MATH 314 Assignment #9

1. Let f be an increasing function on $[a, b]$ with $-\infty < a < b < \infty$.

   (a) Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$. Prove

$$U(f, P) - L(f, P) \le \sum_{i=1}^{n} [f(t_i) - f(t_{i-1})](t_i - t_{i-1}).$$

Proof. Let $m_i := \inf\{f(x) : t_{i-1} \le x \le t_i\}$ and $M_i := \sup\{f(x) : t_{i-1} \le x \le t_i\}$ for $i = 1, \ldots, n$. Since f is an increasing function on $[a, b]$, we have $m_i = f(t_{i-1})$ and $M_i = f(t_i)$. It follows that

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) - \sum_{i=1}^{n} m_i(t_i - t_{i-1}) = \sum_{i=1}^{n} [f(t_i) - f(t_{i-1})](t_i - t_{i-1}).$$

   (b) Prove that $U(f, P) - L(f, P) \le [f(b) - f(a)]\delta$ whenever $\|P\| < \delta$.

Proof. Suppose $\|P\| < \delta$. Then $t_i - t_{i-1} \le \delta$ for $i = 1, \ldots, n$. Since $f(t_i) \ge f(t_{i-1})$, we have $[f(t_i) - f(t_{i-1})](t_i - t_{i-1}) \le [f(t_i) - f(t_{i-1})]\delta$. Consequently,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} [f(t_i) - f(t_{i-1})](t_i - t_{i-1}) \\ &\le \sum_{i=1}^{n} [f(t_i) - f(t_{i-1})]\delta = [f(b) - f(a)]\delta. \end{aligned}$$

   (c) Prove that f is integrable on $[a, b]$.

Proof. If f is a constant function, then f is integrable. Thus we suppose that f is a non-constant increasing function on $[a, b]$. Given $\varepsilon > 0$, choose $\delta := \varepsilon / [f(b) - f(a)]$. Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$ with $\|P\| < \delta$. By part (a) we have

$$U(f, P) - L(f, P) \le [f(b) - f(a)]\|P\| < [f(b) - f(a)]\delta = \varepsilon.$$

Therefore f is integrable on $[a, b]$.

2. Let g be the function on $[0, 1]$ defined by $g(0) := 0$ and

$$g(x) := 2^{-n} \text{ for } 2^{-n-1} < x \le 2^{-n}, \quad n = 0, 1, 2, \ldots.$$

   (a) Prove that g is integrable on $[0, 1]$.

Proof. By what has been proved in problem 1, it suffices to show that g is an increasing function on $[0,1]$. Suppose $0 \le x < y \le 1$. If $x = 0$, then $g(0) = 0 < g(y)$. So we may assume $x > 0$. There exists a unique nonnegative integer n such that $2^{-n-1} < x \le 2^{-n}$. It follows that $2^{-n-1} < x < y$. There exists a unique nonnegative integer m such that $2^{-m-1} < y \le 2^{-m}$. Since $y > 2^{-n-1}$, we must have $m \le n$. By the definition of g we have

$$g(x) = 2^{-n} \le 2^{-m} = g(y).$$

This shows that g is an increasing function on $[0,1]$.

(b) Find $\int_0^1 g(x)\,dx$.

Solution. We have

$$\int_0^1 g(x)\,dx = \int_0^{1/2^n} g(x)\,dx + \int_{1/2^n}^1 g(x)\,dx = t_n + s_n,$$

where $t_n := \int_0^{1/2^n} g(x)\,dx$ and $s_n := \int_{1/2^n}^1 g(x)\,dx$ for $n \in \mathbb{N}$. Since $0 \le g(x) \le 1/2^n$ for $x \in [0, 1/2^n]$, we get

$$0 \le t_n = \int_0^{1/2^n} g(x)\,dx \le \frac{1}{2^n}\frac{1}{2^n} = \frac{1}{4^n}.$$

It follows that $\lim_{n\to\infty} t_n = 0$. Moreover,

$$s_n = \int_{1/2^n}^1 g(x)\,dx = \sum_{k=0}^{n-1} \int_{1/2^{k+1}}^{1/2^k} g(x)\,dx = \sum_{k=0}^{n-1} \int_{1/2^{k+1}}^{1/2^k} \frac{1}{2^k}\,dx.$$

It follows that

$$s_n = \sum_{k=0}^{n-1} \frac{1}{2^k}\left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right) = \sum_{k=0}^{n-1} \frac{1}{2^k}\frac{1}{2^{k+1}} = \sum_{k=0}^{n-1} \frac{1}{2^{2k+1}} = \frac{1}{2}\frac{1-(1/4)^n}{1-1/4}.$$

Since $\int_0^1 g(x)\,dx = t_n + s_n$ for all $n \in \mathbb{N}$, we obtain

$$\int_0^1 g(x)\,dx = \lim_{n\to\infty}(t_n + s_n) = \lim_{n\to\infty} t_n + \lim_{n\to\infty} s_n = \frac{1}{2}\frac{1}{1-1/4} = \frac{2}{3}.$$

3. Let f be the function on $\mathbb{R}$ defined as follows: $f(x) := 0$ for $x < 0$; $f(x) := x$ for $0 \le x \le 1$; $f(x) := 2$ for $x > 1$.

(a) Find an explicit expression of the function $F(x) := \int_0^x f(t)\,dt$, $x \in \mathbb{R}$.

Solution. For $x < 0$ we have

$$F(x) = \int_0^x f(t)\,dt = -\int_x^0 f(t)\,dt = 0.$$

For $0 \le x \le 1$ we have

$$F(x) = \int_0^x f(t)\,dt = \int_0^x t\,dt = \frac{x^2}{2}.$$

For $x > 1$ we have

$$F(x) = \int_0^x f(t)\,dt = \int_0^1 t\,dt + \int_1^x 2\,dt = \frac{1}{2} + 2(x-1) = 2x - \frac{3}{2}.$$

(b) Is F continuous on $\mathbb{R}$?

Answer. By the Fundamental Theorem of Calculus, F is continuous on $\mathbb{R}$. This also can be seen from the explicit expression of F given in part (a).

(c) Where is F differentiable? Calculate F' at the points of differentiability.

Solution. For $x < 0$ we have $F'(x) = 0$. For $0 < x < 1$ we have $F'(x) = x$. For $x > 1$ we have $F'(x) = 2$. At $x = 0$ we have

$$\lim_{h\to 0^-} \frac{F(0+h) - F(0)}{h} = 0 \quad \text{and} \quad \lim_{h\to 0^+} \frac{F(0+h) - F(0)}{h} = \lim_{h\to 0^+} \frac{h^2/2}{h} = 0.$$

Consequently, F is differentiable at 0 and $F'(0) = 0$. At $x = 1$ we have

$$\lim_{h\to 0^-} \frac{F(1+h) - F(1)}{h} = 1 \quad \text{and} \quad \lim_{h\to 0^+} \frac{F(1+h) - F(1)}{h} = 2.$$

Consequently, F is *not* differentiable at 1. We conclude that F is differentiable on $\mathbb{R} \setminus \{1\}$.

4. (a) Let $G(x) := \int_{-x}^{x^2} \sqrt{1+t^2}\,dt$, $x \in \mathbb{R}$. Find $G'(x)$ for $x \in \mathbb{R}$.

Solution. We have

$$G(x) = \int_{-x}^{x^2} \sqrt{1+t^2}\,dt = \int_0^{x^2} \sqrt{1+t^2}\,dt - \int_0^{-x} \sqrt{1+t^2}\,dt.$$

By the Fundamental Theorem of Calculus and the chain rule, we obtain

$$G'(x) = 2x\sqrt{1+(x^2)^2} + \sqrt{1+(-x)^2} = 2x\sqrt{1+x^4} + \sqrt{1+x^2}, \quad x \in \mathbb{R}.$$

(b) Let $H(x) := \int_0^x xe^{t^2}\,dt$ for $x \in \mathbb{R}$. Find $H''(x)$ for $x \in \mathbb{R}$.

Solution. By the product rule and the Fundamental Theorem of Calculus, we obtain

$$H'(x) = \int_0^x e^{t^2}\,dt + xe^{x^2}.$$

It follows that

$$H''(x) = e^{x^2} + e^{x^2} + x(2x)e^{x^2} = 2e^{x^2} + 2x^2e^{x^2}, \quad x \in \mathbb{R}.$$

5. Let

$$F(x) := \int_x^{x+\pi} |\cos t|\,dt, \quad x \in \mathbb{R}.$$

(a) Find $F'(x)$ for $x \in \mathbb{R}$.

Solution. By the fundamental theorem of calculus we have

$$F'(x) = |\cos(x+\pi)| - |\cos x| = |-\cos x| - |\cos x| = 0, \quad x \in \mathbb{R}.$$

(b) Find an explicit expression for $F(x)$, $x \in \mathbb{R}$.

Solution. By part (a) we see that F is constant. Hence, for all $x \in \mathbb{R}$,

$$F(x) = F(-\pi/2) = \int_{-\pi/2}^{\pi/2} |\cos t|\,dt = \int_{-\pi/2}^{\pi/2} \cos t\,dt = \big[\sin t\big]_{-\pi/2}^{\pi/2} = 2.$$