Dependence Relations Among the Shifts of a Multivariate Refinable Distribution

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Abstract. Refinable functions are an intrinsic part of subdivision schemes and wavelet constructions. The relevant properties of such functions must usually be determined from their refinement masks. In this paper, we provide a characterization of linear independence for the shifts of a multivariate refinable vector of distributions in terms of its (finitely supported) refinement mask.

1. Introduction

Suppose $\phi_1, \dots, \phi_r$ are functions on $\mathbf{R}^s$. We say that the vector $\phi = (\phi_1, \dots, \phi_r)^T$ is **refinable** if it satisfies the following refinement equation:

$$(1.1) \quad \phi = \sum_{\alpha \in \mathbf{Z}^s} a(\alpha) \phi(2 \cdot - \alpha),$$

where a is a finitely supported sequence of $r \times r$ complex matrices, called the **refinement mask**. Refinable functions are an intrinsic part of subdivision schemes and wavelet constructions. In general, any relevant properties of the vector ϕ must be determined from the mask a . In Section 2 of this paper, we provide a characterization of linear independence of the shifts of a continuous multivariate refinable vector of functions in terms of its refinement mask. A similar characterization is provided in Section 3 for refinable distributions.

Here, and throughout this paper, for c in $\mathbf{C}^{r \times \mathbf{Z}^s}$ and $\alpha \in \mathbf{Z}^s$, $c(\alpha)$ denotes the α th column of c . More generally, for any subset K of $\mathbf{Z}^s$, any c in $\mathbf{C}^{r \times K}$, and any α in K :

$$c(\alpha) := (c(1, \alpha), \dots, c(r, \alpha))^T \in \mathbf{C}^r.$$

For a vector ϕ of compactly supported functions, define

$$N(\phi) := \left\{ c \in \mathbf{C}^{r \times \mathbf{Z}^s} : \sum_{\alpha \in \mathbf{Z}^s} \langle c(\alpha), \phi(\cdot - \alpha) \rangle = 0 \right\},$$

where, for any two vectors u and v in $\mathbf{C}^r$, $\langle u, v \rangle := \sum_{j=1}^r u(j)v(j)$.

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We say that ϕ has **linearly independent** shifts if $N(\phi)$ is trivial, i.e., $N(\phi) = \{0\}$. When ϕ is a vector of compactly supported distributions, Jia and Micchelli [15] showed that $N(\phi)$ is nontrivial if and only if it contains a sequence c of the form $c(j, \alpha) = \lambda(j)z^\alpha$ ($j = 1, \dots, r; \alpha \in \mathbf{Z}^s$) for some $\lambda \in \mathbf{C}^r \setminus \{0\}$ and some $z \in (\mathbf{C} \setminus \{0\})^s$. Earlier results of this kind were provided for scalar-valued continuous functions by Dahmen and Micchelli in [3] and for scalar-valued distributions by Ron in [22].

Linear independence is a necessary condition for orthogonality, or even biorthogonality, of refinable functions. It is also a sufficient condition for stability. In fact, our results provide a characterization of stability for the shifts of ϕ in terms of a , since we characterize all exponentials in $N(\phi)$ and ϕ has stable shifts if and only if $N(\phi)$ contains an exponential which lies on the s -dimensional unit torus (see [14]). A characterization of stability for the shifts of a refinable vector of functions in $L_2(\mathbf{R}^s)$ was also provided by Shen in [26] (also see [19] and [20]) but, as our examples in Section 2 demonstrate, the results of this paper can be less complicated to apply. Moreover, Section 3 characterizes linear independence for refinable distributions, while the method in [20] does not apply to distributions.

In the univariate case, a useful characterization of linear independence for the shifts of a single function ϕ was given in terms of a by Jia and Wang in [17]. Related results were provided by Ron in [23] and Zhou in [28]. However, their techniques are inherently univariate. Attempts to generalize their results to functions of several variables have been made by Hogan [12] and Zhou [30]. In [12], the conditions of [17] were shown to be necessary in several variables; and they were shown to be also sufficient for functions of a certain type. These results were not satisfactory for two reasons: the proofs do not apply to general multivariate functions; and the conditions, though easy to verify in the univariate case, are more elusive in several variables. In [30], a fairly easily verifiable condition on the mask a was provided which, together with stability, characterizes linear independence. However, no satisfactory means were provided for determining the stability. In the univariate case, a generalization of the results of [17] to vectors of functions was provided by Hogan in [10] (also see [21] and [27]), but the techniques used were again inherently univariate.

If ϕ is a vector of compactly supported functions in $L_1(\mathbf{R}^s)$ with linearly independent shifts, it is known (see [4], [11], [26]) that the matrix

$$(1.2) \quad M := 2^{-s} \sum_{\alpha \in \mathbf{Z}^s} a(\alpha)$$

has a simple eigenvalue 1 and all the other eigenvalues of M are less than 1 in modulus. In fact, this result is valid under the weaker condition that the entries of $\sum_{\alpha \in \mathbf{Z}^s} \phi(\cdot - \alpha)$ are linearly independent. So we assume throughout this paper that M satisfies these conditions, and we let $y = (y_1, \dots, y_r)$ denote some fixed left eigenvector of M corresponding to the eigenvalue 1. That is, $y \neq 0$ and

$$yM = y.$$

Under these conditions, there exists a unique distributional solution ϕ to (1.1) such that $y\hat{\phi}(0) = 1$ (see [9]). We call this solution the **normalized solution** of (1.1).

Given a mask a , there is, in general, no explicit expression for the solution ϕ to (1.1). Instead, the solution is approximated by iterating the **cascade operator**

$$T_a : (C_c(\mathbf{R}^s))^r \rightarrow (C_c(\mathbf{R}^s))^r : f \mapsto T_a f := \sum_{\alpha \in \mathbf{Z}^s} a(\alpha) f(2 \cdot - \alpha)$$

associated with a . Here and throughout, $(C_c(\mathbf{R}^s))^r$ denotes the normed linear space of all r -vectors of compactly supported continuous functions on $\mathbf{R}^s$ equipped with the norm

$$\|f\| := \max_{1 \leq j \leq r} \max_{x \in \mathbf{R}^s} |f_j(x)|, \quad f \in (C_c(\mathbf{R}^s))^r.$$

This process is called a **subdivision scheme**. If $\lim_{n \rightarrow \infty} \|T_a^n f_0 - \phi\| = 0$ for some $f_0 \in (C_c(\mathbf{R}^s))^r$, then one must have

$$y \sum_{\alpha \in \mathbf{Z}^s} f_0(\cdot - \alpha) = 1.$$

(A derivation of this can be found in [16] for the case $s = 1$. The reasoning there is, in fact, valid for any s . We omit it for brevity.) Thus, we say that the subdivision scheme associated with the mask a converges if there exists $\phi \in (C_c(\mathbf{R}^s))^r$ such that, for every $f_0 \in (C_c(\mathbf{R}^s))^r$ satisfying $y \sum_{\alpha \in \mathbf{Z}^s} f_0(\cdot - \alpha) = 1$:

$$\lim_{n \rightarrow \infty} \|T_a^n f_0 - \phi\| = 0.$$

In this case, the limit is the normalized solution of (1.1) and it is continuous.

The results in Section 2 of this paper apply to compactly supported refinable vectors of continuous functions. In particular, they are applicable whenever the associated subdivision scheme converges. The results in Section 3 are for compactly supported refinable vectors of distributions.

2. Continuous Functions

In this section, it is assumed that the entries of ϕ are continuous. We begin with some notation and terminology.

Suppose a is supported on the closed cell

$$\prod_{j=1}^s [M_j, N_j],$$

where M_j and N_j are integers and $M_j < N_j$, $j = 1, \dots, s$. (We point out that ϕ is supported on the same closed cell.) Let

$$K := \mathbf{Z}^s \cap \left(\prod_{j=1}^s [M_j, N_j - 1] \right).$$

For $\alpha \in \mathbf{Z}^s$, we define

$$\Phi_\alpha(x) := \begin{cases} \phi(x + \alpha) & \text{for } x \in [0, 1)^s, \\ 0 & \text{for } x \in \mathbf{R}^s \setminus [0, 1)^s. \end{cases}$$

Clearly, $\Phi_\alpha = 0$ for $\alpha \in \mathbf{Z}^s \setminus K$ and $\phi = \sum_{\alpha \in K} \Phi_\alpha(\cdot - \alpha)$. Let $\Phi := (\Phi_\alpha)_{\alpha \in K}$. Then $\Phi(x) \in \mathbf{C}^{r \times K}$ for all x in $\mathbf{R}^s$, $\Phi(x) = 0$ if $x \notin [0, 1]^s$, and Φ (i.e., each entry of Φ) is continuous on $[0, 1]^s$.

By E we denote the set $\{0, 1\}^s$ of all vertices of the unit cube $[0, 1]^s$. For $\varepsilon \in E$, let A_ε be the linear operator on $\mathbf{C}^{r \times K}$ given by

$$A_\varepsilon v(\alpha) := \sum_{\beta \in K} a(\varepsilon + 2\alpha - \beta) v(\beta), \quad \alpha \in K,$$

where $v \in \mathbf{C}^{r \times K}$. Now, for all $\varepsilon \in E$, if $x \in \varepsilon/2 + [0, \frac{1}{2})^s$ then, by (1.1):

$$\begin{aligned} \Phi_\alpha(x) &= \phi(x + \alpha) = \sum_{\gamma \in \mathbf{Z}^s} a(\gamma) \phi(2x + 2\alpha - \gamma) \\ &= \sum_{\beta \in K} a(\varepsilon + 2\alpha - \beta) \Phi_\beta(2x - \varepsilon), \quad \alpha \in K. \end{aligned}$$

This shows that Φ satisfies the following refinement equation:

$$(2.1) \quad \Phi = \sum_{\varepsilon \in E} A_\varepsilon \Phi(2 \cdot - \varepsilon).$$

Let $\mathcal{A}$ be a finite collection of linear operators on a vector space X . A subspace Y of X is said to be $\mathcal{A}$ -invariant if it is invariant under every operator A in $\mathcal{A}$. Let u be a vector in X . The intersection of all $\mathcal{A}$ -invariant subspaces of X that contain u is itself $\mathcal{A}$ -invariant. We call this the **minimal common invariant subspace** of the operators A in $\mathcal{A}$ generated by u .

Lastly, for any two vectors $u = (u(\alpha))_{\alpha \in K}$ and $v = (v(\alpha))_{\alpha \in K}$ in $\mathbf{C}^{r \times K}$, define

$$\langle u, v \rangle := \sum_{\alpha \in K} \langle u(\alpha), v(\alpha) \rangle.$$

2.1. Results and Proofs

Let V be the minimal common invariant subspace of $\{A_\varepsilon: \varepsilon \in E\}$ generated by the vector $u := \Phi(0) = (\phi(\alpha))_{\alpha \in K} \in \mathbf{C}^{r \times K}$.

Lemma 1. *For a vector $c \in \mathbf{C}^{r \times K}$, the following two conditions are equivalent:*

- (a) $\langle c, \Phi(x) \rangle = 0$ for all $x \in [0, 1]^s$.
- (b) $\langle c, v \rangle = 0$ for all $v \in V$.

Remark. A variation of Lemma 1 has been provided already by Theorem 2.7 of [8]. In fact, it was that theorem which motivated our current work.

Proof. Suppose c is an element of $\mathbf{C}^{r \times K}$ such that $\langle c, \Phi(x) \rangle = 0$ for all $x \in [0, 1]^s$. The space V is spanned by the vectors $A_{\varepsilon_1} \cdots A_{\varepsilon_j} u$, where $j = 0, 1, \dots$ and $\varepsilon_1, \dots, \varepsilon_j \in E$. With the help of refinement equation (1.1), we have $A_{\varepsilon_1} \cdots A_{\varepsilon_j} u = \Phi(x)$ for $x := \sum_{i=1}^j 2^{-i} \varepsilon_i$. Hence, $\langle c, A_{\varepsilon_1} \cdots A_{\varepsilon_j} u \rangle = 0$. In other words, $\langle c, v \rangle = 0$ for all $v \in V$. This shows that (a) implies (b).

To prove that (b) implies (a), let G be the set of those points $(m_1/2^n, \dots, m_s/2^n)$ for which $n = 1, 2, \dots$ and $m_1, \dots, m_s$ are odd integers between 0 and 2^n . Evidently, G is dense in the closed cube $[0, 1]^s$. We claim that $\Phi(x) \in V$ for each $x = (m_1/2^n, \dots, m_s/2^n) \in G$. This will be proved by induction on n .

When $n = 1$ and $x = (m_1/2, \dots, m_s/2)$, we must have $m_1 = \dots = m_s = 1$. Hence $x = \mathbf{1}/2$ and $\Phi(x) = A_{\mathbf{1}} u \in V$, where $\mathbf{1} := (1, \dots, 1) \in E$. Suppose $n > 1$ and our claim has been verified for $n - 1$. Let $x = (m_1/2^n, \dots, m_s/2^n)$, where $m_1, \dots, m_s$ are odd integers between 0 and 2^n . Set $\eta = (\eta_1, \dots, \eta_s)$, where

$$\eta_j := \begin{cases} 0 & \text{if } m_j < 2^{n-1}, \\ 1 & \text{if } m_j > 2^{n-1}, \end{cases}$$

for $j = 1, \dots, s$. By using the vector refinement equation, we have

$$\Phi(x) = \sum_{\varepsilon \in E} A_\varepsilon \Phi(2x - \varepsilon) = A_\eta \Phi(y),$$

where $y = (m_1/2^{n-1} - \eta_1, \dots, m_s/2^{n-1} - \eta_s)$. By the induction hypothesis, we have $\Phi(y) \in V$. Since V is invariant under A_η , it follows that $\Phi(x) = A_\eta \Phi(y) \in V$. This justifies our claim that $\Phi(x) \in V$ for each $x = (m_1/2^n, \dots, m_s/2^n) \in G$.

Let c be an element of $\mathbf{C}^{r \times K}$ such that $\langle c, v \rangle = 0$ for all $v \in V$. By what has been proved, $\langle c, \Phi(x) \rangle = 0$ for all $x \in G$. But G is dense in $[0, 1]^s$ and Φ is continuous on $[0, 1]^s$. Therefore, we have $\langle c, \Phi(x) \rangle = 0$ for all $x \in [0, 1]^s$. This shows that (b) implies (a). $\blacksquare$

If $V = \mathbf{C}^{r \times K}$, then $\langle \lambda, v \rangle = 0$ for all $v \in V$ implies $\lambda = 0$. So, by Lemma 1, a sufficient condition for the shifts of ϕ to be linearly independent is that $V = \mathbf{C}^{r \times K}$. This condition, which is also necessary when $r = s = 1$, is not necessary in general. A simple necessary and sufficient condition is provided by the following theorem.

Now, let $\{v_1, \dots, v_m\}$ be a basis for V . For $j = 1, \dots, r$ and $k = 1, \dots, m$, define

$$p_{j,k}(z) := \sum_{\alpha \in K} v_k(j, \alpha) z^\alpha$$

and let P denote the $r \times m$ matrix of Laurent polynomials

$$P(z) := \begin{bmatrix} p_{1,1}(z) & \dots & p_{1,m}(z) \\ \vdots & \ddots & \vdots \\ p_{r,1}(z) & \dots & p_{r,m}(z) \end{bmatrix}.$$

Then we have the following theorem:

Theorem 1. *The shifts of ϕ are linearly independent if and only if $\text{rank } P(z) = r$ for all $z \in (\mathbf{C} \setminus \{0\})^s$. Moreover, $c \in N(\phi)$ if and only if, for all $v \in V$:*

$$(2.2) \quad \sum_{\beta \in K} \langle c(\alpha - \beta), v(\beta) \rangle = 0 \quad \text{for all } \alpha \in \mathbf{Z}^s.$$

The following immediate corollary shows how the statement of Theorem 1 is simplified when ϕ is not a vector, i.e., when $r = 1$.

Corollary 1. *The shifts of ϕ are linearly independent if and only if the Laurent polynomials*

$$p_v(z) := \sum_{\alpha \in K} v(\alpha) z^\alpha, \quad v \in V,$$

have no common zeros in $(\mathbf{C} \setminus \{0\})^s$. Moreover, $c \in N(\phi)$ if and only if $c * v = 0$ for all $v \in V$, where $c * v \in \mathbf{C}^{\mathbf{Z}^s}$ is defined by

$$c * v(\alpha) := \sum_{\beta \in K} c(\alpha - \beta) v(\beta), \quad \alpha \in \mathbf{Z}^s.$$

Proof of Theorem 1. Let $c \in \mathbf{C}^{r \times \mathbf{Z}^s}$. Then

$$\begin{aligned} \sum_{\alpha \in \mathbf{Z}^s} \langle c(\alpha), \phi(\cdot - \alpha) \rangle &= \sum_{\alpha \in \mathbf{Z}^s} \left\langle c(\alpha), \sum_{\beta \in K} \Phi_\beta(\cdot - \alpha - \beta) \right\rangle \\ &= \sum_{\alpha \in \mathbf{Z}^s} \sum_{\beta \in K} \langle c(\alpha - \beta), \Phi_\beta(\cdot - \alpha) \rangle. \end{aligned}$$

So $c \in N(\phi)$ if and only if $\sum_{\beta \in K} \langle c(\alpha - \beta), \Phi_\beta \rangle = 0$ for all $\alpha \in \mathbf{Z}^s$, since the supports of $\Phi(\cdot - \alpha)$ are pairwise disjoint. By Lemma 1, this is equivalent to (3.3) holding for every $v \in V$.

Now, according to [15], the shifts of ϕ are linearly dependent if and only if there exist $\lambda \in \mathbf{C}^r \setminus \{0\}$ and $z \in (\mathbf{C} \setminus \{0\})^s$ such that $\sum_{\alpha \in \mathbf{Z}^s} z^{-\alpha} \langle \lambda, \phi(\cdot - \alpha) \rangle = 0$. For $\lambda \in \mathbf{C}^r \setminus \{0\}$ and $z \in (\mathbf{C} \setminus \{0\})^s$, define $c_{\lambda, z} \in \mathbf{C}^{r \times K}$ by

$$c_{\lambda, z}(j, \alpha) := \lambda(j) z^{-\alpha}, \quad j = 1, \dots, r, \quad \alpha \in K.$$

Then for each $k = 1, \dots, m$ and each $\alpha \in \mathbf{Z}^s$:

$$\begin{aligned} \sum_{\beta \in K} \langle c_{\lambda, z}(\alpha - \beta), v_k(\beta) \rangle &= \sum_{\beta \in K} \sum_{j=1}^r \lambda(j) z^{\beta - \alpha} v_k(j, \beta) \\ &= z^{-\alpha} \sum_{j=1}^r \lambda(j) \sum_{\beta \in K} v_k(j, \beta) z^\beta \\ &= z^{-\alpha} \sum_{j=1}^r \lambda(j) p_{j, k}(z). \end{aligned}$$

So ϕ has dependent shifts if and only if there exist $\lambda \in \mathbf{C}^r \setminus \{0\}$ and $z \in (\mathbf{C} \setminus \{0\})^s$ such that $\lambda^T P(z) = 0$ which, in turn, is equivalent to saying that $\text{rank } P(z)$ is less than r for this particular z . ■

These results are valid for any compactly supported refinable vector, ϕ , of continuous functions. However, their application requires knowledge of the values

$$\Phi(0) = (\phi(\alpha))_{\alpha \in K}.$$

These values are easily obtained if the subdivision scheme associated with the mask a converges (if the subdivision scheme does not converge, the results of Section 3 may be used).

Suppose the subdivision scheme associated with a converges. By (2.1), we have $\Phi(0) = A_0\Phi(0)$, so 1 is an eigenvalue of A_0 . Let

$$V := \left\{ v \in \mathbf{C}^{r \times K} : \sum_{\alpha \in K} \langle y, v(\alpha) \rangle = 0 \right\}.$$

Then convergence of the subdivision scheme implies that V is invariant under both A_0 and A_1 , and that $\rho(A_0|_V, A_1|_V) < 1$. (For the univariate case, this was proved in [15, Theorem 5.3]. The same reasoning is valid for the multivariate case.) It follows that $\rho(A_0|_V) < 1$. Note that V is a subspace of $\mathbf{C}^{r \times K}$ of codimension 1. Therefore, 1 is a simple eigenvalue of A_0 and the other eigenvalues of A_0 are less than 1 in modulus. In fact, $\Phi(0) = (\phi(\alpha))_{\alpha \in K}$ is the unique eigenvector of A_0 corresponding to the eigenvalue 1 and subject to the condition $y \sum_{\alpha \in K} \phi(\alpha) = 1$.

2.2. Examples

We provide two examples of bivariate functions with linearly dependent shifts and an example of a univariate vector of functions with independent shifts. In each of the bivariate examples, the symbol of the mask is not factorizable—so the characterizations provided in [12] are not applicable. In the first example, the function actually has stable shifts, so the dependence relations would not be identified by the techniques of [20]. It is true, however, that necessary conditions for independence were provided in [12] which would allow one to determine that these functions have dependent shifts, if one could identify the zero set of the symbol of the mask. In our second example, the symbol of the mask has no symmetric zeros. This, along with the fact that the symbol is not factorizable, makes it especially difficult to identify its pertinent zeros. The vector example is provided primarily to illustrate how the matrix $P(z)$ of Theorem 1 is constructed from a basis for V .

To apply our results to the first two examples, we will need to know that the associated subdivision schemes converge. One necessary condition for convergence (see [2, Proposition 2.1]) is that the mask a satisfies the **basic sum rules**:

$$\sum_{\beta \in \mathbf{Z}^s} a(\varepsilon + 2\beta) = 1 \quad \forall \varepsilon \in \mathbf{Z}^s;$$

and the masks in these two examples clearly satisfy this condition. The masks are also nonnegative, i.e., $a(\alpha) \geq 0$ for all $\alpha \in \mathbf{Z}^s$. So, by Theorem 1.2 of [18], convergence of the associated subdivision schemes will be guaranteed if we supply another nonnegative mask whose support is contained in the support of a and whose associated subdivision scheme is known to converge. But, in both examples, the support of a contains the set $\{0, 1, 2\}^2$ which is the support of the nonnegative mask for the refinable function $\chi_{[0,1)^2} * \chi_{[0,1)^2}$. Since the subdivision scheme associated with this mask is known to converge, the subdivision schemes associated with the masks in the following two examples also converge.

Example 1. Let ϕ be a solution of the refinement equation

$$\phi = \sum_{\alpha \in \mathbb{Z}^2} a(\alpha) \phi(2 \cdot - \alpha),$$

where the symbol of the mask a is given by

$$\begin{aligned} \sum_{\alpha \in \mathbb{Z}^2} a(\alpha) z^\alpha &:= \frac{1}{12} \{z_2^3 + z_1^3 + 2(1 + z_2 + z_1) + 3(z_2^2 + z_1^2) \\ &\quad + 4(z_1 z_2 + z_1 z_2^3 + z_1^2 z_2^2 + z_1^2 z_2^3 + z_1^3 z_2 + z_1^3 z_2^2) + 5(z_1 z_2^2 + z_1^2 z_2)\}. \end{aligned}$$

The nonzero terms of the sequence $12a$ are shown in Figure 1(a) along with an outline of the support of ϕ . The lower left corner of this support is at the origin $(0, 0)$. In this case, $K = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}$.

As we have said, the associated subdivision scheme converges. So let $u \in \mathbb{C}^K$ be the eigenvector of $A_{(0,0)}$ corresponding to the eigenvalue 1 such that the sum of its components is equal to 1. Using Maple, we obtain

$$u = \Phi(0) = \frac{1}{15}(0, 0, 0, 0, 3, 4, 0, 4, 4).$$

Define

$$V_0 := \{u\}, \quad V_n := V_{n-1} \cup \bigcup_{\varepsilon \in E} A_\varepsilon V_{n-1} \quad \text{for } n = 1, 2, 3, \dots$$

Then the minimal common invariant subspace of $\{A_\varepsilon: \varepsilon \in E\}$ generated by u is $V = \bigcup_{n \in \mathbb{Z}_+} \text{span } V_n = \text{span } V_N$, with

$$N := \min\{n \in \mathbb{Z}_+: \dim(\text{span } V_n) = \dim(\text{span } V_{n+1})\}.$$

Fig. 1. The nonzero terms of the mask a along with an outline of the support of the corresponding refinable function ϕ . (a) $12a$ from Example 1; (b) $6a$ from Example 2.

(Note that N is necessarily less than or equal to $(\#K) \cdot r$.) Using `Maple` again, we find that $\dim(\text{span } V_2) = \dim(\text{span } V_3) = 8$, and that a basis for $V (= \text{span } V_2)$ is provided by the vectors $v_j \in \mathbb{C}^K$, $j = (j_1, j_2) \in J := K \setminus \{(2, 2)\}$ defined by

$$v_j(k) := \begin{cases} 0 & \text{for } k \in J \setminus \{j\}, \\ 2 & \text{for } k = j, \\ (-2)^{5-j_1-j_2} & \text{for } k = (2, 2). \end{cases}$$

Thus, by Corollary 1, the shifts of ϕ are linearly independent if and only if the polynomials

$$q_j(z_1, z_2) := p_{v_j}(z_1, z_2) = 2z_1^{j_1}z_2^{j_2} + (-2)^{5-j_1-j_2}z_1^2z_2^2, \quad j = (j_1, j_2) \in J,$$

have no common zeros in $(\mathbb{C} \setminus \{0\})^2$.

For all nonzero z_1 and z_2 , if $q_{1,2}(z_1, z_2) = 0$ then $z_1 = -\frac{1}{2}$, and if $q_{2,1}(z_1, z_2) = 0$ then $z_2 = -\frac{1}{2}$. On the other hand, $q_{j_1, j_2}(-\frac{1}{2}, -\frac{1}{2}) = 0$ for all $(j_1, j_2) \in J$. So these polynomials have exactly one common zero in $(\mathbb{C} \setminus \{0\})^2$, namely $(-\frac{1}{2}, -\frac{1}{2})$. Therefore, we conclude that the shifts of ϕ are linearly dependent. We have also determined all possible linear dependence relations of the shifts of ϕ . That is, $N(\phi)$ is spanned by the sequence c given by $c(\alpha_1, \alpha_2) := (-\frac{1}{2})^{-\alpha_1-\alpha_2}$, $(\alpha_1, \alpha_2) \in \mathbb{Z}^2$.

Note that since $(-\frac{1}{2}, -\frac{1}{2})$ is not on the two-dimensional torus

$$\mathbf{T}^2 := \{z \in \mathbb{C}^2: |z_1| = |z_2| = 1\},$$

and $\phi \in L_\infty(\mathbb{R}^2)$, Theorem 1 of [14] implies that ϕ has ℓ_p -stable shifts for any $0 < p \leq \infty$.

Example 2. Let ϕ be a solution of the refinement equation

$$\phi = \sum_{\alpha \in \mathbb{Z}^2} a(\alpha) \phi(2 \cdot - \alpha),$$

where the symbol of the mask a is

$$\begin{aligned} & \frac{1}{6}(z_2^2 + z_1z_2^3 + z_1^2 + z_1^3z_2) \\ & + \frac{1}{3}(1 + z_2 + z_1 + z_1z_2 + z_1z_2^2 + z_1^2z_2 + z_1^2z_2^2 + z_1^2z_2^3 + z_1^3z_2^2 + z_1^3z_2^3). \end{aligned}$$

The nonzero terms of the sequence $6a$ are shown in Figure 1(b) along with an outline of the support of the function ϕ . K is the same as in Example 1.

The subdivision scheme again converges and, with the same notation as in Example 1, we find that

$$u = (0, 0, 0, 0, \frac{1}{3}, \frac{1}{6}, 0, \frac{1}{6}, \frac{1}{3}),$$

and $\dim(\text{span } V_2) = \dim(\text{span } V_3) = 7$. The rows of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

form a basis for $\text{span } V_2$. Thus, the shifts of ϕ are linearly independent if and only if the polynomials

$$\begin{aligned} 1 - z_1 z_2^2, \quad 1 - z_1^2 z_2, \quad 1 + z_2^2 + z_1^2 z_2^2, \quad 1 + z_1 z_2 + z_1^2 z_2^2, \quad 1 + z_1^2 + z_1^2 z_2^2, \\ z_1^2 z_2^2 - z_2, \quad z_1^2 z_2^2 - z_1, \end{aligned}$$

have no common zeros. These polynomials have two common zeros: $\frac{1}{2}(-1 + \sqrt{3}i, -1 + \sqrt{3}i)$ and $\frac{1}{2}(-1 - \sqrt{3}i, -1 - \sqrt{3}i)$. Therefore, we conclude that the shifts of ϕ are linearly dependent.

Since these common zeros actually lie on $\mathbf{T}^2$, Theorem 1 of [14] implies that the shifts of ϕ are in fact not stable.

Example 3. In this example, we apply Theorem 1 to a univariate refinable vector of functions. The main point of the example is to illustrate how $P(z)$ is constructed from a basis for V .

Let ϕ be a solution of the refinement equation

$$\phi = \sum_{\alpha \in \mathbf{Z}} a(\alpha) \phi(2 \cdot - \alpha),$$

where the symbol of the mask a is given by

$$\begin{aligned} \sum_{\alpha \in \mathbf{Z}} a(\alpha) z^\alpha &:= \begin{bmatrix} -\frac{1}{4} & \frac{2}{3} \\ -\frac{9}{16} & 1 \end{bmatrix} + \begin{bmatrix} -\frac{11}{4} & \frac{32}{9} \\ -\frac{51}{16} & 4 \end{bmatrix} z \\ &+ \begin{bmatrix} -\frac{61}{12} & \frac{16}{3} \\ -\frac{21}{4} & \frac{11}{2} \end{bmatrix} z^2 + \begin{bmatrix} -\frac{21}{4} & \frac{16}{3} \\ -\frac{21}{4} & \frac{16}{3} \end{bmatrix} z^3 + \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} z^4. \end{aligned}$$

In this case, $r = 2$ and $K = \{0, 1, 2, 3\}$.

The results of [6] and [16] could be used to show that the subdivision scheme associated with a converges and, hence, that ϕ is continuous. But, for this mask, a solution can be given explicitly and is easily seen to be continuous.

The matrix M defined by (1.2) is

$$M = \begin{bmatrix} -\frac{23}{3} & \frac{76}{9} \\ -\frac{65}{8} & \frac{107}{12} \end{bmatrix}.$$

One can check that the eigenvalues of M are 1 and $\frac{1}{4}$, and that $(38, 39)^T$ is a right eigenvector corresponding to the eigenvalue 1. This implies that there is a unique distributional solution ϕ to (1.1) satisfying $\hat{\phi}(0) = (38, 39)^T$. In fact, the solution is $\phi := (\phi_1, \phi_2)^T$, where

$$\begin{aligned} \phi_1(x) &:= \begin{cases} 24x + 9x^2 & \text{for } x \in [0, 1), \\ 69 - 42x + 6x^2 & \text{for } x \in [1, 2), \\ 81 - 54x + 9x^2 & \text{for } x \in [2, 3), \\ 0 & \text{otherwise;} \end{cases} \\ \phi_2(x) &:= \begin{cases} 27x + \frac{27}{4}x^2 & \text{for } x \in [0, 1), \\ 72 - 45x + \frac{27}{4}x^2 & \text{for } x \in [1, 2), \\ 81 - 54x + 9x^2 & \text{for } x \in [2, 3), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It is easy to check that this ϕ is continuous. Moreover, although it is not as straightforward, one can also directly check that ϕ satisfies the given refinement equation.

Let $u \in \mathbf{C}^{r \times K}$ be an eigenvector of A_0 corresponding to the eigenvalue 1. Using Maple (or directly from above), we obtain (for example)

$$u = \frac{4}{3}\Phi(0) = \begin{bmatrix} 0 & 44 & 12 & 0 \\ 0 & 45 & 12 & 0 \end{bmatrix}.$$

Then $\dim(\text{span } V_2) = \dim(\text{span } V_3)$, and one basis for $V = \text{span } V_2$ is provided by the vectors

$$v_1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad v_2 := \begin{bmatrix} 0 & 12 & 0 & 0 \\ 1 & 12 & 0 & 0 \end{bmatrix}, \quad v_3 := \begin{bmatrix} 0 & 44 & 12 & 0 \\ 0 & 45 & 12 & 0 \end{bmatrix}.$$

It follows that

$$P(z) = \begin{bmatrix} 1 & 12z & 44z + 12z^2 \\ 1 & 1 + 12z & 45z + 12z^2 \end{bmatrix}.$$

Since $\text{rank } P(z) = 2$ for all z , the shifts of ϕ are linearly independent by Theorem 1.

3. Distributions

Throughout this section, for any distribution ϕ and any suitable test function f , the notation (ϕ, f) denotes the value of ϕ at f . That is, the linear functional ϕ maps the test function f to (ϕ, f) :

$$\phi : f \mapsto (\phi, f).$$

The definitions of refinability, refinement mask, linearly independent shifts, and the set $N(\phi)$ remain unchanged when ϕ is a compactly supported distribution. However, the arguments and results of the previous section only apply to continuous functions and their use essentially requires a convergent subdivision scheme. In this section, we provide a characterization of linear independence of the shifts of a vector ϕ of compactly supported refinable distributions in terms of its mask a under the assumption that the matrix M defined in (1.2) has a simple eigenvalue 1 and all other eigenvalues of M are less than 1 in modulus. Recall that, under this assumption, the refinement equation (1.1) has a one-dimensional space of compactly supported distributional solutions.

Let W be a finite-dimensional normed linear space with norm $\|\cdot\|$. As usual, the norm of a linear operator A on W is defined by

$$\|A\| := \max_{\|v\|=1} \{\|Av\|\}.$$

Let $\mathcal{A}$ be a finite collection of linear operators on W . For a positive integer n , we denote by $\mathcal{A}^n$ the n -fold Cartesian product of $\mathcal{A}$ with itself:

$$\mathcal{A}^n := \{(A_1, \dots, A_n) : A_1, \dots, A_n \in \mathcal{A}\}.$$

By convention, $\mathcal{A}^0 := \{I\}$, where I is the identity operator on W . Define

$$\|\mathcal{A}^n\| := \max\{\|A_1 \cdots A_n\| : (A_1, \dots, A_n) \in \mathcal{A}^n\}.$$

The **(uniform) joint spectral radius** of $\mathcal{A}$ was defined in [24] to be

$$\rho(\mathcal{A}) := \limsup_{n \rightarrow \infty} \|\mathcal{A}^n\|^{1/n}.$$

It is well known that $\rho(\mathcal{A})$ is independent of which norm is used in W and that

$$(3.1) \quad \rho(\mathcal{A}) = \inf_{n \geq 1} \|\mathcal{A}^n\|^{1/n}.$$

In the case $s = 1$, the joint spectral radius was used in [5] to derive sufficient conditions to ensure continuity of the normalized solution to (1.1).

Define the operator

$$\mu: (\mathbf{C}^{r \times r})^{\mathbf{Z}^s} \rightarrow (\mathbf{C}^{r \times r})^{\mathbf{Z}^s} : c \mapsto 2^{-s} \sum_{\varepsilon \in E} c(\cdot - \varepsilon),$$

where $(\mathbf{C}^{r \times r})^{\mathbf{Z}^s}$ denotes the set of all sequences $c = (c(\alpha))_{\alpha \in \mathbf{Z}^s}$ of $r \times r$ complex matrices. The method used to prove Theorem 3.3 of [13] can be used to show that for any mask a for which M has a simple eigenvalue 1 and all other eigenvalues of M are less than 1 in modulus, if n is a positive integer such that $2^n > \rho(\{A_\varepsilon : \varepsilon \in E\})$ then the subdivision scheme associated with $b := \mu^n a$ converges. In fact, there is a unique vector $\psi = (\psi_1, \dots, \psi_r)^T$ that satisfies

$$(3.2) \quad \psi = \sum_{\alpha \in \mathbf{Z}^s} b(\alpha) \psi(2 \cdot - \alpha) \quad \text{and} \quad \hat{\psi}(0) = \hat{\phi}(0).$$

Moreover, the entries of ψ are necessarily continuous.

We modify our definitions of the set K and the operators A_ε slightly for this section. Let

$$K := \mathbf{Z}^s \cap \left(\prod_{j=1}^s [M_j, N_j + n - 1] \right)$$

(where M_j and N_j are as in Section 2). Let A_ε ($\varepsilon \in E$) and B be the linear operators on $\mathbf{C}^{r \times K}$ given by

$$A_\varepsilon v(\alpha) := \sum_{\beta \in K} a(\varepsilon + 2\alpha - \beta) v(\beta) \quad \text{and} \quad Bv(\alpha) := \sum_{\beta \in K} b(2\alpha - \beta) v(\beta), \quad \alpha \in K,$$

where $v \in \mathbf{C}^{r \times K}$.

Since the subdivision scheme associated with b converges, 1 is a simple eigenvalue of B and all other eigenvalues of B are strictly less than 1 in modulus. In fact, the eigenspace of B corresponding to the eigenvalue 1 is spanned by the vector $\Psi(0) := (\psi(\alpha) : \alpha \in K)$.

Let V be the minimal common invariant subspace of $\{A_\varepsilon : \varepsilon \in E\}$ generated by the vector $\Psi(0)$ (so V is invariant under each A_ε , but is generated by an eigenvector of B , not A_0). As in Section 2, let $\{v_1, \dots, v_m\}$ be a basis for V ; and, for $j = 1, \dots, r$ and $k = 1, \dots, m$, define

$$p_{j,k}(z) := \sum_{\alpha \in K} v_k(j, \alpha) z^\alpha$$

and let P denote the $r \times m$ matrix of Laurent polynomials

$$P(z) := \begin{bmatrix} p_{1,1}(z) & \cdots & p_{1,m}(z) \\ \vdots & \ddots & \vdots \\ p_{r,1}(z) & \cdots & p_{r,m}(z) \end{bmatrix}.$$

Then we have the following theorem:

Theorem 2. *The shifts of ϕ are linearly independent if and only if $\text{rank } P(z) = r$ for all $z \in (\mathbf{C} \setminus \{0\})^s$. Moreover, $c \in N(\phi)$ if and only if, for all $v \in V$:*

$$(3.3) \quad \sum_{\beta \in K} \langle c(\alpha - \beta), v(\beta) \rangle = 0 \quad \text{for all } \alpha \in \mathbf{Z}^s.$$

The proof of Theorem 2 requires further notation and terminology. Define

$$\mathcal{M}_1 := \chi_{\{0,1\}^s} \quad \text{and} \quad \mathcal{M}_{\ell+1} := \mathcal{M}_1 * \mathcal{M}_\ell \quad \text{for each } \ell \in \mathbf{N}.$$

By convention, $\mathcal{M}_0$ is the Dirac functional, $(\mathcal{M}_0, f) := f(0)$ for all continuous functions f on $\mathbf{R}^s$. Then, with ψ defined by (3.2), $\psi = \phi * \mathcal{M}_n$, i.e., $\psi_j = \phi_j * \mathcal{M}_n$ for each $j = 1, \dots, r$. It follows that $N(\phi) \subset N(\psi)$.

In the single function ($r = 1$) univariate ($s = 1$) case, $N(\phi) = N(\psi)$. Thus, for univariate scalar-valued distributions, one could simply apply the results of Section 2 directly to the mask b . When $r > 1$, however, $N(\phi)$ is usually a *proper* subset of $N(\psi)$ as we will see in Example 4. This is because $\sum_{\alpha \in \mathbf{Z}^s} \psi_j(\cdot - \alpha)$ is a constant function for each j , while the shifts of ϕ would not, in general, satisfy such dependence relations. $N(\phi)$ may also be a proper subset of $N(\psi)$ for multivariate scalar-valued distributions. For this reason, Theorem 2 is an appropriate modification of Theorem 1.

For each nonnegative integer N , we denote by $C_c^N := C_c^N(\mathbf{R}^s)$ the normed linear space of all compactly supported functions on $\mathbf{R}^s$ whose derivatives $D^\alpha f$ are continuous for each multi-index α with $|\alpha| \leq N$; and $C_c^\infty := \bigcap_{N \in \mathbf{N}} C_c^N$. Each C_c^N is equipped with the norm

$$\|f\|_N := \max\{|D^\alpha f(x)| : x \in \mathbf{R}^s, |\alpha| \leq N\}, \quad f \in C_c^N.$$

Note that, for each $j \geq 2$, $\mathcal{M}_j \in C_c^{j-2}$.

We will also make use of the **semidiscrete convolution**

$$f *' c := \sum_{\alpha \in \mathbf{Z}^s} \langle c(\alpha), f(\cdot - \alpha) \rangle,$$

where f is an r -vector of distributions on $\mathbf{R}^s$ and $c \in \mathbf{C}^{r \times \mathbf{Z}^s}$; as well as the **subdivision operator** $S_a : \mathbf{C}^{r \times \mathbf{Z}^s} \rightarrow \mathbf{C}^{r \times \mathbf{Z}^s}$ given by

$$S_a c(\alpha) := \sum_{\beta \in \mathbf{Z}^s} a(\alpha - 2\beta)^T c(\beta), \quad \alpha \in \mathbf{Z}^s,$$

where a is the refinement mask and $c \in \mathbf{C}^{r \times \mathbf{Z}^s}$. It follows, by iterating the refinement equation (1.1), that

$$\phi *' c = \phi *' S_a^j c(2^j \cdot)$$

for all $c \in \mathbf{C}^{r \times \mathbf{Z}^s}$, $j \in \mathbf{N}$.

See [25, pp. 155–161] for a thorough discussion of the convolution of two distributions. Most importantly, $f * g$ is well defined for any two distributions f and g as long as at least one of them has compact support. When $f = (f_1, \dots, f_r)^T$ is a vector of distributions and g is a scalar-valued distribution, $f * g$ denotes the vector $(f_1 * g, \dots, f_r * g)^T$. Among the properties most relevant here are

$$f * g(2 \cdot) = 2^s (f(2 \cdot)) * (g(2 \cdot)) \quad \text{and} \quad (f *' c) * g = (f * g) *' c,$$

for any r -vector f of distributions, any scalar-valued distribution g , and any $c \in \mathbf{C}^{r \times \mathbf{Z}^s}$.

Proof of Theorem 2. Our proof of Theorem 2 is based on the following several lemmas:

Lemma 2. *Suppose $f = \varphi *' c$, where φ is an r -vector of compactly supported distributions on $\mathbf{R}^s$ and c is a sequence in $\mathbf{C}^{r \times \mathbf{Z}^s}$. If there exists a positive integer ℓ such that*

$$(3.4) \quad f * (\mathcal{M}_\ell(2^m \cdot)) = 0 \quad \text{for all } m \in \mathbf{N},$$

then $f = 0$.

Proof. Since $\varphi = (\varphi_1, \dots, \varphi_r)^T$ is a vector of compactly supported distributions, there exist a constant $C_\varphi < \infty$ and a nonnegative integer N such that

$$(3.5) \quad |(\varphi_j, g)| \leq C_\varphi \|g\|_N \quad \text{for each } j = 1, \dots, r \text{ and all } g \in C_c^N.$$

We denote the smallest such N by N_φ . Of course inequality (3.5) holds for all $N \geq N_\varphi$. Let $k := \max\{\ell, N_\varphi + 2\}$. Then it follows from (3.4) that

$$(3.6) \quad f * (\mathcal{M}_k(2^m \cdot)) = 0 \quad \text{for all } m \in \mathbf{N},$$

since $\mathcal{M}_k = \mathcal{M}_\ell * \mathcal{M}_{k-\ell}$.

Now, let g be an arbitrary test function in C_c^∞ and let R be a positive integer such that

$$\text{supp } g + [-1, 1]^s \subset [-R, R]^s.$$

Then it is known from the theory of spline approximation (see [1, Chap. III]) that there exists a sequence of functions $(g_m: m = 1, 2, \dots)$ such that:

- (a) each g_m is a finite linear combination of $\mathcal{M}_k(\alpha - 2^m \cdot)$, $\alpha \in \mathbf{Z}^s$;
- (b) each g_m is supported on $[-R, R]^s$; and
- (c) $\|g - g_m\| := \|g - g_m\|_0 = O(2^{-mk})$ as $m \rightarrow \infty$.

In fact, as shown in the proof of [29, Theorem 3.1], $\|g - g_m\|_{k-j} = O(2^{-mj})$ for each $j = 0, \dots, k$. In particular, we have:

$$(c') \quad \lim_{m \rightarrow \infty} \|g - g_m\|_{k-2} = 0.$$

By (3.5) and the compact support of φ , f extends to a linear functional on C_c^{k-2} . Moreover,

$$(f, \mathcal{M}_k(\alpha - 2^m \cdot)) = f * (\mathcal{M}_k(2^m \cdot))(2^{-m}\alpha) \quad \text{for all } \alpha \in \mathbf{Z}^s, \quad m \in \mathbf{N}.$$

So, by property (a) and (3.6):

$$(3.7) \quad (f, g) = (f, g - g_m) + (f, g_m) = (f, g - g_m).$$

Since g and g_m have support in $[-R, R]^s$:

$$(3.8) \quad (f, g - g_m) = \left(\sum_{\alpha \in S} \langle c(\alpha), \varphi(\cdot - \alpha) \rangle, g - g_m \right),$$

where

$$S := \{\alpha \in \mathbf{Z}^s : (\alpha + \text{supp } \varphi) \cap [-R, R]^s \neq \emptyset\}$$

is a finite set. Consequently, it follows from (3.5), (3.7), and (3.8) that

$$|(f, g)| \leq C_\varphi \|g - g_m\|_{k-2} \sum_{\alpha \in S} \sum_{j=1}^r |c(j, \alpha)|.$$

Taking property (c') into account and letting $m \rightarrow \infty$ in the above inequality, we obtain $(f, g) = 0$. Since $g \in C_c^\infty$ was arbitrary, $f = 0$. ■

Lemma 3. Suppose $f = \phi *' c$, where $c \in \mathbf{C}^{r \times \mathbf{Z}^s}$. If $f * (\mathcal{M}_n(2^m \cdot + \alpha))(0) = 0$ for all $m \in \mathbf{N}$ and all $\alpha \in \mathbf{Z}^s$, then $f = 0$.

Proof. By assumption, the subdivision scheme associated with $\mu^n a$ converges, so $\phi * \mathcal{M}_n$ is continuous. It follows that, for each $m \in \mathbf{N}$:

$$f * (\mathcal{M}_n(2^m \cdot)) = (\phi *' S_a^m c(2^m \cdot)) * (\mathcal{M}_n(2^m \cdot)) = 2^{-ms} (\phi * \mathcal{M}_n) *' S_a^m c(2^m \cdot)$$

is also continuous.

Fix a multi-integer β in $\mathbf{Z}^s$ and natural numbers ℓ and m for the time being. Since $\mathcal{M}_n$ is refinable, there exists a finitely supported sequence d such that

$$\mathcal{M}_n = \sum_{\alpha \in \mathbf{Z}^s} d(\alpha) \mathcal{M}_n(2^\ell \cdot - \alpha).$$

It follows that

$$\begin{aligned} f * (\mathcal{M}_n(2^m \cdot))(2^{-\ell} \beta) &= f * (\mathcal{M}_n(2^m \cdot + 2^{m-\ell} \beta))(0) \\ &= \sum_{\alpha \in \mathbf{Z}^s} d(\alpha) f * (\mathcal{M}_n(2^{m+\ell} \cdot + 2^m \beta - \alpha))(0). \end{aligned}$$

Now suppose $f * (\mathcal{M}_n(2^m \cdot + \alpha))(0) = 0$ for all $m \in \mathbf{N}$ and all $\alpha \in \mathbf{Z}^s$. Then $f * (\mathcal{M}_n(2^m \cdot))(x) = 0$ for all $m \in \mathbf{N}$ and all $x \in \{2^{-\ell} \beta : \beta \in \mathbf{Z}^s, \ell \in \mathbf{N}\}$. Since this set is dense in $\mathbf{R}^s$ and $f * (\mathcal{M}_n(2^m \cdot))$ is continuous, (3.4) hold and, by Lemma 2, $f = 0$. ■

Lemma 4. Let $v = \Psi(0)$. For all nonnegative integers j :

$$(\phi(\cdot + \alpha)) * \left(\mathcal{M}_n \left(2^j \cdot + \sum_{i=0}^{j-1} 2^i \varepsilon_i \right) \right) (0) = 2^{-js} (A_{\varepsilon_{j-1}} A_{\varepsilon_{j-2}} \cdots A_{\varepsilon_0} v)(\alpha)$$

for every $\alpha \in K$ and each $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{j-1}) \in E^j$.

Proof. When $j = 0$, each expression is simply the α th entry of v . Now, suppose $m \geq 0$ and the statement is true when $j = m$. Let $\alpha \in K$ and let $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_m) \in E^{m+1}$ be given. Then

$$\begin{aligned}
 & (\phi(\cdot + \alpha)) * \left(\mathcal{M}_n \left(2^{m+1} \cdot + \sum_{i=0}^m 2^i \varepsilon_i \right) \right) (0) \\
 &= 2^{-s} (\phi((\cdot + \varepsilon_m + 2\alpha)/2)) * \left(\mathcal{M}_n \left(2^m \cdot + \sum_{i=0}^{m-1} 2^i \varepsilon_i \right) \right) (0) \\
 &= 2^{-s} \sum_{\beta \in \mathbf{Z}^s} a(\beta) (\phi(\cdot + \varepsilon_m + 2\alpha - \beta)) * \left(\mathcal{M}_n \left(2^m \cdot + \sum_{i=0}^{m-1} 2^i \varepsilon_i \right) \right) (0) \\
 &= 2^{-s} \sum_{\beta \in K} a(\varepsilon_m + 2\alpha - \beta) (\phi(\cdot + \beta)) * \left(\mathcal{M}_n \left(2^m \cdot + \sum_{i=0}^{m-1} 2^i \varepsilon_i \right) \right) (0) \\
 &= 2^{-(m+1)s} \sum_{\beta \in K} a(\varepsilon_m + 2\alpha - \beta) (A_{\varepsilon_{m-1}} \cdots A_{\varepsilon_0} v)(\beta) \\
 &= 2^{-(m+1)s} (A_{\varepsilon_m} \cdots A_{\varepsilon_0} v)(\alpha),
 \end{aligned}$$

where the induction hypothesis has been used to derive the fourth equality. The switch from a summation over $\mathbf{Z}^s$ to a summation over K in the third equality is justified by the facts that $\text{supp } \phi(\cdot + \beta) \subset \prod_{j=1}^s [M_j, N_j] - \beta$ and $\text{supp } \mathcal{M}_n(2^m \cdot + \sum_{i=0}^{m-1} 2^i \varepsilon_i) \subset (-1, n]^s$, so their convolution vanishes outside the *open* cube $\prod_{j=1}^s (M_j - 1, N_j + n) - \beta$ which does not contain 0 if $\beta \in \mathbf{Z}^s \setminus K$. (The convolution vanishes on the boundary of the cube because it is continuous.) ■

Proof of Theorem 2. Let $f = \phi *' c$ for some sequence $c \in \mathbf{C}^{r \times \mathbf{Z}^s}$, and let j be any nonnegative integer and α be any multi-integer. Then $\alpha = 2^j \gamma + \sum_{i=0}^{j-1} 2^i \varepsilon_i$ for some multi-integer γ and some sequence $(\varepsilon_i: i = 0, \dots, j-1)$ in E . So

$$\begin{aligned}
 f * (\mathcal{M}_n(2^j \cdot + \alpha))(0) &= (\phi *' c) * \left(\mathcal{M}_n \left(2^j (\cdot + \gamma) + \sum_{i=0}^{j-1} 2^i \varepsilon_i \right) \right) (0) \\
 &= \left(\phi * \left(\mathcal{M}_n \left(2^j \cdot + \sum_{i=0}^{j-1} 2^i \varepsilon_i \right) \right) \right) *' c(\gamma) \\
 &= \sum_{\beta \in K} \left\langle c(\gamma - \beta), \phi(\cdot + \beta) * \left(\mathcal{M}_n \left(2^j \cdot + \sum_{i=0}^{j-1} 2^i \varepsilon_i \right) \right) (0) \right\rangle \\
 &= 2^{-js} \sum_{\beta \in K} \langle c(\gamma - \beta), (A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_0} \Psi(0))(\beta) \rangle.
 \end{aligned}$$

So, by Lemma 3, $f = 0$ if and only if (3.3) are satisfied for all $v \in V$.

The rest of this proof is identical to the final paragraph in the proof of Theorem 1. ■

Example 4. Let ϕ be a solution of the refinement equation

$$\phi = \sum_{\alpha \in \mathbb{Z}^2} a(\alpha) \phi(2 \cdot - \alpha),$$

where the symbol of the mask a is

$$\sum_{\alpha \in \mathbb{Z}^2} a(\alpha) z_1^{\alpha_1} z_2^{\alpha_2} = 2(1 - z_1 + z_1 z_2 + z_1^2).$$

We claim that ϕ is not in L_1 and has linearly independent shifts.

Clearly, the subdivision scheme associated with mask a does not converge, since a does not satisfy the basic sum rules. Let us consider the mask $b := \mu a$. Its symbol is given by

$$\begin{aligned} \sum_{\alpha \in \mathbb{Z}^2} b(\alpha) z_1^{\alpha_1} z_2^{\alpha_2} &= 2(1 - z_1 + z_1 z_2 + z_1^2)(1 + z_1)(1 + z_2)/4 \\ &= (1 + z_2 + z_1 z_2 + z_1 z_2^2 + z_1^2 z_2 + z_1^2 z_2^2 + z_1^3 + z_1^3 z_2)/2. \end{aligned}$$

Let ψ be the normalized solution to the refinement equation associated with mask b . Then $\psi = \phi * \mathcal{M}_1$. In this example,

$$K = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\}.$$

For each $\varepsilon \in E$, let B_ε be the linear operators on $\mathbb{C}^K$ given by

$$B_\varepsilon v(j) := \sum_{k \in K} b(\varepsilon + 2j - k) v(k), \quad j \in K,$$

where $v \in \mathbb{C}^K$. We observe that the mask b is nonnegative. By computation we find that, for any pair of ε_1 and ε_2 in E , the matrix $B_{\varepsilon_1} B_{\varepsilon_2}$ has a positive row. Therefore, by Theorem 1.1 of [18], we conclude that the subdivision scheme associated with mask b converges.

Let $w \in \mathbb{C}^K$ be the eigenvector of $B_{(0,0)}$ corresponding to the eigenvalue 1 such that the sum of its components is equal to 1. Then

$$w = [0, 0, 0, \frac{1}{2}, 0, \frac{1}{2}].$$

Let W be the minimal common invariant subspace of $\{B_{(0,0)}, B_{(0,1)}, B_{(1,0)}, B_{(1,1)}\}$ generated by w . Using Maple, we find that $\dim(W) = 5$ and the rows of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

form a basis for W . The polynomials

$$z_1 z_2 + z_1^2 z_2, \quad z_1 + z_1 z_2 + z_1^2 + z_1^2 z_2, \quad z_2 + 2z_1 z_2 + z_1^2 z_2, \quad 1 + 2z_1 z_2 + z_1^2, \quad 1 + z_1 + z_1 z_2 + z_1^2,$$

have one and only one common zero: $(-1, 1)$. Therefore, by Theorem 2, the shifts of ψ are linearly dependent.

Now, we invoke Theorem 2 to determine the linear independence of ϕ . For each $\varepsilon \in E$, let A_ε be the linear operators on $\mathbf{C}^K$ given by

$$A_\varepsilon v(j) := \sum_{k \in K} a(\varepsilon + 2j - k)v(k), \quad j \in K,$$

where $v \in \mathbf{C}^K$. We have

$$u := A_{11}w = [1, 0, 0, 0, 0, 0].$$

Hence, $p_u(z) = 1$ has no zeros. By Theorem 2, the shifts of ϕ are linearly independent. Consequently, ϕ does not lie in $L_1(\mathbf{R})$, for otherwise a would satisfy the basic sum rules (see [7]).

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