x + a_1/x^2 + a_0/x^3) = -\infty.$$

Hence, there exists some $x_2 \in \mathbb{R}$ such that $p(x_2) < 0$. By the Intermediate Value Theorem, there exists a real number c between x_1 and x_2 such that $p(c) = 0$. If $a_3 < 0$, then $\lim_{x\to\infty} p(x) = -\infty$ and $\lim_{x\to-\infty} p(x) = \infty$. A similar argument shows that p has at least one real root.

3. (a) Suppose that f is a continuous function from $[0, \infty)$ to $\mathbb{R}$. Moreover, there exists some $a > 0$ such that f is uniformly continuous on $[a, \infty)$. Prove that f is uniformly continuous on $[0, \infty)$.

Proof. Since f is continuous on $[0, a]$, f is uniformly continuous on $[0, a]$, by Theorem 4.2. Let $\varepsilon > 0$ be given. Since f is uniformly continuous on $[0, a]$, there exists some $\delta_1 > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [0, a]$ and $|x - y| < \delta_1$. Since f is uniformly continuous on $[a, \infty)$, there exists some $\delta_2 > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [a, \infty)$ and $|x - y| < \delta_2$. Since f is continuous at a, there exists some $\delta_0 > 0$ such that $|f(x) - f(a)| < \varepsilon/2$ whenever $x \in [0, \infty)$ and $|x - a| < \delta_0$. Let $\delta := \min\{\delta_0, \delta_1, \delta_2\}$. Then $\delta > 0$. Now let $x, y \in [0, \infty)$ satisfy $|x - y| < \delta$. If both x and y lie in $[0, a]$, or both lie in $[a, \infty)$, we have $|f(x) - f(y)| < \varepsilon$. Suppose that one of x and y, say x, is in $[0, a]$ and the other is in $[a, \infty)$. Then $x \le a \le y$. It follows that $|x - a| = a - x \le y - x < \delta \le \delta_0$ and $|y - a| = y - a < \delta_0$. Consequently, $|f(x) - f(y)| \le |f(x) - f(a)| + |f(a) - f(y)| < \varepsilon$. Therefore, $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [0, \infty)$ and $|x - y| < \delta$. This shows that f is uniformly continuous on $[0, \infty)$.

(b) Let g be the function from $[0, \infty)$ to $\mathbb{R}$ given by $g(x) = \sqrt{x}$, $x \ge 0$. Prove that g is uniformly continuous on $[0, \infty)$.

Proof. Since g is continuous on $[0, 1]$, g is uniformly continuous on $[0, 1]$, by Theorem 4.2. In order to show that g is uniformly continuous on $[0, \infty)$, it suffices, by part (a), to prove that g is uniformly continuous on $[1, \infty)$. Let $x, y \in [1, \infty)$. We have

$$\big|g(x) - g(y)\big| = \big|\sqrt{x} - \sqrt{y}\big| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \le |x - y|,$$

because $\sqrt{x} + \sqrt{y} \ge 1$ for $x, y \in [1, \infty)$. Thus g is a Lipschitz function on $[1, \infty)$. Therefore g is uniformly continuous on $[1, \infty)$.

4. Let f be a real-valued function defined by

$$f(x) = \begin{cases} 2^x & \text{for } 0 \le x \le 1, \\ 3 - 1/x^2 & \text{for } 1 < x \le 2. \end{cases}$$

(a) Show that f is continuous and strictly increasing on $[0, 2]$.

Proof. Let $c \in [0, 2]$. If $c \in [0, 1)$ we have $\lim_{x \to c} f(x) = f(c)$, since the exponential function is continuous. For $c \in (1, 2]$, we also have $\lim_{x \to c} f(x) = f(c)$. Let us consider the case $c = 1$. We have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2^x = 2 = f(1) \quad \text{and} \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left(3 - \frac{1}{x^2}\right) = 2 = f(1).$$

This shows that f is also continuous at 1.

Suppose that $x_1, x_2 \in [0,2]$ and $x_1 < x_2$. If $x_2 \le 1$, then $f(x_1) = 2^{x_1} < 2^{x_2} = f(x_2)$. If $x_1 \le 1 < x_2$, then $f(x_1) \le f(1) = 2 < 3 - 1/x_2^2 = f(x_2)$. If $1 < x_1$, then $f(x_1) = 3 - 1/x_1^2 < 3 - 1/x_2^2 = f(x_2)$. This shows that f is strictly increasing on $[0,2]$.

(b) Find an explicit expression for the inverse function f^{-1} including its domain and range.

Solution. The function f maps the interval $[0,2]$ one-to-one and onto the interval $[1, 11/4]$. Consequently, the inverse function f^{-1} maps the interval $[1, 11/4]$ one-to-one and onto the interval $[0,2]$. We have

$$f^{-1}(y) = \begin{cases} \log_2 y & \text{for } 1 \le y \le 2, \\ 1/\sqrt{3-y} & \text{for } 2 < y \le 11/4. \end{cases}$$

(c) Is f^{-1} continuous on its domain? Justify your answer.

Answer. By the inverse function theorem, f^{-1} is continuous on $[1, 11/4]$. This conclusion can also be derived directly from the above expression of f^{-1}.

5. (a) Let f be the function given by

$$f(x) := \begin{cases} 3^{-1/|x|} & \text{for } x \in (-\infty, 0) \cup (0, \infty), \\ 0 & \text{for } x = 0. \end{cases}$$

Find $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$. Is f continuous on $(-\infty, \infty)$? Justify your answer.

Solution. Clearly, f is continuous at any point in $(-\infty, 0) \cup (0, \infty)$. We claim that f is also continuous at 0. Let $y := -1/|x|$ for $x \ne 0$. We have

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} \frac{-1}{x} = -\infty \quad \text{and} \quad \lim_{x \to 0^-} y = \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

It follows that

$$\lim_{x \to 0^-} 3^{-1/|x|} = \lim_{y \to -\infty} 3^y = 0 \quad \text{and} \quad \lim_{x \to 0^+} 3^{-1/|x|} = \lim_{y \to -\infty} 3^y = 0.$$

Consequently, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0 = f(0)$. Therefore, f is continuous on $(-\infty, \infty)$.

(b) Let $g(x) := \log_2 x$ for $0 < x < \infty$. Prove that g is *not* uniformly continuous on the interval $(0,1)$.

Proof. For $n = 1, 2, \ldots$, let $x_n := 2^{-n}$ and $y_n := 2^{-n-1}$. Then $\lim_{n \to \infty}(x_n - y_n) = 0$. But

$$g(x_n) - g(y_n) = \log_2 2^{-n} - \log_2 2^{-n-1} = (-n) - (-n-1) = 1.$$

By Theorem 4.1, the function g is not uniformly continuous on $(0,1)$.