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Approximation with scaled shift-invariant spaces by means of quasi-projection operators

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Dedicated to Professor Carl de Boor on the occasion of his 65th birthday

Abstract

The work of de Boor and Fix on spline approximation by quasiinterpolants has had far-reaching influence in approximation theory since publication of their paper in 1973. In this paper, we further develop their idea and investigate quasi-projection operators. We give sharp estimates in terms of moduli of smoothness for approximation with scaled shift-invariant spaces by means of quasi-projection operators. In particular, we provide error analysis for approximation of quasi-projection operators with Lipschitz spaces. The study of quasi-projection operators has many applications to various areas related to approximation theory and wavelet analysis.

Keywords: Approximation order; Cascade algorithms; Moduli of smoothness; Quasi-interpolation; Quasi-projection; Shift-invariant spaces; Wavelet Analysis; Sobolev spaces; Lipschitz spaces

1. Introduction

The work of de Boor and Fix on spline approximation by quasiinterpolants has had far-reaching influence in approximation theory since publication of their paper [7] in 1973. For L_p approximation ($1 \leqslant p \leqslant \infty$), de Boor [3] proposed an approximation scheme using linear projectors induced by dual functionals. See [5] for a comprehensive survey on quasi-

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interpolation schemes. In [6], de Boor et al., used L_2 projectors to give a characterization for the L_2 approximation order of shift-invariant spaces.

The idea of de Boor [3] for L_p approximation was further developed by Jia and Lei [15] and applied to shift-invariant spaces. On the basis of the work of Lei [20] on L_p approximation by linear projectors, Lei et al. [21] investigated approximation with scaled shift-invariant spaces by means of certain integral operators.

In [18,19] Kyriazis investigated the approximation properties of principal shift-invariant spaces in various function spaces. His approach was based on an analysis of the boundedness of certain kernel operators on the concerned function spaces. For L_2 approximation, Jetter and Zhou [11] also employed a projection method to realize the optimal approximation order as given in [6].

A linear space of functions from $\mathbb{R}^s$ to $\mathbb{C}$ is called *shift-invariant* if it is invariant under multi-integer translates. For a finite set $\Phi = \{\phi_1, \dots, \phi_N\}$ of compactly supported functions on $\mathbb{R}^s$, we use $\mathbb{S}(\Phi)$ to denote the shift-invariant space generated by Φ. In other words, a function f lies in $\mathbb{S}(\Phi)$ if and only if there exist sequences b_j $(j = 1, \dots, N)$ such that

$$f = \sum_{j=1}^{N} \sum_{\alpha \in \mathbb{Z}^s} b_j(\alpha) \phi_j(\cdot - \alpha).$$

For a complex-valued (Lebesgue) measurable function f on a measurable subset E of $\mathbb{R}^s$, let

$$\|f\|_p(E) := \left(\int_E |f(x)|^p \, dx \right)^{1/p} \qquad \text{for } 1 \leqslant p < \infty$$

and let $\|f\|_\infty(E)$ denote the essential supremum of $|f|$ on E. When $E = \mathbb{R}^s$, we omit the reference to E. For $1 \leqslant p \leqslant \infty$, by $L_p(\mathbb{R}^s)$ we denote the Banach space of all measurable functions f on $\mathbb{R}^s$ such that $\|f\|_p < \infty$.

Suppose $\Phi = \{\phi_1, \dots, \phi_N\}$ is a finite set of compactly supported functions in $L_p(\mathbb{R}^s)$ $(1 \leqslant p \leqslant \infty)$. Let $S := \mathbb{S}(\Phi) \cap L_p(\mathbb{R}^s)$. It was proved in [12] that S is closed in $L_p(\mathbb{R}^s)$. For $h > 0$, by σ_h we denote the scaling operator given by $\sigma_h f = f(\cdot/h)$. Let $S_h := \sigma_h(S)$. We are interested in approximation with scaled shift-invariant spaces $(S_h)_{h>0}$ by means of quasi-projection operators.

Let $\tilde{\phi}_1, \dots, \tilde{\phi}_N$ be compactly supported functions in $L_{\tilde{p}}(\mathbb{R}^s)$, where $1/\tilde{p} + 1/p = 1$. Let Q be the linear operator on $L_p(\mathbb{R}^s)$ defined by

$$Qf = \sum_{j=1}^{N} \sum_{\alpha \in \mathbb{Z}^s} \langle f, \tilde{\phi}_j(\cdot - \alpha) \rangle \, \phi_j(\cdot - \alpha), \qquad f \in L_p(\mathbb{R}^s), \tag{1.1}$$

where

$$\langle f, g \rangle := \int_{\mathbb{R}^s} f(x) \, \overline{g(x)} \, dx.$$

It is known (see, e.g., [16, Theorems 2.1 and 3.1]) that Q is a bounded operator on $L_p(\mathbb{R}^s)$, that is,

$$\|Qf\|_p \leqslant C\|f\|_p \quad \forall f \in L_p(\mathbb{R}^s),$$

where C depends only on Φ and $\tilde{\Phi} = \{\tilde{\phi}_1, \ldots, \tilde{\phi}_N\}$. If the shifts of $\{\phi_1, \ldots, \phi_N\}$ and $\{\tilde{\phi}_1, \ldots, \tilde{\phi}_N\}$ are biorthogonal, then the linear operator Q is a projector from $L_p(\mathbb{R}^s)$ onto S. However, as far as approximation is concerned, Q is not required to be a projector. Thus, we call Q a *quasi-projection operator*. Quasi-projection can be viewed as an extension of quasi-interpolation.

For $h > 0$, let $Q_h = \sigma_h Q \sigma_{1/h}$. Evidently,

$$Q_h f = \sum_{j=1}^{N} \sum_{\alpha \in \mathbb{Z}^s} \langle f, h^{-s}\tilde{\phi}_j(\cdot/h - \alpha)\rangle \, \phi_j(\cdot/h - \alpha), \qquad f \in L_p(\mathbb{R}^s).$$

For a nontrivial function $f \in L_p(\mathbb{R}^s)$, we have

$$\frac{\|Q_h(\sigma_h f)\|_p}{\|\sigma_h f\|_p} = \frac{\|\sigma_h(Qf)\|_p}{\|\sigma_h f\|_p} = \frac{\|Qf\|_p}{\|f\|_p}.$$

Hence, $\|Q_h\| = \|Q\|$ for all $h > 0$. For $f \in L_p(\mathbb{R}^s)$, $Q_h f$ provides an approximation from $S_h = \sigma_h(S)$ to f. The main purpose of this paper is to estimate the error $Q_h f - f$ in Sobolev spaces or Lipschitz spaces as h tends to 0.

Before going further we introduce some notation. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the set of positive integers. An element of $\mathbb{N}_0^s$ is called a *multi-index*. The length of a multi-index $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ is given by $|\mu| := \mu_1 + \cdots + \mu_s$, and the factorial of μ is $\mu! := \mu_1! \cdots \mu_s!$. By $\nu \leqslant \mu$ we mean $\nu_j \leqslant \mu_j$ for all $j = 1, \ldots, s$. For $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ and $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$, define

$$x^\mu := x_1^{\mu_1} \cdots x_s^{\mu_s}.$$

The function $x \mapsto x^\mu$ ($x \in \mathbb{R}^s$) is called a monomial and its (total) degree is $|\mu|$. A polynomial is a linear combination of monomials. The degree of a polynomial $q = \sum_\mu c_\mu x^\mu$ is defined to be $\deg q := \max\{|\mu| : c_\mu \neq 0\}$. By Π_k we denote the linear space of all polynomials of degree at most k.

For a vector $y = (y_1, \ldots, y_s) \in \mathbb{R}^s$, its norm is defined as $|y| := \max_{1 \leqslant j \leqslant s} |y_j|$. We use D_y to denote the differential operator given by

$$D_y f(x) := \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}, \qquad x \in \mathbb{R}^s.$$

Moreover, we use ∇_y to denote the difference operator given by $\nabla_y f = f - f(\cdot - y)$. Let $e_1, \ldots, e_s$ be the unit coordinate vectors in $\mathbb{R}^s$. For $j = 1, \ldots, s$, we write D_j for D_{e_j}. For a multi-index $\mu = (\mu_1, \ldots, \mu_s)$, D^μ stands for the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$. Similarly, we write ∇_j for ∇_{e_j} and use ∇^μ to denote the difference operator $\nabla_1^{\mu_1} \cdots \nabla_s^{\mu_s}$. For $h > 0$, we use ∇_h^μ to denote the difference operator $\nabla_{he_1}^{\mu_1} \cdots \nabla_{he_s}^{\mu_s}$.

For $k \in \mathbb{N}_0$, let

$$\Delta_k := \{(\alpha_1, \ldots, \alpha_s) \in \mathbb{N}_0^s : \alpha_1 + \cdots + \alpha_s = k.\}. \tag{1.2}$$

For $\alpha = (\alpha_1, \ldots, \alpha_s) \in \Delta_k$ and $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$ we have

$$(\alpha \cdot x)^k = (\alpha_1 x_1 + \cdots + \alpha_s x_s)^k = \sum_{\mu \in \Delta_k} \alpha^\mu \binom{k}{\mu} x^\mu.$$

Since the matrix $(\alpha^\mu)_{\alpha, \mu \in \Delta_k}$ is invertible (see, e.g., [4]), each x^μ ($\mu \in \Delta_k$) is a linear combination of $(\alpha \cdot x)^k$, $\alpha \in \Delta_k$. Note that

$$D_\alpha = \alpha_1 D_1 + \cdots + \alpha_s D_s.$$

Hence, each D^μ ($\mu \in \Delta_k$) is a linear combination of D_α^k, $\alpha \in \Delta_k$.

By $C(\mathbb{R}^s)$ we denote the space of all continuous functions on $\mathbb{R}^s$. For a nonnegative integer k, we use $C^k(\mathbb{R}^s)$ to denote the linear space of those functions $f \in C(\mathbb{R}^s)$ for which $D^\mu f \in C(\mathbb{R}^s)$ for all $|\mu| \leqslant k$. Moreover, by $C_c^k(\mathbb{R}^s)$ we denote the linear space of all functions in $C^k(\mathbb{R}^s)$ with compact support. For $1 \leqslant p \leqslant \infty$, the Sobolev space $W_p^k(\mathbb{R}^s)$ consists of all functions $f \in L_p(\mathbb{R}^s)$ such that $\|f\|_{k,p} < \infty$, where

$$\|f\|_{k,p} := \sum_{j=0}^{k} |f|_{j,p} \quad \text{with} \quad |f|_{j,p} := \sum_{|\mu|=j} \|D^\mu f\|_p.$$

For a nonzero vector u in $\mathbb{R}^s$, let I_u be the linear operator on $C_c(\mathbb{R}^s)$ given by

$$I_u g := \int_0^1 g(x - tu)\, dt, \quad g \in C_c(\mathbb{R}^s). \tag{1.3}$$

We claim that

$$D_u(I_u g) = \nabla_u g \quad \forall g \in C_c(\mathbb{R}^s). \tag{1.4}$$

Indeed, for $x \in \mathbb{R}^s$, we have

$$\begin{aligned} D_u(I_u g)(x) &= \lim_{\lambda \to 0} \frac{I_u g(x + \lambda u) - I_u g(x)}{\lambda} \\ &= \lim_{\lambda \to 0} \frac{1}{\lambda} \left[\int_{-\lambda}^{1-\lambda} g(x - tu)\, du - \int_0^1 g(x - tu)\, du \right] \\ &= \lim_{\lambda \to 0} \left[\frac{1}{\lambda} \int_{-\lambda}^{0} g(x - tu)\, du - \frac{1}{\lambda} \int_{1-\lambda}^{1} g(x - tu)\, du \right] \\ &= g(x) - g(x - u) = (\nabla_u g)(x). \end{aligned}$$

The *modulus of continuity* of a function f in $L_p(\mathbb{R}^s)$ is defined by

$$\omega(f, h)_p := \sup_{|y| \leqslant h} \left\| \nabla_y f \right\|_p, \quad h \geqslant 0.$$

For a positive integer k, the kth *modulus of smoothness* of $f \in L_p(\mathbb{R}^s)$ is defined by

$$\omega_k(f,h)_p := \sup_{|y|\leqslant h} \left\| \nabla_y^k f \right\|_p, \qquad h \geqslant 0.$$

For $1 \leqslant p \leqslant \infty$ and $0 < \lambda \leqslant 1$, the Lipschitz space $\mathrm{Lip}(\lambda, L_p(\mathbb{R}^s))$ consists of all functions $f \in L_p(\mathbb{R}^s)$ ($f \in C(\mathbb{R}^s)$ in the case $p = \infty$) for which

$$\omega(f,h)_p \leqslant Ch^\lambda \quad \forall\, h > 0,$$

where C is a positive constant independent of h. For $\lambda > 0$ we write $\lambda = r + \eta$, where r is an integer and $0 < \eta \leqslant 1$. The Lipschitz space $\mathrm{Lip}(\lambda, L_p(\mathbb{R}^s))$ consists of those functions $f \in W_p^r(\mathbb{R}^s)$ ($f \in C^r(\mathbb{R}^s)$ in the case $p = \infty$) for which $D^\mu f \in \mathrm{Lip}(\eta, L_p(\mathbb{R}^s))$ for all multi-indices μ with $|\mu| = r$. The semi-norm in $\mathrm{Lip}(\lambda, L_p) = \mathrm{Lip}(\lambda, L_p(\mathbb{R}^s))$ is given by

$$|f|_{\mathrm{Lip}(\lambda,L_p)} := \max_{|\nu|=r} \sup_{y\in\mathbb{R}^s\setminus\{0\}} \frac{\|\nabla_y D^\nu f\|_p}{|y|^\eta}.$$

See [9, Chapter 2] for a discussion on Lipschitz spaces.

Section 2 will be devoted to some preliminary results concerning moduli of smoothness. We will give an explicit construction of linear operators $(A_h)_{h>0}$ from $L_p(\mathbb{R}^s)$ ($C(\mathbb{R}^s)$ in the case $p = \infty$) to $C^k(\mathbb{R}^s)$ such that

$$\|f - A_h f\|_p \leqslant C\omega_k(f,h)_p \quad \text{and} \quad |A_h f|_{k,p} \leqslant C\omega_k(f,h)_p/h^k,$$

where C is a constant independent of h, f, and p.

In Section 3, we will investigate approximation properties of the quasi-projection operators $(Q_h)_{h>0}$. Suppose $0 \leqslant j < k$ and $\phi_1, \ldots, \phi_N$ are compactly supported functions in $W_p^j(\mathbb{R}^s)$. Let Q be the quasi-projection operator given in (1.1) such that $Qq = q$ for all $q \in \Pi_{k-1}$. We will show that there exists a constant C independent of h, f, and p such that

$$|f - Q_h f|_{j,p} \leqslant C \sum_{|\nu|=j} \omega_{k-j}(D^\nu f, h)_p \quad \forall\, f \in W_p^j(\mathbb{R}^s). \tag{1.5}$$

In Section 4, we will study quasi-projection operators on Lipschitz spaces. Suppose $0 < \tau < \lambda \leqslant k$. We will give the following estimate:

$$|f - Q_h f|_{\mathrm{Lip}(\tau,L_p)} \leqslant Ch^{\lambda-\tau} |f|_{\mathrm{Lip}(\lambda,L_p)} \quad \forall\, f \in \mathrm{Lip}(\lambda, L_p), \tag{1.6}$$

where C is a constant independent of h, f, and p.

Finally, in Section 5, we will discuss quasi-projection schemes with shift-invariant spaces scaled by powers of an isotropic expansive matrix.

Quasi-projection operators have many applications to various problems in approximation theory and wavelet analysis. For instance, in [13] the author applied the quasi-projection scheme to the investigation of convergence rates of cascade algorithm and confirmed a conjecture of Ron [23] on this subject. In [10], Han further used such an idea in his study of vector cascade algorithms and refinable vectors in Sobolev spaces. Quasi-projection schemes also play a significant role in the recent work on sampling theory by Aldroubi et al. [1], and by Blu and Unser [2].

2. Preliminaries

In this section, we review some basic inequalities related to moduli of smoothness.

Let ψ be an element of $C_c^k(\mathbb{R}^s)$ such that $\int_{\mathbb{R}^s} \psi(x)\,dx = 1$. For $h > 0$, let $A_{\psi,h}$ be the linear operator on $L_p(\mathbb{R}^s)$ $(1 \leqslant p \leqslant \infty)$ given by

$$(A_{\psi,h} f)(x) := \int_{\mathbb{R}^s} \big(f - \nabla_u^k f\big)(x)\psi_h(u)\,du, \quad f \in L_p(\mathbb{R}^s),\ x \in \mathbb{R}^s. \tag{2.1}$$

where $\psi_h := \psi(\cdot/h)/h^s$. If there is no ambiguity about ψ, $A_{\psi,h}$ will be abbreviated as A_h. When the dimension $s = 1$ and ψ is a properly normalized B-spline, these operators were studied in classical approximation theory under the name "generalized Steklov functions" (see [22, p. 50; 8, pp. 33–35]). These operators were also used to study K-functionals (see [17; 9, Chapter 6]).

We observe that

$$f - \nabla_u^k f = \sum_{m=1}^{k} (-1)^{m-1} \binom{k}{m} f(\cdot - mu).$$

Hence,

$$A_{\psi,h} f(x) = \sum_{m=1}^{k} (-1)^{m-1} \binom{k}{m} \int_{\mathbb{R}^s} f(x - mhu)\psi(u)\,du, \quad x \in \mathbb{R}^s. \tag{2.2}$$

Since $\psi \in C_c^k(\mathbb{R}^s)$, we have $A_{\psi,h} f \in C^k(\mathbb{R}^s)$.

Let $u \in \mathbb{R}^s \setminus \{0\}$. Then the following inequality is valid for $1 \leqslant p \leqslant \infty$:

$$\|\nabla_u^k f\|_p \leqslant \|D_u^k f\|_p \quad \forall\, f \in W_p^k(\mathbb{R}^s). \tag{2.3}$$

Indeed, for $f \in W_p^1(\mathbb{R}^s)$, the relation

$$\nabla_u f(x) = \int_0^1 D_u f(x - tu)\,dt$$

is true for almost every $x \in \mathbb{R}^s$. Applying the Minkowski inequality to the above integral, we see that (2.3) is true for $k = 1$. Consequently, (2.3) is verified by induction on k. It follows immediately from (2.3) that

$$\omega_k(f, h)_p \leqslant C h^k |f|_{k,p} \quad \forall\, f \in W_p^k(\mathbb{R}^s),$$

where C is a constant depending only on k and s.

Lemma 2.1. *Suppose* $0 \leqslant j < k$. *If* $f \in W_p^j(\mathbb{R}^s)$ *for* $1 \leqslant p < \infty$ *or* $f \in C^j(\mathbb{R}^s)$ *for the case* $p = \infty$, *then*

$$\|f - A_{\psi,h} f\|_p \leqslant C h^j \sum_{|v|=j} \omega_{k-j}(D^v f, h)_p, \qquad (2.4)$$

where C is a constant independent of h and f.

Proof. It follows from (2.1) that

$$f(x) - A_{\psi,h} f(x) = \int_{\mathbb{R}^s} (\nabla_u^k f)(x) \psi_h(u)\, du = \int_{\mathbb{R}^s} (\nabla_{hu}^k f)(x) \psi(u)\, du, \quad x \in \mathbb{R}^s.$$

By Minkowski's inequality for integrals,

$$\|f - A_{\psi,h} f\|_p \leqslant \int_{\mathbb{R}^s} \|\nabla_{hu}^k f\|_p |\psi(u)|\, du.$$

By (2.3) we have

$$\left\|\nabla_{hu}^k f\right\|_p = \left\|\nabla_{hu}^j \nabla_{hu}^{k-j} f\right\|_p \leqslant \left\|D_{hu}^j \nabla_{hu}^{k-j} f\right\|_p = h^j \left\|\nabla_{hu}^{k-j} D_u^j f\right\|_p,$$

where the fact $D_{hu} = h D_u$ has been used to derive the last equality. Since ψ is compactly supported, there exists a positive constant K such that $\psi(u) = 0$ for all $u \notin [-K, K]^s$. Therefore, for $|u| \leqslant K$, there exists a constant C independent of h such that

$$\left\|\nabla_{hu}^{k-j} D_u^j f\right\|_p \leqslant C \sum_{|v|=j} \omega_{k-j}(D^v f, h)_p.$$

This completes the proof of (2.4). □

Lemma 2.2. *Suppose* $f \in L_p(\mathbb{R}^s)$ *for* $1 \leqslant p < \infty$ *or* $f \in C(\mathbb{R}^s)$ *for* $p = \infty$. *Then*

$$|A_{\psi,h} f|_{k,p} \leqslant C \omega_k(f, h)_p \big/ h^k, \quad h > 0, \qquad (2.5)$$

where C is a constant independent of h.

Proof. First, consider the case when ψ has the form

$$\psi = \Big(\prod_{\alpha \in \varDelta_k} I_\alpha^k\Big)\rho \qquad \text{with} \quad \rho \in C_c(\mathbb{R}^s) \quad \text{and} \quad \int_{\mathbb{R}^s} \rho(x)\, dx = 1,$$

where $\varDelta_k$ is given in (1.2) and I_u is the linear operator defined in (1.3). It is easily seen that $\psi \in C_c^k(\mathbb{R}^s)$ and $\int_{\mathbb{R}^s} \psi(x)\, dx = 1$. Moreover, taking (1.4) into account, for each $\alpha \in \varDelta_k$ we have $D_\alpha^k \psi = \nabla_\alpha^k \psi_\alpha$ for some $\psi_\alpha \in C_c(\mathbb{R}^s)$.

In light of expression (2.2), in order to estimate $|A_{\psi,h} f|_{k,p}$ it suffices to estimate $\|D^\mu g_m\|_p$ for $|\mu| = k$ and $m = 1, \ldots, k$, where

$$g_m(x) := \int_{\mathbb{R}^s} f(x - mu)\psi_h(u)\,du = \int_{\mathbb{R}^s} f(x - mhu)\psi(u)\,du, \quad m = 1, \ldots, k. \tag{2.6}$$

For $\alpha \in \Delta_k$, we observe that

$$D_\alpha^k g_m(x) = \frac{1}{(mh)^k} \int_{\mathbb{R}^s} f(x - mhu)(D_\alpha^k \psi)(u)\,du = \frac{1}{(mh)^k} \int_{\mathbb{R}^s} f(x - mhu)\left(\nabla_\alpha^k \psi_\alpha\right)(u)\,du.$$

Consequently,

$$D_\alpha^k g_m(x) = \frac{1}{(mh)^k} \int_{\mathbb{R}^s} \left(\nabla_{\alpha h}^k f\right)(x - mhu)\psi_\alpha(u)\,du.$$

Applying Minkowski's inequality for integrals to the above integral, we obtain

$$\left\| D_\alpha^k g_m \right\|_p \leqslant \frac{1}{(mh)^k} \int_{\mathbb{R}^s} \left\| \left(\nabla_{\alpha h}^k f\right)(\cdot - mhu) \right\|_p |\psi_\alpha(u)|\,du \leqslant C_1 \omega_k(f, h)_p / h^k,$$

where C_1 is a constant independent of h. But each D^μ ($\mu \in \Delta_k$) is a linear combination of D_α^k, $\alpha \in \Delta_k$. Hence, (2.5) is valid for this case.

Next, consider the general case when $\psi \in C_c^k(\mathbb{R}^s)$ and $\int_{\mathbb{R}^s} \psi(x)\,dx = 1$. For $h > 0$, let $f_h := A_{\phi,h} f$, where $\phi := \left(\prod_{\alpha \in \Delta_k} I_\alpha^k\right)\rho$ with $\rho \in C_c(\mathbb{R}^s)$ and $\int_{\mathbb{R}^s} \rho(x)\,dx = 1$. For $m = 1, \ldots, k$, let g_m be the function given in (2.6). Write $g_m = v_m + w_m$, where

$$v_m(x) := \int_{\mathbb{R}^s} f_h(x - mhu)\psi(u)\,du \quad \text{and} \quad w_m(x) := \int_{\mathbb{R}^s} (f - f_h)(x - mhu)\psi(u)\,du.$$

For $|\mu| = k$, we have

$$D^\mu v_m(x) = \int_{\mathbb{R}^s} D^\mu f_h(x - mhu)\psi(u)\,du.$$

By what has been proved for $f_h = A_{\phi,h} f$, we obtain

$$\left\| D^\mu v_m \right\|_p \leqslant \int_{\mathbb{R}^s} \left\| D^\mu f_h(\cdot - mhu) \right\|_p |\psi(u)|\,du \leqslant C_2 \omega_k(f, h)_p / h^k,$$

where C_2 is a constant independent of h. Moreover, integration by parts gives

$$D^\mu w_m(x) = \frac{1}{(mh)^k} \int_{\mathbb{R}^s} (f - f_h)(x - mhu) D^\mu \psi(u)\,du.$$

Hence, there exists a constant C_3 such that

$$\left\| D^\mu w_m \right\|_p \leqslant \frac{1}{(mh)^k} \int_{\mathbb{R}^s} \|(f - f_h)(\cdot - mhu)\|_p |D^\mu \psi(u)|\,du \leqslant C_3 \omega_k(f, h)_p / h^k,$$

where Lemma 2.1 has been used to derive the last estimate. Combining the above estimates together, we conclude that (2.5) is true for the general case. □

3. Quasi-projection operators

Let Q be the quasi-projection operator given in (1.1). If $Qq = q$ for all $q \in \Pi_{k-1}$, then $\Phi = \{\phi_1, \dots, \phi_N\}$ satisfies the Strang–Fix conditions of order k (see [24]). Conversely, if $\Phi = \{\phi_1, \dots, \phi_N\}$ satisfies the Strang–Fix conditions of order k, then there exist compactly supported functions $\tilde{\phi}_1, \dots, \tilde{\phi}_N$ in $L_{\tilde{p}}(\mathbb{R}^s)$ $(1/\tilde{p} + 1/p = 1)$ such that the corresponding quasi-projection operator has the property that $Qq = q$ for all $q \in \Pi_{k-1}$. See [14] for a recent survey on Strang–Fix conditions and their applications to the study of approximation power of refinable vectors of functions.

In order to establish estimate (1.5) we only need to deal with the case when Φ consists of only one function, since the proof for the general case will be analogous. Thus, we consider the quasi-projection operator Q given by

$$Qf = \sum_{\alpha \in \mathbb{Z}^s} \langle f, \tilde{\phi}(\cdot - \alpha) \rangle \, \phi(\cdot - \alpha), \quad f \in L_p(\mathbb{R}^s), \tag{3.1}$$

where ϕ is a compactly supported function in $L_p(\mathbb{R}^s)$ $(1 \leqslant p \leqslant \infty)$, and $\tilde{\phi}$ is a compactly supported function in $L_{\tilde{p}}(\mathbb{R}^s)$ $(1/\tilde{p} + 1/p = 1)$. It was proved in [13, Lemma 3.2] that $Qq = q$ for all $q \in \Pi_{k-1}$ if

$$D^\mu (1 - \hat{\tilde{\phi}} \hat{\phi})(0) = 0 \quad \forall\, |\mu| < k,$$

where $\hat{\phi}$ and $\hat{\tilde{\phi}}$ denote the Fourier transforms of ϕ and $\tilde{\phi}$, respectively.

In what follows we shall use C to denote a positive constant independent of h, f, and p, whose value may vary from time to time.

Theorem 3.1. *Suppose $0 \leqslant j < k$ and ϕ is a compactly supported function in $W_p^j(\mathbb{R}^s)$. Let Q be the quasi-projection operator given in* (3.1), *and let $Q_h := \sigma_h Q \sigma_{1/h}$ for $h > 0$. If $Qq = q$ for all $q \in \Pi_{k-1}$, then*

$$|f - Q_h f|_{j,p} \leqslant C h^{k-j} |f|_{k,p} \quad \forall\, f \in W_p^k(\mathbb{R}^s). \tag{3.2}$$

Proof. Let ν be a multi-index with $|\nu| = j$. In order to estimate $\|D^\nu (f - Q_h f)\|_p$, we use the following decomposition:

$$\begin{aligned} D^\nu (f - Q_h f) &= D^\nu (f - A_h f) + D^\nu (A_h f - Q_h A_h f) \\ &\quad + D^\nu (Q_h A_h f - Q_h f), \end{aligned} \tag{3.3}$$

where $A_h = A_{\psi,h}$ is the linear operator on $L_p(\mathbb{R}^s)$ given by (2.1).

The first term on the right-hand side of (3.3) can be easily estimated by noting that $D^\nu (A_h f) = A_h (D^\nu f)$. Indeed, by Lemma 2.1 we have

$$\left\| D^\nu (f - A_h f) \right\|_p = \left\| D^\nu f - A_h (D^\nu f) \right\|_p \leqslant C h^{k-j} |f|_{k,p}.$$

In order to estimate the third term on the right-hand side of (3.3), we introduce the linear operator $Q^{(\nu)}$ defined by

$$Q^{(\nu)} f = \sum_{\alpha\in\mathbb{Z}^s} \langle f, \tilde{\phi}(\cdot - \alpha)\rangle\, D^\nu \phi(\cdot - \alpha), \qquad f \in L_p(\mathbb{R}^s). \tag{3.4}$$

For $h > 0$, let $Q_h^{(\nu)} := \sigma_h Q^{(\nu)} \sigma_{1/h}$. It is easily seen that $D^\nu(Q_h f) = h^{-j} Q_h^{(\nu)} f$. Hence, by Lemma 2.1 we have

$$\begin{aligned}\left\| D^\nu(Q_h A_h f - Q_h f)\right\|_p &= h^{-j} \left\| Q_h^{(\nu)}(A_h f - f)\right\|_p \leqslant h^{-j} \|Q^{(\nu)}\| \|A_h f - f\|_p \\ &\leqslant C h^{k-j} |f|_{k,p}.\end{aligned}$$

It remains to estimate $\|D^\nu(A_h f - Q_h A_h f)\|_p$. Let $f_h := A_h f$ and $g_h := Q_h f_h$. For $\alpha \in \mathbb{Z}^s$, let $G_{\alpha,h} := (\alpha + (-1,1)^s)h$. We shall estimate $|D^\nu(f_h - g_h)(x)|$ for $x \in G_{\alpha,h}$. Let q be the Taylor polynomial of f_h of degree $k-1$ about αh. Write

$$D^\nu(A_h f - Q_h A_h f) = D^\nu(f_h - q) + D^\nu(q - g_h).$$

Taylor's theorem gives the following estimate:

$$\left| D^\nu(f_h - q)(x)\right| \leqslant C h^{k-j} \sum_{|\mu|=k} \|D^\mu f_h\|_\infty(G_{\alpha,h}) \quad \forall x \in G_{\alpha,h}. \tag{3.5}$$

Since $Q_h q = q$, we have

$$D^\nu(q - g_h) = D^\nu(Q_h(q - f_h)) = h^{-j} Q_h^{(\nu)}(q - f_h).$$

But

$$Q_h^{(\nu)}(q - f_h)(x) = \sum_{\beta\in\mathbb{Z}^s} \langle q - f_h, h^{-s}\tilde{\phi}(\cdot/h - \beta)\rangle\, D^\nu\phi(x/h - \beta).$$

It follows that

$$\left| Q_h^{(\nu)}(q - f_h)(x)\right| \leqslant \sum_{\beta\in\mathbb{Z}^s} |D^\nu\phi(x/h - \beta)| \int_{\mathbb{R}^s} |(q - f_h)(y)|\, |h^{-s}\tilde{\phi}(y/h - \beta)|\, dy.$$

Since ϕ and $\tilde{\phi}$ are compactly supported, there exists a positive constant K such that

$$\phi(x/h - \beta)\tilde{\phi}(y/h - \beta) \neq 0 \implies |y - x| \leqslant Kh.$$

By Taylor's theorem, we get

$$|(q - f_h)(y)| \leqslant C h^k \sum_{|\mu|=k} \|D^\mu f_h\|_\infty(G_{\alpha,h} + K[-h,h]^s) \quad \text{for } |y - x| \leqslant Kh.$$

Moreover,

$$\int_{\mathbb{R}^s} |h^{-s}\tilde{\phi}(y/h - \beta)|\, dy = \int_{\mathbb{R}^s} |\tilde{\phi}(y)|\, dy < \infty.$$

Hence, for $x \in G_{\alpha,h}$ we have

$$\left|D^\nu(q - g_h)(x)\right| \leqslant Ch^{k-j} |D^\nu \phi|^\circ(x/h) \sum_{|\mu|=k} \|D^\mu f_h\|_\infty (G_{\alpha,h} + K[-h,h]^s), \quad (3.6)$$

where $|D^\nu \phi|^\circ$ denotes the 1-periodization of $|D^\nu \phi|$:

$$|D^\nu \phi|^\circ(x) := \sum_{\beta \in \mathbb{Z}^s} |D^\nu \phi(x - \beta)|, \quad x \in \mathbb{R}^s.$$

Combining (3.5) and (3.6) together, we obtain the following estimate for $x \in G_{\alpha,h}$:

$$\begin{aligned} &\left|D^\nu(f_h - g_h)(x)\right| \\ &\quad \leqslant Ch^{k-j}\left[1 + |D^\nu \phi|^\circ(x/h)\right] \sum_{|\mu|=k} \|D^\mu f_h\|_\infty (G_{\alpha,h} + K[-h,h]^s). \end{aligned} \quad (3.7)$$

Since $f_h = A_{\psi,h} f$, it follows from (2.2) that

$$D^\mu f_h(x) = \sum_{m=1}^{k} (-1)^{m-1} \binom{k}{m} \int_{\mathbb{R}^s} D^\mu f(x - mu) \psi_h(u)\, du.$$

Applying Hölder's inequality to the above integrals, we get

$$\|D^\mu f_h\|_\infty (G_{\alpha,h} + K[-h,h]^s) \leqslant Ch^{-s/p} \|D^\mu f\|_p (G_{\alpha,h} + K'[-h,h]^s),$$

where K' is a constant independent of h and $K' \geqslant K$. In particular, for the case $p = \infty$, this in connection with (3.7) gives the desired estimate:

$$\left\|D^\nu(f_h - g_h)\right\|_\infty \leqslant Ch^{k-j} |f|_{k,\infty}.$$

Similarly, for the case $1 \leqslant p < \infty$, we see that the estimate

$$\begin{aligned} &\left|D^\nu(f_h - g_h)(x)\right|^p \\ &\quad \leqslant (Ch^{k-j})^p\, h^{-s}\left[1 + |D^\nu \phi|^\circ(x/h)\right]^p \sum_{|\mu|=k} \|D^\mu f\|_p^p (G_{\alpha,h} + K'[-h,h]^s) \end{aligned}$$

is valid for all $x \in G_{\alpha,h}$. Note that

$$\begin{aligned} &\int_{G_{\alpha,h}} h^{-s}\left[1 + |D^\nu \phi|^\circ(x/h)\right]^p dx \\ &\quad = \int_{(-1,1)^s} \left[1 + |D^\nu \phi|^\circ(x)\right]^p dx < \infty \quad \forall\, \alpha \in \mathbb{Z}^s. \end{aligned}$$

It follows that

$$\begin{aligned} \left\|D^\nu(f_h - g_h)\right\|_p^p &\leqslant \sum_{\alpha \in \mathbb{Z}^s} \int_{G_{\alpha,h}} \left|D^\nu(f_h - g_h)(x)\right|^p dx \\ &\leqslant (Ch^{k-j})^p \sum_{|\mu|=k} \sum_{\alpha \in \mathbb{Z}^s} \|D^\mu f\|_p^p (G_{\alpha,h} + K'[-h,h]^s). \end{aligned}$$

But

$$\sum_{\alpha\in\mathbb{Z}^s} \|D^\mu f\|_p^p(G_{\alpha,h} + K'[-h,h]^s) \leqslant C \int_{\mathbb{R}^s} |D^\mu f(y)|^p\, dy.$$

Therefore,

$$\left\| D^\nu (f_h - g_h) \right\|_p \leqslant C h^{k-j} |f|_{k,p}.$$

This completes the proof of the theorem. □

For the case $j = 0$, Theorem 3.1 was already established in [15] and [21]. In fact, in these two papers, the functions ϕ and $\tilde{\phi}$ are only required to lie in the space $\mathcal{L}_k$ of all functions g with

$$\text{ess sup}_{x\in[-1,1]^s} \sum_{\beta\in\mathbb{Z}^s} |g(x+\beta)|(1+|x+\beta|)^k < \infty.$$

Clearly, if there is some $\varepsilon > 0$ such that $|g(x)| \leqslant C(1+|x|)^{-k-s-\varepsilon}$ for all $x \in \mathbb{R}^s$, then f lies in $\mathcal{L}_k$.

Kyriazis [19] investigated approximation schemes associated with a pair of Sobolev spaces. Estimate (3.2) was established under the assumption that there exists some $\varepsilon > 0$ such that

$$\begin{aligned} &\left|D^\nu \phi(x)\right| \\ &\quad \leqslant C(1+|x|)^{-k-s-\varepsilon}\ (|\nu|\leqslant j) \quad \text{and} \quad \left|\tilde{\phi}(x)\right| \leqslant C(1+|x|)^{-k-s-\varepsilon} \quad \forall\, x\in\mathbb{R}^s. \end{aligned}$$

A careful examination of the proof of Theorem 3.1 reveals that estimate (3.2) is still valid if ϕ, $D^\nu\phi$ ($|\nu| = j$), and $\tilde{\phi}$ lie in the space $\mathcal{L}_k$. Our proof of Theorem 3.1 was conducted exclusively in the time domain, while the error analysis in [19] was given in terms of the frequency domain. Each approach has its own merit. Our method can be easily adapted to approximation associated with spaces of functions on general domains without the shift-invariant structure.

Simultaneous approximation in derivatives was also studied by Zhao [25]. But his method only works for local shift-invariant spaces. In other words, his method only applies to the case when ϕ is compactly supported.

We are in a position to give an error estimate for the quasi-projection scheme in terms of moduli of smoothness.

Theorem 3.2. *Let ϕ be a compactly supported function in $W_p^j(\mathbb{R}^s)$, $0 \leqslant j < k$, and let Q be the quasi-projection operator given in* (3.1) *with $\tilde{\phi}$ being a compactly supported function* *in* $L_{\tilde{p}}(\mathbb{R}^s)$ $(1/\tilde{p}+1/p=1)$. *If* $Qq=q$ *for all* $q\in\Pi_{k-1}$, *then the following estimate*

$$|f-Q_hf|_{j,p}\leqslant C\sum_{|\nu|=j}\omega_{k-j}(D^\nu f,h)_p$$

is valid for $f\in W_p^j(\mathbb{R}^s)$ *in the case* $1\leqslant p<\infty$ *or* $f\in C^j(\mathbb{R}^s)$ *in the case* $p=\infty$.

Proof. Let $A_h=A_{\psi,h}$ be the linear operator on $L_p(\mathbb{R}^s)$ given in (2.1). In order to prove the theorem we shall employ the decomposition given in (3.3) with $|\nu|=j$. By Lemma 2.1 we have

$$\left\|D^\nu(f-A_hf)\right\|_p=\left\|D^\nu f-A_h(D^\nu f)\right\|_p\leqslant C\omega_{k-j}(D^\nu f,h)_p. \tag{3.8}$$

Moreover, applying Theorem 3.1 to A_hf, we obtain

$$\left\|D^\nu(A_hf-Q_hA_hf)\right\|_p\leqslant Ch^{k-j}|A_hf|_{k,p}=Ch^{k-j}\sum_{|\mu|=k}\left\|D^\mu A_hf\right\|_p.$$

For each multi-index μ with $|\mu|=k$, we can find a multi-index γ with $\gamma\leqslant\mu$ and $|\gamma|=j$. Hence, by Lemma 2.2 we have

$$\begin{aligned}\|D^\mu(A_hf)\|_p&=\|D^{\mu-\gamma}D^\gamma(A_hf)\|_p\\&=\left\|D^{\mu-\gamma}(A_h(D^\gamma f))\right\|_p\leqslant C\omega_{k-j}(D^\gamma f,h)_p/h^{k-j}.\end{aligned}$$

This shows that

$$\left\|D^\nu(A_hf-Q_hA_hf)\right\|_p\leqslant C\sum_{|\nu|=j}\omega_{k-j}(D^\nu f,h)_p. \tag{3.9}$$

In order to estimate $\|D^\nu(Q_hA_hf-Q_hf)\|_p$ we use the linear operator $Q^{(\nu)}$ defined in (3.4). With $Q_h^{(\nu)}=\sigma_hQ^{(\nu)}\sigma_{1/h}$ we get

$$\left\|D^\nu(Q_hA_hf-Q_hf)\right\|_p=\left\|h^{-j}Q_h^{(\nu)}(A_hf-f)\right\|_p\leqslant h^{-j}\|Q^\nu\|\|A_hf-f\|_p.$$

This in connection with (2.4) gives

$$\left\|D^\nu(Q_hA_hf-Q_hf)\right\|_p\leqslant C\sum_{|\nu|=j}\omega_{k-j}(D^\nu f,h)_p. \tag{3.10}$$

The combination of (3.8)–(3.10) completes the proof of the theorem. □

4. Approximation in Lipschitz spaces

In this section we study quasi-projection operators on Lipschitz spaces. In order to establish estimate (1.6) for the quasi-projection operator given in (1.1), we only need to consider the case when Φ consists of only one function.

If $f \in \mathrm{Lip}(\lambda, L_p(\mathbb{R}^s))$ and k is an integer greater than or equal to λ, then

$$\omega_k(f,h)_p \leqslant Ch^\lambda |f|_{\mathrm{Lip}(\lambda,L_p)} \qquad \forall\, h>0,$$

where C is a positive constant independent of h and f.

Theorem 4.1. *Suppose* $0<\tau<\lambda\leqslant k$. *Let* ϕ *be a compactly supported function in* $W_p^j(\mathbb{R}^s)$, *where j is the integer such that* $j-1<\tau\leqslant j$. *Let Q be the quasi-projection operator given in* (3.1) *such that* $Qq=q$ *for all* $q\in\Pi_{k-1}$. *Then*

$$|f-Q_hf|_{\mathrm{Lip}(\tau,L_p)} \leqslant Ch^{\lambda-\tau}|f|_{\mathrm{Lip}(\lambda,L_p)} \quad \forall\, f\in \mathrm{Lip}(\lambda,L_p). \tag{4.1}$$

Proof. Suppose $\tau=r+\eta$, where r is an integer and $0<\eta\leqslant 1$. Then $j=r+1$. In order to establish (4.1), it suffices to show that there exists a constant C independent of h, y and f such that

$$\max_{|v|=r}\left\|\nabla_y D^v(f-Q_hf)\right\|_p \leqslant C|y|^\eta h^{\lambda-\tau}|f|_{\mathrm{Lip}(\lambda,L_p)} \quad \forall y\in\mathbb{R}^s. \tag{4.2}$$

Let us first consider the case $|y|\geqslant h$. Since $f\in\mathrm{Lip}(\lambda,L_p)$, for $|v|=r$ we have $D^vf\in \mathrm{Lip}(\lambda-r,L_p)$. By Theorem 3.2 we get the following estimates:

$$\begin{aligned}\left\|\nabla_y D^v(f-Q_hf)\right\|_p &\leqslant \left\|D^v(f-Q_hf)\right\|_p + \left\|D^v(f-Q_hf)(\cdot-y)\right\|_p\\ &\leqslant Ch^{\lambda-r}|f|_{\mathrm{Lip}(\lambda,L_p)}.\end{aligned}$$

But $h^{\lambda-r}=h^\eta h^{\lambda-r-\eta}\leqslant |y|^\eta h^{\lambda-\tau}$ for $|y|\geqslant h$. Hence, (4.2) is valid for this case.

Next, let us deal with the case $|y|<h$ and $\lambda>j$. Suppose $|v|=r=j-1$. We observe that

$$\nabla_y D^v(f-Q_hf)(x)=\int_0^1 D_yD^v(f-Q_hf)(x-ty)\,dt, \quad x\in\mathbb{R}^s.$$

Applying the Minkowski inequality to the above integral, we obtain

$$\left\|\nabla_y D^v(f-Q_hf)\right\|_p \leqslant C|y||f-Q_hf|_{j,p},$$

where C is a constant independent of h and y. This in connection with Theorem 3.2 gives

$$\left\|\nabla_y D^v(f-Q_hf)\right\|_p \leqslant C|y|h^{\lambda-j}|f|_{\mathrm{Lip}(\lambda,L_p)} \leqslant C|y|^\eta h^{\lambda-\tau}|f|_{\mathrm{Lip}(\lambda,L_p)},$$

thereby verifying (4.2) for the case $\lambda>j$.

Finally, let us investigate the case $|y|<h$ and $\lambda\leqslant j$. For this case, we write

$$\begin{aligned}\nabla_y D^v(f-Q_hf) &= \nabla_y D^v(f-f_h)+\nabla_y D^v(f_h-Q_hf_h)\\ &\quad +\nabla_y D^v(Q_hf_h-Q_hf),\end{aligned} \tag{4.3}$$

where $f_h:=A_hf$ with $A_h:=A_{\psi,h}$ being the linear operator given in (2.1).

For the first term on the right-hand side of (4.3) we have

$$\begin{aligned}\nabla_y D^v(f-f_h)(x) &= \nabla_y(D^vf-A_h(D^vf))(x)\\ &= \int_{\mathbb{R}^s}\nabla_u^k\nabla_y D^vf(x)\psi_h(u)\,du, \quad x\in\mathbb{R}^s.\end{aligned}$$

Applying the Minkowski inequality to the above integral, we obtain

$$\begin{aligned}\big\|\nabla_y D^\nu (f - f_h)\big\|_p &\leqslant \int_{\mathbb{R}^s} \big\|\nabla_u^k \nabla_y D^\nu f\big\|_p |\psi_h(u)|\, du \\ &\leqslant 2^k \big\|\nabla_y D^\nu f\big\|_p \int_{\mathbb{R}^s} |\psi(u)|\, du.\end{aligned}$$

Since $f \in \mathrm{Lip}(\lambda, L_p)$, we have $D^\nu f \in \mathrm{Lip}(\lambda - r, L_p)$, and hence

$$\|\nabla_y D^\nu f\|_p \leqslant C|y|^{\lambda-r} |f|_{\mathrm{Lip}(\lambda,L_p)}$$

for some constant C independent of h, y and f. Therefore,

$$\big\|\nabla_y D^\nu (f - f_h)\big\|_p \leqslant C|y|^{\lambda-r} |f|_{\mathrm{Lip}(\lambda,L_p)} \leqslant C|y|^\eta h^{\lambda-\tau} |f|_{\mathrm{Lip}(\lambda,L_p)}. \tag{4.4}$$

For the second term on the right-hand side of (4.3), by (2.3) we have

$$\big\|\nabla_y D^\nu (f_h - Q_h f_h)\big\|_p \leqslant \big\|D_y D^\nu (f_h - Q_h f_h)\big\|_p \leqslant C|y||f_h - Q_h f_h|_{j,p}.$$

Theorem 3.1 tells us

$$|f_h - Q_h f_h|_{j,p} \leqslant C h^{k-j} |f_h|_{k,p}.$$

But $f \in \mathrm{Lip}(\lambda, L_p)$ implies $|f_h|_{k,p} \leqslant C h^\lambda |f|_{\mathrm{Lip}(\lambda,L_p)}/h^k$, by Lemma 2.2. Consequently,

$$\big\|\nabla_y D^\nu (f_h - Q_h f_h)\big\|_p \leqslant C|y|h^{\lambda-j} |f|_{\mathrm{Lip}(\lambda,L_p)} \leqslant C|y|^\eta h^{\lambda-\tau} |f|_{\mathrm{Lip}(\lambda,L_p)}. \tag{4.5}$$

For the third term on the right-hand side of (4.3) we have

$$\big\|\nabla_y D^\nu (Q_h f_h - Q_h f)\big\|_p \leqslant |y||Q_h(f - f_h)|_{j,p}.$$

For $|\mu| = j$, recall that $D^\mu Q_h(f - f_h) = h^{-j} Q_h^{(\mu)}(f - f_h)$, where $Q_h^{(\mu)} = \sigma_h Q^{(\mu)} \sigma_{1/h}$ with $Q^{(\mu)}$ being the linear operator defined in (3.4). Hence,

$$|Q_h(f - f_h)|_{j,p} = \sum_{|\mu|=j} \big\|D^\mu Q_h(f - f_h)\big\|_p \leqslant h^{-j} \sum_{|\mu|=j} \|Q^{(\mu)}\| \|f - f_h\|_p.$$

But $f \in \mathrm{Lip}(\lambda, L_p)$ implies $\|f - f_h\|_p \leqslant C h^\lambda |f|_{\mathrm{Lip}(\lambda,L_p)}$, by Lemma 2.1. Consequently,

$$\big\|\nabla_y D^\nu (Q_h f_h - Q_h f)\big\|_p \leqslant C|y|h^{\lambda-j} |f|_{\mathrm{Lip}(\lambda,L_p)} \leqslant C|y|^\eta h^{\lambda-\tau}. \tag{4.6}$$

Combining estimates (4.4), (4.5), and (4.6) together, we obtain the desired result (4.2) for the case $|y| < h$ and $\lambda \leqslant j$. The proof of the theorem is complete. □

In [19] Kyriazis investigated quasi-projection schemes for pairs of Triebel–Lizorkin spaces. But the case $p = \infty$ was excluded in his study.

5. Approximation under isotropic scaling

Now let us discuss approximation with shift-invariant spaces scaled by an expansive matrix. Let M be an $s \times s$ integer matrix with $\xi_1, \ldots, \xi_s$ being its eigenvalues. If $|\xi_i| > 1$ for all $j = 1, \ldots, s$, then M is said to be *expansive*. We say that M is *isotropic* if M is similar to the diagonal matrix diag $(\xi_1, \ldots, \xi_s)$ with $|\xi_1| = \cdots = |\xi_s|$. Let $m := |\det M|$. Then $m = |\xi_1 \cdots \xi_s|$. In particular, if M is isotropic, then $|\xi_1| = \cdots = |\xi_s| = m^{1/s}$. Refinement equations associated with expansive matrices play a vital role in wavelet analysis.

Suppose M is an isotropic expansive matrix. Let $\Phi = \{\phi_1, \ldots, \phi_N\}$ be a finite set of compactly supported functions in $L_p(\mathbb{R}^s)$, and let $S := \mathbb{S}(\Phi) \cap L_p(\mathbb{R}^s)$ $(1 \leqslant p \leqslant \infty)$. For $n = 0, 1, \ldots$, let $S_n := \{g(M^n \cdot) : g \in S\}$. Then $(S_n)_{n=0,1,\ldots}$ is a family of shift-invariant spaces scaled by powers of matrix M. We are interested in the approximation properties of $(S_n)_{n=0,1,\ldots}$. Again, quasi-projection operators induce good approximation schemes.

For $n = 0, 1, \ldots$, let Q_n be the linear operator on $L_p(\mathbb{R}^s)$ given by

$$Q_n f = \sum_{j=1}^{N} \sum_{\alpha \in \mathbb{Z}^s} \langle f, m^n \tilde{\phi}_j(M^n \cdot - \alpha) \rangle \, \phi_j(M^n \cdot - \alpha), \quad f \in L_p(\mathbb{R}^s),$$

where $\tilde{\phi}_1, \ldots, \tilde{\phi}_N$ are compactly supported functions in $L_{\tilde{p}}(\mathbb{R}^s)$ $(1/\tilde{p} + 1/p = 1)$. In particular, Q_0 is the quasi-projection operator Q given in (1.1).

Theorem 5.1. *Suppose $\phi_1, \ldots, \phi_N$ are compactly supported functions in $W_p^j(\mathbb{R}^s)$, where $0 \leqslant j < k$. If $Qq = q$ for all $q \in \Pi_{k-1}$, then the following estimate*

$$|f - Q_n f|_{j,p} \leqslant C \sum_{|\nu|=j} \omega_{k-j}(D^\nu f, m^{-n/s})_p$$

is valid for $f \in W_p^j(\mathbb{R}^s)$ in the case $1 \leqslant p < \infty$ or $f \in C^j(\mathbb{R}^s)$ in the case $p = \infty$. Furthermore, suppose $0 < \tau < \lambda \leqslant k$ and $j - 1 < \tau \leqslant j$. If $Qq = q$ for all $q \in \Pi_{k-1}$, then

$$|f - Q_n f|_{\mathrm{Lip}(\tau, L_p)} \leqslant C (m^{-n/s})^{\lambda - \tau} \quad |f|_{\mathrm{Lip}(\lambda, L_p)} \forall f \in \mathrm{Lip}(\lambda, L_p).$$

This theorem can be proved by following the procedure of the proofs of Theorems 3.1, 3.2, and 4.1. It is not necessary to repeat the details.

References

[1] A. Aldroubi, Q. Sun, W.-S. Tang, Non-uniform sampling in multiply generated shift-invariant subspaces of $L^p(\mathbb{R}^d)$, in: D. Deng, D. Huang, R.-Q. Jia, W. Lin, J. Wang (Eds.), Wavelet Analysis and Applications, the American Mathematical Society and International Press, 2002, pp. 1–8.

[2] T. Blu, M. Unser, Approximation error for quasi-interpolators and (multi)-wavelet expansions, Appl. Comput. Harmonic Anal. 6 (1999) 219–251.

[3] C. de Boor, Splines as linear combinations of B-splines: A survey, in: G.G. Lorentz, C.K. Chui, L.L. Schumaker (Eds.), Approximation Theory II, Academic Press, New York, 1976, pp. 1–47.

[4] C. de Boor, Topics in multivariate approximation theory, in: P. Turner (Ed.), Topics in Numerical Analysis, Lecture Notes in Mathematics, vol. 965, Springer, 1982, pp. 39–78.

[5] C. de Boor, Quasiinterpolants and approximation power of multivariate splines, in: W. Dahmen, M. Gasca, C.A. Micchelli (Eds.), Computation of Curves and Surfaces, Kluwer Academic Publishers, Netherlands, 1990, pp. 313–345.
[6] C. de Boor, R. DeVore, A. Ron, Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$, Trans. Amer. Math. Soc. 341 (1994) 787–806.
[7] C. de Boor, G. Fix, Spline approximation by quasiinterpolants, J. Approx. Theory 8 (1973) 19–45.
[8] P.L. Butzer, R.J. Nessel, Fourier Analysis and Approximation, vol. 1, One-Dimensional Theory, Birkhäuser Verlag, Basel, 1971.
[9] R.A. DeVore, G.G. Lorentz, Constructive Approximation, Springer, Berlin, 1993.
[10] Bin Han, Vector cascade algorithms and refinable function vectors in Sobolev spaces, J. Approx. Theory 124 (2003) 44–88.
[11] K. Jetter, D.X. Zhou, Order of linear approximation from shift-invariant spaces, Constr. Approx. 11 (1995) 423–438.
[12] R.Q. Jia, Shift-invariant spaces and linear operator equations, Israel J. Math. 103 (1998) 259–288.
[13] R.Q. Jia, Convergence rates of cascade algorithms, Proc. Amer. Math. Soc. 131 (2003) 1739–1749.
[14] R.Q. Jia, Q.T. Jiang, Approximation power of refinable vectors of functions, in: D. Deng, D. Huang, R.-Q. Jia, W. Lin, J. Wang (Eds.), Wavelet Analysis and Applications, the American Mathematical Society and International Press, 2002, pp. 153–176.
[15] R.Q. Jia, J.J. Lei, Approximation by multiinteger translates of functions having global support, J. Approx. Theory 72 (1993) 2–23.
[16] R.Q. Jia, C.A. Micchelli, Using the refinement equations for the construction of pre-wavelets II: Powers of two, in: P.J. Laurent, A. Le Méhauté, L.L. Schumaker (Eds.), Curves and Surfaces, Academic Press, New York, 1991, pp. 209–246.
[17] H. Johnen, K. Scherer, On the equivalence of the K-functional and moduli of continuity and some applications, in: W. Schempp, K. Zeller (Eds.), Constructive Theory of Functions of Several Variables, Lecture Notes in Mathematics, Vol. 571, Springer, Berlin, 1977, pp. 119–140.
[18] G.C. Kyriazis, Approximation from shift-invariant spaces, Constr. Approx. 11 (1993) 141–164.
[19] G.C. Kyriazis, Approximation of distribution spaces by means of kernel operators, J. Fourier Anal. Appl. 3 (1996) 261–286.
[20] J.J. Lei, $L_p(\mathbb{R}^d)$-Approximation by certain projection operators, J. Math. Anal. Appl. 185 (1994) 1–14.
[21] J.J. Lei, R.Q. Jia, E.W. Cheney, Approximation from shift-invariant spaces by integral operators, SIAM J. Math. Anal. 28 (1997) 481–498.
[22] P.P. Petrushev, V.A. Popov, Rational Approximation of Real Functions, Cambridge University Press, Cambridge, 1987.
[23] A. Ron, Wavelets and their associated operators, in: C.K. Chui, L.L. Schumaker (Eds.), Approximation Theory IX, vol. 2: Computational Aspects, Vanderbilt University, 1998, pp. 283–317.
[24] G. Strang, G. Fix, A Fourier analysis of the finite-element variational method, in: G. Geymonat (Ed.), Constructive Aspects of Functional Analysis, C.I.M.E., 1973, pp. 793–840.
[25] K. Zhao, Approximation from locally finite-dimensional shift-invariant spaces, Proc. Amer. Math. Soc 124 (1996) 1857–1867.