MATH 314 Assignment #8

1. For each of the following functions, determine the interval(s) where the function is increasing or decreasing, and find all maxima and minima.

   (a) $f(x) := 4x - x^4$, $x \in \mathbb{R}$.

   Solution. We have $f'(x) = 4(1 - x^3)$, $x \in \mathbb{R}$. Clearly, $f'(x) > 0$ for $x < 1$, $f'(x) = 0$ for $x = 1$, and $f'(x) < 0$ for $x > 1$. Hence, f is strictly increasing on the interval $(-\infty, 1]$ and strictly decreasing on the interval $[1, \infty)$. Moreover, $f(1) = 3$ is the (absolute) maximum of f.

   (b) $g(x) := \frac{x^2}{1+x^2}$, $x \in \mathbb{R}$.

   Solution. We have

$$g'(x) = \frac{2x(1 + x^2) - x^2(2x)}{(1 + x^2)^2} = \frac{2x}{(1 + x^2)^2}, \quad x \in \mathbb{R}.$$

   Clearly, $g'(x) < 0$ for $x < 0$, $g'(x) = 0$ for $x = 0$, and $g'(x) > 0$ for $x > 0$. Hence g is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$. Moreover, $g(0) = 0$ is the (absolute) minimum of g.

   (c) $u(x) := \sqrt{x} - x/2$, $x \geq 0$.

   Solution. We have $u'(x) = 1/(2\sqrt{x}) - 1/2$, $x > 0$. Clearly, $u'(x) > 0$ for $0 < x < 1$, $u'(x) = 0$ for $x = 1$, and $u'(x) < 0$ for $x > 1$. Hence, u is strictly increasing on the interval $(0, 1]$ and strictly decreasing on the interval $[1, \infty)$. Moreover, $u(1) = 1/2$ is the (absolute) maximum of u.

   (d) $v(x) := \frac{x}{1+|x|}$, $x \in \mathbb{R}$.

   Solution. For $x < 0$ we have $v(x) = x/(1 - x)$ and $v'(x) = 1/(1 - x)^2$. For $x > 0$ we have $v(x) = x/(1 + x)$ and $v'(x) = 1/(1 + x)^2$. For $x = 0$ we have

$$v'(0) = \lim_{h \to 0} \frac{v(h) - v(0)}{h} = \lim_{h \to 0} \frac{1}{1 + |h|} = 1.$$

   It follows that

$$v'(x) = \frac{1}{(1 + |x|)^2} > 0 \quad \text{for all } x \in (-\infty, \infty).$$

   Thus v is strictly increasing on $(-\infty, \infty)$. It has no maximum or minimum.

2. Establish the following inequalities.

   (a) For $0 < t < 1$, prove that $x^t \leq tx + (1 - t)$ for all $x > 0$.

   Proof. Let $g(x) := x^t - [tx + (1 - t)]$ for $x > 0$. Then $g'(x) = tx^{t-1} - t = t[x^{t-1} - 1]$ for $x > 0$. Since $t - 1 < 0$, we have $x^{t-1} > 1$ for $0 < x < 1$ and $x^{t-1} < 1$ for $x > 1$.

Hence $g'(x) > 0$ for $0 < x < 1$ and $g'(x) < 0$ for $x > 1$. Thus, g is increasing on $(0, 1]$ and decreasing on $[1, \infty)$. Consequently, $g(x) \le g(1) = 0$ for all $x > 0$. This shows that $x^t \le tx + (1 - t)$ for all $x > 0$.

(b) Prove that $a^t b^{1-t} \le ta + (1 - t)b$ for $a \ge 0$, $b \ge 0$, and $0 < t < 1$.

Proof. The inequality is obvious if $a = 0$ or $b = 0$. Suppose that $a > 0$ and $b > 0$. Then the inequality holds if and only if

$$\frac{a^t b^{1-t}}{b} \le \frac{ta + (1-t)b}{b}, \quad i.e., \quad \left(\frac{a}{b}\right)^t \le t\left(\frac{a}{b}\right) + (1 - t).$$

Choosing $x = a/b$ in the inequality in part (a), we see that the above inequality is valid.

3. Let g be the function given by $g(x) := \ln[(1 + x)/(1 - x)]$ for $-1 < x < 1$.

(a) Find the Taylor series of g about 0.

Solution. Note that $g(x) = \ln(1 + x) - \ln(1 - x)$. We have

$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n, \quad x \in (-1, 1).$$

Substituting $-x$ for x into the above equation, we obtain

$$\ln(1 - x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (-1)^n x^n = \sum_{n=1}^{\infty} \frac{-1}{n} x^n, \quad x \in (-1, 1).$$

It follows that

$$g(x) = \ln(1 + x) - \ln(1 - x) = \sum_{n=1}^{\infty} \frac{1 + (-1)^{n-1}}{n} x^n, \quad x \in (-1, 1).$$

If $n = 2k$ is an even number, then $1 + (-1)^{n-1} = 0$. If $n = 2k - 1$ is an odd number, then $1 + (-1)^{n-1} = 2$. Hence

$$g(x) = \sum_{k=1}^{\infty} \frac{2}{2k - 1} x^{2k-1}, \quad x \in (-1, 1).$$

(b) Find the interval of convergence of the power series in (a).

Solution. The radius of convergence of the above power series is 1. When $x = 1$ we have

$$\sum_{k=1}^{\infty} \frac{2}{2k - 1} x^{2k-1} = \sum_{k=1}^{\infty} \frac{2}{2k - 1}.$$

We observe that $2/(2k-1) \geq 1/k$ for all $k \in \mathbb{N}$. Since the harmonic series $\sum_{k=1}^{\infty} 1/k$ diverges, the series $\sum_{k=1}^{\infty} \frac{2}{2k-1}$ diverges, by the comparison test. Similarly, when $x = -1$, the series $\sum_{k=1}^{\infty} \frac{2}{2k-1} x^{2k-1} = -\sum_{k=1}^{\infty} \frac{2}{2k-1}$ diverges. This shows that the interval of convergence of the series $\sum_{k=1}^{\infty} \frac{2}{2k-1} x^{2k-1}$ is $(-1, 1)$.

(c) Use the power series in (a) to evaluate $\ln 2 = g(1/3)$ accurate to four decimal places.

Solution. We have

$$\ln 2 = g(1/3) = \sum_{k=1}^{\infty} \frac{2}{2k-1} \left(\frac{1}{3}\right)^{2k-1}.$$

For a positive integer n, we use s_n to denote the nth partial sum of the above seris. A choice of $n = 4$ gives

$$0 < \ln 2 - s_4 = \sum_{k=5}^{\infty} \frac{2}{2k-1} \left(\frac{1}{3}\right)^{2k-1} \leq \frac{2}{9} \sum_{k=5}^{\infty} \left(\frac{1}{3}\right)^{2k-1} = \frac{2}{9} \left(\frac{1}{3}\right)^{9} \frac{1}{1 - 1/9} < 0.00002.$$

Thus we obtain

$$\ln 2 \approx s_4 = 2\left(\frac{1}{3}\right) + \frac{2}{3}\left(\frac{1}{3}\right)^3 + \frac{2}{5}\left(\frac{1}{3}\right)^5 + \frac{2}{7}\left(\frac{1}{3}\right)^7 \approx 0.69313.$$

This approximation is accurate to four decimal places.

4. Let f be the function on $\mathbb{R}$ defined by

$$f(x) := \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \in \mathbb{R} \setminus \{0\}, \\ 0 & \text{for } x = 0. \end{cases}$$

(a) Find $f'(x)$ for for $x \in \mathbb{R} \setminus \{0\}$.

Solution. For $x \neq 0$ we have

$$f'(x) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \left(-\frac{1}{x^2}\right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

(b) Prove that f is differentiable at 0 and that $f'(0) = 0$.

Solution. For $h \neq 0$ we have

$$\frac{f(0+h) - f(0)}{h} = \frac{h^2 \sin(1/h)}{h} = h \sin \frac{1}{h}.$$

Note that $|\sin(1/h)| \leq 1$ for all $h \in \mathbb{R} \setminus \{0\}$. It follows that $|h \sin(1/h)| \leq |h|$. By the squeeze theorem for limits we obtain

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

Therefore, f is differentiable at 0 and $f'(0) = 0$.

(c) Show that f' is not continuous at 0.

Proof. Choose $x_n := 1/(2n\pi)$ for $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} x_n = 0$. On the other hand, by part (a) we have

$$f'(x_n) = \frac{2}{2n\pi}\sin(2n\pi) - \cos(2n\pi) = -1 \quad \forall n \in \mathbb{N}.$$

Hence $\lim_{n\to\infty} f'(x_n) = -1 \neq f'(0)$. This shows that f' is not continuous at 0.

5. Let $u(x) := \arctan x$ and $v(x) := 1/(1 + x^2)$ for $x \in (-\infty, \infty)$.

(a) Find the Taylor series of v about 0 and its interval of convergence.

Solution. We have

$$v(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

The interval of convergence of the above power series is $(-1, 1)$.

(b) Find the Taylor series of u about 0 and its interval of convergence.

Solution. Since $u' = v$ and $u(0) = 0$, we have

$$u(x) = \arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}.$$

At $x = 1$ the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

is an alternating series. Note that $\lim_{n\to\infty} 1/(2n+1) = 0$ and $1/(2n+1) > 1/(2n+3)$ for all $n \in \mathbb{N}$. By the alternating series test, $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ converges. For the same reason, at $x = -1$ the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(-1)^{2n+1}$ converges. Therefore, the interval of convergence of the above power series is $[-1, 1]$.

(c) Compute $v^{(6)}(0)$ and $v^{(7)}(0)$.

Solution. By the power series expansion of v we have

$$\frac{v^{(6)}(0)}{6!} = (-1)^3 \quad \text{and} \quad \frac{v^{(7)}(0)}{7!} = 0.$$

It follows that $v^{(6)}(0) = -720$ and $v^{(7)}(0) = 0$.

(d) Compute $u^{(6)}(0)$ and $u^{(7)}(0)$.

Solution. By the power series expansion of u we have

$$\frac{u^{(6)}(0)}{6!} = 0 \quad \text{and} \quad \frac{u^{(7)}(0)}{7!} = \frac{(-1)^3}{7}.$$

It follows that $u^{(6)}(0) = 0$ and $u^{(7)}(0) = -720$.