MATH 314 Assignment #7

1. Let $f(x) := x^2$ for $x \geq 0$ and $f(x) := 0$ for $x < 0$.

   (a) Use the definition of derivative to show that f is differentiable at 0.

   Proof. We have

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x^2}{x} = \lim_{x\to 0^+} x = 0$$

   and

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = 0.$$

   This shows that f is differentiable at 0 and $f'(0) = 0$.

   (b) Find an explicit expression of $f'(x)$ for $x \in \mathbb{R}$.

   Solution. We have $f'(x) = 2x$ for $x \geq 0$ and $f'(x) = 0$ for $x < 0$.

   (c) Is f' continuous on $\mathbb{R}$? Is f' differentiable on $\mathbb{R}$?

   Answer. f' is continuous on $\mathbb{R}$. But f' is not differentiable at 0, since

$$\lim_{x\to 0^+} \frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0^+} \frac{2x}{x} = 2 \quad \text{and} \quad \lim_{x\to 0^-} \frac{f'(x)-f'(0)}{x-0} = 0.$$

2. Find the derivative of each of the following functions.

   (a) $g(x) := \sqrt[3]{x^2} - \frac{1}{x}$, $x \neq 0$.

   Solution. We have $g'(x) = \frac{2}{3}x^{2/3-1} - (-1)x^{-1-1} = \frac{2}{3\sqrt[3]{x}} + \frac{1}{x^2}$.

   (b) $h(x) := \frac{1-x^2}{1+x^2}$, $x \in \mathbb{R}$.

   Solution. By the quotient rule we have

$$h'(x) = \frac{(1-x^2)'(1+x^2) - (1-x^2)(1+x^2)'}{(1+x^2)^2} = \frac{-4x}{(1+x^2)^2}.$$

   (c) $u(x) := \ln(x + \sqrt{a^2+x^2})$, $x \in \mathbb{R}$.

   Solution. By the chain rule we have

$$u'(x) = \frac{1 + (\sqrt{a^2+x^2})'}{x + \sqrt{a^2+x^2}} = \frac{1 + \frac{x}{\sqrt{a^2+x^2}}}{x+\sqrt{a^2+x^2}} = \frac{1}{\sqrt{a^2+x^2}}.$$

   (d) $v(x) := x^x$, $x > 0$.

   Solution. We have $v(x) = e^{x\ln x}$. Hence, by the chain rule we have

$$v'(x) = e^{x\ln x}(x\ln x)' = x^x(1+\ln x).$$

3. Let f be a real-valued function on an open interval I, and let c be a point in I. Prove the following statements.

(a) If f is differentiable at c, then $\lim_{n\to\infty} n\big[f\big(c+\frac{1}{n}\big)-f(c)\big] = f'(c)$.

Proof. Since f is differentiable at c, we have

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = f'(c).$$

Let $x_n := 1/n$ for $n \in \mathbb{N}$. Then $\lim_{n\to\infty} x_n = 0$. It follows that

$$\lim_{n\to\infty} n\Big[f\Big(c+\frac{1}{n}\Big)-f(c)\Big] = \lim_{x_n\to 0}\frac{f(c+x_n)-f(c)}{x_n} = f'(c).$$

(b) If f is differentiable at c, then

$$\lim_{h\to 0}\frac{f(c+h)-f(c-h)}{2h} = f'(c).$$

Proof. The desired relation follows from

$$\frac{f(c+h)-f(c-h)}{2h} = \frac{1}{2}\frac{f(c+h)-f(c)}{h} + \frac{1}{2}\frac{f(c)-f(c-h)}{h}$$

and

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = f'(c) \quad\text{and}\quad \lim_{h\to 0}\frac{f(c)-f(c-h)}{h} = f'(c).$$

4. Let $f(x) := x\ln\big(1+\frac{1}{x}\big)$, $0 < x < \infty$.

(a) Compute $\lim_{x\to\infty} f(x)$.

Proof. Since $\lim_{x\to\infty}(1+1/x)^x = \lim_{y\to 0}(1+y)^{1/y} = e$, we have

$$\lim_{x\to\infty} x\ln\Big(1+\frac{1}{x}\Big) = \lim_{x\to\infty}\ln\Big(1+\frac{1}{x}\Big)^x = \ln e = 1.$$

(b) Find $f'(x)$ and $f''(x)$ for $0 < x < \infty$.

Proof. For $x > 0$, we have

$$f'(x) = \ln\Big(1+\frac{1}{x}\Big) + x\frac{1}{1+\frac{1}{x}}\Big(-\frac{1}{x^2}\Big) = \ln\Big(1+\frac{1}{x}\Big) - \frac{1}{x+1}$$

and

$$f''(x) = -\frac{1}{x(1+x)} + \frac{1}{(x+1)^2} = -\frac{1}{x(x+1)^2}.$$

(c) Prove that f' is strictly decreasing on $(0, \infty)$ and f is strictly increasing on $(0, \infty)$.

Proof. Since $f''(x) < 0$ for $0 < x < \infty$, the function f' is strictly decreasing on $(0, \infty)$. But $\lim_{x\to\infty} f'(x) = 0$. Therefore, $f'(x) > 0$ for all $x > 0$. This shows that f is strictly increasing on $(0, \infty)$.

5. Let f be the function on $\mathbb{R}$ given by

$$f(x) := \begin{cases} e^{-1/x} & \text{for } x \in (0, \infty), \\ 0 & \text{for } x \in (-\infty, 0]. \end{cases}$$

(a) Compute $f'(x)$ and $f''(x)$ for $x > 0$.

Solution. For $x > 0$ we have

$$f'(x) = \frac{1}{x^2} e^{-1/x} \quad \text{and} \quad f''(x) = \frac{-2x + 1}{x^4} e^{-1/x}.$$

(b) Show that f is differentiable at 0 and find $f'(0)$.

Proof. We have

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = 0 \quad \text{and} \quad \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{e^{-1/x}}{x} = 0$$

Hence, f is differentiable at 0 and $f'(0) = 0$.

(c) Show that f' is differentiable at 0 and find $f''(0)$.

Proof. We have

$$\lim_{x\to 0^-} \frac{f'(x) - f'(0)}{x - 0} = 0 \quad \text{and} \quad \lim_{x\to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0^+} \frac{e^{-1/x}}{x^3} = 0$$

Hence, f' is differentiable at 0 and $f''(0) = 0$.

Let k be a fixed positive integer. In order to find $\lim_{x\to 0^+}(e^{-1/x}/x^k)$, we make change of variables: $y := 1/x$. Then $\lim_{x\to 0^+} y = +\infty$. Hence,

$$\lim_{x\to 0^+} \frac{e^{-1/x}}{x^k} = \lim_{y\to\infty} \frac{e^{-y}}{(1/y)^k} = \lim_{y\to\infty} \frac{y^k}{e^y} = 0.$$