APPROXIMATION BY PIECEWISE EXPONENTIALS*

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Abstract. A function is called an exponential if it is a linear combination of products of polynomials with pure exponentials. In this paper lower and upper bounds for families of spaces of piecewise exponentials are established. In particular, the exact L_p -approximation order ($1 \leq p \leq \infty$) is found for a family $\{S_h\}_{h>0}$ of function spaces when each S_h is generated by an exponential box spline and its multi-integer translates.

Key words. multivariate approximation, order of approximation, exponentials, exponential box splines

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1. Introduction. Spaces spanned by multi-integer translates of compactly supported functions have recently attracted much attention. The general setup can be described as follows: Let $\{S_h\}_h$ be a family of function spaces each of which is spanned by the $h\mathbb{Z}^n$ -translates of one or several compactly supported functions on $\mathbb{R}^n$. We want to investigate the approximation order of $\{S_h\}_h$ and the ways to realize this approximation order. Here the approximation order of $\{S_h\}_h$ is defined to be the largest real number k for which

$$\text{dist}(f, S_h) = O(h^k)$$

for all sufficiently smooth complex-valued functions on a domain $G \subseteq \mathbb{R}^n$, where dist is measured by some norm (usually an L_p -norm).

A case of particular interest is the scaling case that occurs when the refined spaces S_h are dilations of S_1 , i.e.,

$$S_h = \sigma_h S_1$$

with σ_h being the scaling operator

$$\sigma_h : f \mapsto f(\cdot/h).$$

As early as 1946, Schoenberg [19] considered the scaling case where S_1 is spanned by the integer translates of a single compactly supported function ϕ on $\mathbb{R}$. In the late sixties and early seventies Schoenberg's work was extended to the case where S_1 is spanned by several compactly supported functions on $\mathbb{R}^n$. In particular, in the setting of the finite element method, Strang and Fix [20] successfully characterized the so-called controlled approximation order of $\{\sigma_h S_1\}_h$ when S_1 is spanned by a single compactly supported function. However, when S_1 is spanned by several compactly supported functions, their attempt at characterizing the controlled approximation order of $\{\sigma_h S_1\}_h$ failed, as was demonstrated by Jia's counterexample [13]. Nevertheless, the conditions formulated by them to ensure a certain approximation power of $\{\sigma_h S_1\}_h$ have been widely used, and these conditions are now called the Strang-Fix

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conditions. In [4] de Boor and Jia gave a characterization of the local approximation order of $\{\sigma_h S_1\}_h$ in terms of the Strang–Fix conditions.

Both the work of Strang and Fix [20] and that of de Boor and Jia [4] put some restriction on the approximation from $\{\sigma_h S_1\}_h$ (either “controlled” or “local”), and hence do not give a characterization for the (unconditional) approximation order. However, when S_1 is spanned by the multi-integer translates of a box spline, the approximation order of $\{\sigma_h S_1\}_h$ was already established by de Boor and Höllig in [3]. Their work was extended by Ron [18] to the case where S_1 is spanned by the multi-integer translates of a compactly supported function on $\mathbb{R}^n$ under an additional condition. Earlier, Jia [14] characterized the approximation order of $\{\sigma_h S_1\}_h$ when S_1 is spanned by the integer translates of several compactly supported functions on $\mathbb{R}$.

Examples of the nonscaling case were given by Dyn and Ron in [12]. In particular, they emphasized that the Strang–Fix conditions are not applicable to approximation by translates of exponential box splines, which were first introduced by Ron [17]. Using quasi-interpolant schemes based on the Neumann series approach (see [7]), Dyn and Ron [12] established a lower bound for the L_∞ -approximation order, but they did *not* show that this lower bound is, in fact, the exact approximation order. This is in sharp contrast to the case where $S_h = \sigma_h S_1$ and S_1 is spanned by the multi-integer translates of a box spline. In such a case, the approximation order is relatively easy to determine (see [3]). Even though a characterization of the approximation order was given by Ron [18], it is not clear how his characterization can be applied to actually find the exact order of approximation from $\{S_h\}_h$, where S_h are spaces spanned by the translates of exponential box splines.

The primary goal of this paper is to fill this gap. For this purpose we need to introduce some terminology and notation first. We shall use the standard multi-index notation. Let $\mathbb{N}$ be the set of nonnegative integers. An element $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is called a multi-index, and the length of α is defined to be $|\alpha| := \sum_{j=1}^n \alpha_j$. The factorial of α is $\alpha! := \alpha_1! \cdots \alpha_n!$. Let $\mathbb{Z}$ be the set of integers. An element of $\mathbb{Z}^n$ is called a multi-integer. A mapping from $\mathbb{Z}^n$ to $\mathbb{C}$ is called a sequence on $\mathbb{Z}^n$.

Let $\mathbb{R}^n$ be the n -dimensional real space equipped with the uniform norm, i.e.,

$$\|x\| := \max_{1 \leq j \leq n} |x_j|, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

If $\Omega \subseteq \mathbb{R}^n$ and $r \geq 0$, we denote by $B_r(\Omega)$ the *closed* ball of radius r around Ω , that is,

$$B_r(\Omega) := \{x \in \mathbb{R}^n : \text{dist}(x, \Omega) \leq r\}$$

with

$$\text{dist}(x, \Omega) := \inf_{y \in \Omega} \|x - y\|.$$

When $\Omega = \{x\}$, we write $B_r(x)$ for $B_r(\Omega)$, and simply write B_r for $B_r(0)$. Evidently, if Ω is a closed set, then

$$B_r(\Omega) = \Omega + B_r := \{x + y : x \in \Omega, y \in B_r\}.$$

If f is a measurable function on a measurable subset Ω of $\mathbb{R}^n$, we denote by $\|f\|_p(\Omega)$ the quantity $(\int_{\Omega} |f|^p dx)^{1/p}$. Similarly,

$$|f|_{k,p}(\Omega) := \sum_{|\alpha|=k} \|D^{\alpha} f\|_p(\Omega) \quad \text{and} \quad \|f\|_{k,p}(\Omega) := \sum_{|\alpha| \leq k} \|D^{\alpha} f\|_p(\Omega).$$

When Ω is omitted, the norm is understood to be taken over $\mathbb{R}^n$. For a subspace H of $L_p(\Omega)$ and an element $f \in L_p(\Omega)$, let

$$\text{dist}_p(f, H)(\Omega) := \inf_{g \in H} \|f - g\|_p(\Omega).$$

Let $W_p^k = W_p^k(\mathbb{R}^n)$ be the usual Sobolev space equipped with the norm $\|\cdot\|_{k,p}$.

We denote by $\Pi = \Pi(\mathbb{R}^n)$ the linear space of all polynomials on $\mathbb{R}^n$. For a non-negative integer k , we denote by Π_k (respectively, $\Pi_{<k}$) its subspace of all polynomials of (total) degree at most k (respectively, less than k). In particular, the monomials given by

$$(\cdot)^{\alpha} : x \mapsto x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad x \in \mathbb{R}^n,$$

are elements of Π . If $p(x) = \sum a_{\alpha} x^{\alpha}$ is a polynomial, then $p(D)$ denotes the differential operator induced by p , i.e., $p(D) = \sum a_{\alpha} D^{\alpha}$. In particular, $D^{\alpha} = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ with D_j being the j th partial derivative operator, $j = 1, \dots, n$.

Following de Boor and Ron [6], we call a function on $\mathbb{R}^n$ an exponential if it is a linear combination of products of polynomials with the pure exponentials

$$e_{\theta} : x \mapsto e^{\theta \cdot x}, \quad \theta \in \mathbb{C}^n,$$

where $\theta \cdot x$ denotes the inner product of θ and x . Note that any finite-dimensional D -invariant (i.e., invariant under differentiation) space of distributions is a space of exponentials (see [1]).

Let A_0 be the linear space of all functions analytic at the origin. An element $f \in A_0$ can be expanded into a power series in a neighborhood of the origin:

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} D^{\alpha} f(0) x^{\alpha} / \alpha!.$$

For $j \in \mathbb{N}$, let f_j be the j th homogeneous part of f , i.e.,

$$f_j = \sum_{|\alpha|=j} D^{\alpha} f(0) (\cdot)^{\alpha} / \alpha!.$$

The least term of f , denoted by $f_{\downarrow}$, is defined as f_j with j being the smallest integer for which $f_j \neq 0$ (see [5]). For a subspace H of A_0 we denote by $H_{\downarrow}$ the space spanned by all $f_{\downarrow}$ for $f \in H$.

Now let H be a finite-dimensional space of exponentials. In §3, we will prove our main result, which states that if each S_h consists of piecewise H -functions only, then

the approximation order of $\{S_h\}_h$ does not exceed any integer k for which $\Pi_k \not\subseteq H_\downarrow$. This result was already obtained by de Boor and Höllig [3] for the case where H itself is a space of polynomials.

Lower bounds for the approximation order of $\{S_h\}_h$ are usually established by a concrete approximation scheme using quasi-interpolant methods. In the scaling case, the Strang–Fix conditions and their various equivalent forms are the core of such a quasi-interpolant scheme. As pointed out by Dyn and Ron in [12], the original form of the Strang–Fix conditions is not applicable to the nonscaling case. However, a modified version of the Strang–Fix conditions is still available if we are content with spaces of exponentials, as demonstrated by Jia in [15]. In §2, based on de Boor’s survey paper [2], and on our recent work [16], we shall give a construction of L_p -approximation ($1 \leq p \leq \infty$) from $\{S_h\}_h$ when each S_h is spanned by $h\mathbb{Z}^n$ -translates of a compactly supported function on $\mathbb{R}^n$.

In [6], [12], and [18], only L_∞ -approximation was considered. In this paper we deal not only with L_∞ -approximation, but also L_p -approximation ($1 \leq p < \infty$). We contend that L_p -approximation is important. Indeed, Strang and Fix concentrated on L_2 -approximation, since their main concern was the finite element method. In the work of DeVore and Popov [11] on approximation by multivariate splines with free knots, L_p -approximation ($0 < p < \infty$) played an essential role. Furthermore, L_p -approximation ($1 \leq p < \infty$) has a nature different from that of L_∞ -approximation. We can say that L_∞ -approximation is essentially local, while L_p -approximation ($1 \leq p < \infty$) is global. This point will be made clear in the following sections.

2. Lower bounds for the approximation order. In this section, using a quasi-interpolation scheme, we provide lower bounds for the L_p -approximation order ($1 \leq p < \infty$) of a family $\{S_h\}_h$ of approximating spaces, each of which is spanned by the $h\mathbb{Z}^n$ -translates of a single compactly supported function.

Let ϕ be a complex-valued Lebesgue-measurable function on $\mathbb{R}^n$. We say that ϕ is a normal function (see [15]), if for any $x \in \mathbb{R}^n$

$$\phi(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{m(B_\varepsilon(x))} \int_{B_\varepsilon(x)} \phi(y) dy,$$

where m denotes the Lebesgue measure.

Let ϕ be a compactly supported normal function. For a sequence b on $\mathbb{Z}^n$, the semidiscrete convolution product $\phi *' b$ is the function given by

$$\phi *' b := \sum_{\nu \in \mathbb{Z}^n} \phi(\cdot - \nu) b(\nu).$$

More generally, for a sequence b on $h\mathbb{Z}^n$, the h -scaled semidiscrete convolution product $\phi *'_h b$ is given by

$$\phi *'_h b := \sum_{\nu \in \mathbb{Z}^n} \phi(\cdot - h\nu) b(h\nu).$$

In this section, we assume that $\{\phi_h\}_h$ is a collection of normal functions on $\mathbb{R}^n$ satisfying the following three conditions:

- (i) $\text{supp } \phi_h \subseteq hB_r$,

(ii) $\|\phi_h\|_\infty \leq c$, and

(iii) $h^{-n}|\hat{\phi}_h(0)| \geq b$,

where r , b and c are positive constants independent of h . In the third condition above, $\hat{\phi}$ denotes the Fourier transform of ϕ :

$$\hat{\phi}(\xi) := \int_{\mathbb{R}^n} \phi(x) e^{-i\xi \cdot x} dx.$$

These conditions were first formulated in [6].

Let H be a finite-dimensional D -invariant space of exponentials. From [1] we know that H has the form $\sum_{\theta \in T} e_\theta P_\theta$, where T is a finite subset of $\mathbb{C}^n$ and each P_θ is a D -invariant polynomial space.

THEOREM 2.1. *Let $H = \sum_{\theta \in T} e_\theta P_\theta$ be a finite-dimensional D -invariant exponential space such that $H_\perp \supseteq \Pi_{<k}$ for some positive integer k . Assume that $\{\phi_h\}_{h>0}$ is a collection of normal functions on $\mathbb{R}^n$ satisfying the above conditions (i), (ii), and (iii). If, in addition, each ϕ_h satisfies the Strang–Fix conditions for H , i.e., for each $\theta \in T$,*

$$(2.1) \quad q(-iD)\hat{\phi}_h(2\pi\nu/h - i\theta) = 0 \quad \text{for all } q \in P_\theta \text{ and } \nu \in \mathbb{Z}^n \setminus \{0\},$$

then with $S_h := \text{range}(\phi_h *'_h)$, $\{S_h\}_{h>0}$ provides L_p -approximation of order at least k ($1 \leq p \leq \infty$).

Proof. This theorem is a consequence of Lemma 2.2 and Theorem 2.4, which will be proved later. $\square$

Remark. The Strang–Fix conditions as given in (2.1) are equivalent to the statement that $\phi_h *'_h$ maps $e_\theta P_\theta$ to itself for each $\theta \in T$ (see [15]). Obviously, the latter implies that $\phi_h *'_h$ maps H to itself, while the converse is true if $h(T-T) \cap 2\pi i\mathbb{Z}^n = \{0\}$ (see [6]). Since T is fixed (independent of h), this condition is satisfied for sufficiently small h . Also see [8] for some related results.

LEMMA 2.2. *Let $\{\phi_h\}_{h>0}$ be a collection of compactly supported normal functions on $\mathbb{R}^n$ satisfying the Strang–Fix conditions for $H = \sum_{\theta \in T} e_\theta P_\theta$ as given in (2.1). Then for each $\theta \in T$,*

$$(2.2) \quad \phi_h *'_h(e_\theta p) = h^{-n} e_\theta(p(\cdot - iD)\hat{\phi}_h)(-i\theta) \quad \text{for all } p \in P_\theta.$$

If, in addition, $\{\phi_h\}_{h>0}$ satisfies the condition (i), (ii), and (iii), then for each $h > 0$ there exists a linear combination ψ_h of ϕ_h and its $h\mathbb{Z}^n$ -translates such that $\psi_h *'_h$ is an identity on H , and the family $\{\psi_h\}_{h>0}$ satisfies the same conditions (i) and (ii) with possibly different constants r and c .

Proof. The proof of Theorem 3.2 in [15] can be carried over verbatim to prove the first statement of this lemma. The second statement was proved by de Boor and Ron in [6]. Here we sketch a proof, taking the Neumann series approach as introduced by Chui and Diamond in [7], and developed by Dyn and Ron in [12]. In the following we denote by 1 the identity mapping.

Conditions (i) and (ii) imply that for $\xi \in \mathbb{C}^n$,

$$\begin{aligned} h^{-n}|\hat{\phi}_h(\xi) - \hat{\phi}_h(0)| &= h^{-n} \left| \int_{B_{rh}} \phi_h(x)(e^{-i\xi \cdot x} - 1) dx \right| \\ &\leq h^{-n} c \int_{B_{rh}} |e^{-i\xi \cdot x} - 1| dx \leq h^{-n} c (2rh)^n \max_{\|x\| \leq rh} |e^{-i\xi \cdot x} - 1|; \end{aligned}$$

hence $h^{-n}|\hat{\phi}_h(\xi) - \hat{\phi}_h(0)| \rightarrow 0$ as $h \rightarrow 0$. Thus, if we set $b_{\theta,h} := h^{-n}\hat{\phi}_h(-i\theta)$ for $\theta \in T$ and $h > 0$, then by condition (iii), $|b_{\theta,h}| \geq b/2$ for all $\theta \in T$ and sufficiently small $h > 0$. It follows from (2.2) that for any $\theta \in T$, $p \in P_\theta$, and sufficiently small $h > 0$,

$$(1 - \phi_h *'_h / b_{\theta,h})(e_\theta p) = e_\theta q$$

for some polynomial $q \in P_\theta$ of degree less than $\deg p$. Let $d_\theta := \max\{\deg p : p \in P_\theta\}$ for each $\theta \in T$. We see that the operator $(1 - \phi_h *'_h / b_{\theta,h})^{d_\theta+1}$ annihilates $e_\theta P_\theta$. For each $h > 0$ let V_h be the polynomial in one variable given by

$$V_h(t) := \left\{ 1 - \prod_{\theta \in T} (1 - t/b_{\theta,h})^{d_\theta+1} \right\} / t, \quad t \in \mathbb{R},$$

and set

$$\psi_h := V_h(\phi_h *'_h) \phi_h.$$

Then $\psi_h *'_h$ is an identity on H , since $1 - \psi_h *'_h = \prod_{\theta \in T} (1 - \phi_h *'_h / b_{\theta,h})^{d_\theta+1}$ annihilates H . Furthermore, we see from the construction of ψ_h that the family $\{\psi_h\}_{h>0}$ satisfies conditions (i) and (ii) with possibly different constants r and c . $\square$

Now let $\{\phi_h\}_{h>0}$ be a collection of normal functions satisfying the conditions (i) and (ii). Suppose that H is a finite-dimensional D -invariant space of exponentials such that $H_\downarrow \supseteq \Pi_{<k}$ and $\phi_h *'_h$ is the identity on H for every $h > 0$. Let $S_h := \text{range}(\phi_h *'_h)$. Given $f \in W_p^k(\mathbb{R}^n)$ ($1 \leq p < \infty$), we want to construct an L_p -approximation scheme from $\{S_h\}_{h>0}$. In the case $p = \infty$, this was done by de Boor and Ron in [6]. For $f \in W_\infty^k$ and $h > 0$, let

$$(2.3) \quad s_h(x) := \sum_{\nu \in \mathbb{Z}^n} \phi_h(x - \nu h) f(\nu h), \quad x \in \mathbb{R}^n.$$

Then there exists a positive constant C_1 independent of f and h such that

$$|f(x) - s_h(x)| \leq C_1 \text{dist}_\infty(f, H)(B_{rh}(x)) \quad \text{for all } x \in \mathbb{R}^n,$$

where r is the constant appearing in the condition (i). Since $H_\downarrow \supseteq \Pi_{<k}$, for given $f \in W_\infty^k$ there exists a constant C_2 independent of h and f such that

$$\text{dist}_\infty(f, H)(B_{rh}(x)) \leq C_2 h^k \|f\|_{k,\infty}(B_{rh}(x))$$

(see [12, Thm. 3.1]). Thus with $C = C_1 C_2$ we have

(2.4) $|f(x) - s_h(x)| \leq Ch^k \|f\|_{k,\infty}(B_{rh}(x)) \quad \text{for } x \in \mathbb{R}^n.$

To deal with L_p -approximation ($1 \leq p < \infty$) we shall apply a smoothing technique as employed in [16]. Choose a function $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\text{supp } \chi \subseteq B_1(0)$, $\chi \geq 0$ and $\int \chi = 1$. Set

$$\chi_h := \chi(\cdot/h)/h^n, \quad h > 0.$$

For a given function $f \in W_p^k$ and $h > 0$, consider the following function:

(2.5)
$$f_h(x) := \int_{\mathbb{R}^n} (f - \nabla_u^k f)(x) \chi_h(u) \, du, \quad x \in \mathbb{R}^n,$$

where ∇_u denotes the difference operator given by

$$\nabla_u f := f - f(\cdot - u).$$

In the univariate case, such a smoothing technique was first introduced by DeVore [10].

The following lemma was proved in [16].

LEMMA 2.3. *The functions f_h are C^∞ -smooth. Moreover, there exists a constant C depending only on k such that for any measurable set $\Omega \subseteq \mathbb{R}^n$*

- (a) $\|f_h\|_p(\Omega) \leq C \|f\|_p(B_{kh}(\Omega));$
- (b) $\|f_h\|_\infty(\Omega) \leq Ch^{-n/p} \|f\|_p(B_{kh}(\Omega));$
- (c) $\|f - f_h\|_p(\Omega) \leq Ch^k |f|_{k,p}(B_{kh}(\Omega)).$

We are now in a position to construct an L_p -approximation scheme for $1 \leq p < \infty$.

THEOREM 2.4. *Let $\{\phi_h\}_{h>0}$ be a collection of normal functions satisfying the conditions (i) and (ii). Suppose that H is a finite-dimensional D -invariant space of exponentials such that $H_\downarrow \supseteq \Pi_{<k}$ and $\phi_h *'_h$ is the identity on H for every $h > 0$. For a given function $f \in W_p^k$ ($1 \leq p \leq \infty$) and any $h > 0$, set*

$$s_h(x) := \sum_{\nu \in \mathbb{Z}^n} \phi_h(x - h\nu) f_h(h\nu), \quad x \in \mathbb{R}^n,$$

where f_h are given by (2.5). Then for sufficiently small $h > 0$,

$$\|f - s_h\|_p \leq C \|f\|_{k,p} h^k,$$

where C is a constant independent of f , h , and p .

Proof. By Lemma 2.3(c),

$$\|f - f_h\|_p \leq Ch^k |f|_{k,p}.$$

Hence it remains to show that

$$\|f_h - s_h\|_p \leq Ch^k \|f\|_{k,p}.$$

When $p = \infty$, this follows from (2.4), if we replace f by f_h in (2.3). In what follows we assume that $1 \leq p < \infty$. Our argument is motivated by the work of Dahmen and Micchelli [9]. For $\alpha \in \mathbb{Z}^n$ and $h > 0$, let

$$G_{\alpha,h} := (\alpha + [0, 1]^n)h.$$

By (2.4), there exists a constant C_1 such that

$$|f_h(x) - s_h(x)| \leq C_1 \|f_h\|_{k,\infty}(B_{rh}(x)) \quad \text{for all } x \in \mathbb{R}^n.$$

Since the volume of each $G_{\alpha,h}$ is h^n , it follows that

$$(2.6) \quad \begin{aligned} \int_{G_{\alpha,h}} |f_h(x) - s_h(x)|^p dx &\leq h^n \sup_{x \in G_{\alpha,h}} |f_h(x) - s_h(x)|^p \\ &\leq C_1^p h^n \|f_h\|_{k,\infty}^p (G_{\alpha,h} + B_{rh}). \end{aligned}$$

By Lemma 2.3(b), there exists a constant C_2 such that

$$(2.7) \quad \|f_h\|_{k,\infty}(G_{\alpha,h} + B_{rh}) \leq C_2 h^{-n/p} \|f\|_{k,p}(G_{\alpha,h} + B_{(r+k)h}).$$

Now, (2.6) and (2.7) together imply

$$(2.8) \quad \begin{aligned} \|f_h - s_h\|_p^p &= \sum_{\alpha \in \mathbb{Z}^n} \int_{G_{\alpha,h}} |f_h(x) - s_h(x)|^p dx \\ &\leq (C_1 C_2)^p \sum_{\alpha \in \mathbb{Z}^n} \|f\|_{k,p}^p (G_{\alpha,h} + B_{(r+k)h}). \end{aligned}$$

Let $\beta \in \mathbb{N}^n$, $|\beta| \leq k$. Then

$$(2.9) \quad \sum_{\alpha \in \mathbb{Z}^n} \|D^\beta f\|_p^p (G_{\alpha,h} + B_{(r+k)h}) = \int_{\mathbb{R}^n} |D^\beta f(x)|^p \sum_{\alpha \in \mathbb{Z}^n} \rho_\alpha(x) dx,$$

where ρ_α is the characteristic function of the set $G_{\alpha,h} + B_{(r+k)h}$. For any fixed $x \in \mathbb{R}^n$, the number of $\alpha \in \mathbb{Z}^n$ such that $\rho_\alpha(x) \neq 0$ does not exceed $(2r + 2k + 2)^n$; hence

$$(2.10) \quad \int_{\mathbb{R}^n} |D^\beta f(x)|^p \sum_{\alpha \in \mathbb{Z}^n} \rho_\alpha(x) dx \leq (2r + 2k + 2)^n \|D^\beta f\|_p^p.$$

Since

$$\|f\|_{k,p}^p = \sum_{|\beta| \leq k} \|D^\beta f\|_p^p,$$

it follows from (2.8)–(2.10) that there is a constant C independent of f , h , and p such that

$$\|f_h - s_h\|_p^p \leq C^p \|f\|_{k,p}^p,$$

as desired. $\square$

3. Upper bounds for the approximation order. In this section we provide upper bounds for the L_p -approximation order ($1 \leq p \leq \infty$) of a family $\{S_h\}_{h>0}$ of approximating spaces, each of which is $h\mathbb{Z}^n$ -translation invariant and consists of piecewise exponentials.

THEOREM 3.1. *Let H be a finite-dimensional D -invariant space of exponentials, and let Ω be a nonempty open subset of the open unit cube $(0,1)^n \subset \mathbb{R}^n$. Assume that $\{S_h\}_{h>0}$ is a family of linear spaces of functions such that*

$$(3.1) \quad S_h|_{h\Omega+h\nu} \subseteq H|_{h\Omega+h\nu} \quad \text{for all } \nu \in \mathbb{Z}^n.$$

Then for $1 \leq p \leq \infty$ the L_p -approximation order of $\{S_h\}_{h>0}$ on any domain $G \subseteq \mathbb{R}^n$ does not exceed k , where k is the largest integer such that $\Pi_{<k} \subseteq H_\downarrow$.

Proof. Let $\{q_1, \dots, q_N\}$ be a basis of homogeneous polynomials for $H_\downarrow$. There exists a basis $\{f_1, \dots, f_N\}$ for H such that $q_j = f_{j\downarrow}$, $j = 1, \dots, N$ (see [5]). Moreover, since $\Pi_{<k} \subseteq H_\downarrow$, we can choose f_j ($j = 1, \dots, N$) so that

$$f_j(x) - q_j(x) = O(x^k) \quad \text{as } x \rightarrow 0.$$

Let

$$d_j := \max\{k-1, \deg q_j\}.$$

Since $f_1, \dots, f_N$ are exponentials, there exists a positive constant M such that for all $\|x\| \leq 1$,

$$(3.2) \quad |f_j(x) - q_j(x)| \leq M\|x\|^{d_j+1}, \quad j = 1, \dots, N.$$

Since $\Pi_k \not\subseteq H_\downarrow$, there exists $\beta \in \mathbb{N}^n$, $|\beta| = k$, such that $q := (\cdot)^\beta \notin H_\downarrow$. Let G be a bounded domain in $\mathbb{R}^n$. Our goal is to estimate $\text{dist}_p(q, S_h)(G)$ from below, i.e., to prove that for some positive constant C independent of h ,

$$(3.3) \quad \text{dist}_p(q, S_h)(G) \geq Ch^k.$$

Without loss of generality, we may assume that G has Lebesgue measure less than 1. Then by Hölder's inequality,

$$\text{dist}_p(q, S_h)(G) \geq \text{dist}_1(q, S_h)(G) \quad \text{for all } p \geq 1.$$

Hence it suffices to prove (3.3) for $p = 1$.

Note that $q, q_1, \dots, q_N$ are linearly independent, hence the function given by

$$(a_0, a_1, \dots, a_N) \mapsto \left\| a_0 q + \sum_{j=1}^N a_j q_j \right\|_1 (\Omega) \quad \text{for } (a_0, a_1, \dots, a_N) \in \mathbb{C}^{N+1}$$

induces a norm on $\mathbb{C}^{N+1}$. Since any two norms on $\mathbb{C}^{N+1}$ are equivalent, there exists a positive constant C_1 such that

$$(3.4) \quad \left\| a_0 q + \sum_{j=1}^N a_j q_j \right\|_1 (\Omega) \geq C_1 \sum_{j=0}^N |a_j| \quad \text{for all } a_0, a_1, \dots, a_N \in \mathbb{C}.$$

For any $w \in \mathbb{R}^n$, $q(\cdot + w) - q \in \Pi_{<k} \subseteq H_\downarrow$; hence it is a linear combination of $q_1, \dots, q_N$:

$$q(\cdot + w) - q = \sum_{j=1}^N b_j(w) q_j,$$

where b_j are continuous functions on $\mathbb{R}^n$ with $b_j(0) = 0$ ($j = 1, \dots, N$).

In what follows we assume that $0 < h < 1$. For any $a_1, \dots, a_N \in \mathbb{C}$, we have

$$(3.5) \quad \begin{aligned} \left\| q(\cdot + h\nu) - \sum_{j=1}^N a_j f_j \right\|_1 (h\Omega) &= \left\| q + \sum_{j=1}^N b_j(h\nu) q_j - \sum_{j=1}^N a_j f_j \right\|_1 (h\Omega) \\ &\geq \left\| q + \sum_{j=1}^N (b_j(h\nu) - a_j) q_j \right\|_1 (h\Omega) \\ &\quad - \left\| \sum_{j=1}^N a_j (f_j - q_j) \right\|_1 (h\Omega). \end{aligned}$$

To estimate the first term of the far right side of (3.5) we note that

$$\|g\|_1(h\Omega) = h^n \|g(h\cdot)\|_1(\Omega).$$

This together with (3.4) implies that

$$(3.6) \quad \begin{aligned} J_1 &:= \left\| q + \sum_{j=1}^N (b_j(h\nu) - a_j) q_j \right\|_1 (h\Omega) \\ &= h^n \left\| h^k q + \sum_{j=1}^N (b_j(h\nu) - a_j) h^{\deg q_j} q_j \right\|_1 (\Omega) \\ &\geq C_1 h^n \left(h^k + \sum_{j=1}^N |b_j(h\nu) - a_j| h^{d_j} \right). \end{aligned}$$

To estimate the second term of the far right side of (3.5) we note that for any $g \in L_1(h\Omega)$

$$(3.7) \quad \|g\|_1(h\Omega) \leq h^n \sup_{x \in h\Omega} |g(x)|,$$

since $h\Omega \subset h(0, 1)^n$. It follows from (3.2) and (3.7) that

$$(3.8) \quad J_2 := \left\| \sum_{j=1}^N a_j (f_j - q_j) \right\|_1(h\Omega) \leq \sum_{j=1}^N |a_j| \|f_j - q_j\|_1(h\Omega) \leq M h^n \sum_{j=1}^N |a_j| h^{d_j+1}.$$

For sufficiently small h , say $h \leq C_1/M$, we have

$$C_1 |b_j(h\nu) - a_j| \geq M h (|a_j| - |b_j(h\nu)|).$$

This together with (3.6) and (3.8) yields

$$J_1 - J_2 \geq h^n \left(C_1 h^k - M \sum_{j=1}^N |b_j(h\nu)| h^{d_j+1} \right).$$

Since $d_j + 1 \geq k$, while $b_j(h\nu)$ can be made small if $h\nu$ is small, we conclude that there exists $\delta > 0$ such that

$$\left\| q(\cdot + h\nu) - \sum_{j=1}^N a_j f_j \right\|_1(h\Omega) \geq J_1 - J_2 \geq (C_1/2) h^{k+n} \quad \text{for } \|\nu\| \leq \delta/h.$$

The above estimate holds for any $a_1, \dots, a_N \in \mathbb{C}$; hence

$$\text{dist}_1(q(\cdot + h\nu), H)(h\Omega) \geq (C_1/2) h^{k+n} \quad \text{for } \|\nu\| \leq \delta/h.$$

Since H is translation invariant, it follows that

$$(3.9) \quad \begin{aligned} \text{dist}_1(q, H)(h\Omega + h\nu) &= \text{dist}_1(q(\cdot + h\nu), H)(h\Omega) \\ &\geq (C_1/2) h^{k+n} \quad \text{for } \|\nu\| \leq \delta/h. \end{aligned}$$

We are now ready to estimate $\text{dist}_1(q, S_h)(B_\delta) =: \varepsilon(h)$. Observe that

$$B_\delta \supseteq \bigcup_{\|\nu\| \leq \delta/h-1} (h\Omega + h\nu).$$

It follows that

$$(3.10) \quad \varepsilon(h) \geq \sum_{\|\nu\| \leq \delta/h-1} \text{dist}_1(q, S_h)(h\Omega + h\nu).$$

But by (3.1), $S_h|_{h\Omega+h\nu} \subseteq H|_{h\Omega+h\nu}$; hence by (3.9) we have

$$(3.11) \quad \begin{aligned} \text{dist}_1(q, S_h)(h\Omega + h\nu) &\geq \text{dist}_1(q, H)(h\Omega + h\nu) \\ &\geq (C_1/2)h^{k+n} \quad \text{for } \|\nu\| \leq \delta/h. \end{aligned}$$

For sufficiently small $h > 0$, the set $\{\nu \in \mathbb{Z}^n : \|\nu\| \leq \delta/h - 1\}$ has cardinality $\geq (\delta/h)^n$; therefore (3.10) and (3.11) yield

$$\varepsilon(h) \geq (\delta/h)^n (C_1/2) h^{n+k} = (C_1/2) \delta^n h^k.$$

This shows that for sufficiently small $h > 0$,

$$\text{dist}_1(q, S_h)(B_\delta) \geq Ch^k,$$

where $C = (C_1/2)\delta^n$. This proves (3.3) for $p = 1$, as desired. $\square$

Finally, we apply the previous results to the problem of approximation by exponential box splines. Following Dyn and Ron [12], we introduce the h -scaled EB -splines as follows. Let Γ be a finite set of pairs (not necessarily distinct) of the form

$$\gamma = (x_\gamma, \lambda_\gamma), \quad x_\gamma \in \mathbb{R}^n \setminus \{0\}, \quad \lambda_\gamma \in \mathbb{C}.$$

Hereafter we always assume that $X := \{x_\gamma : \gamma \in \Gamma\} \subseteq \mathbb{Z}^n$ and that X spans $\mathbb{R}^n$. The h -scaled EB -spline $B_h(\Gamma | \cdot)$ is defined by the equation

$$\int_{\mathbb{R}^n} B_h(\Gamma | x) \psi(x) dx = h^{n-\#\Gamma} \int_{[0, h]^\Gamma} \left(\prod_{\gamma \in \Gamma} e^{\lambda_\gamma t_\gamma} \right) \psi \left(\sum_{\gamma \in \Gamma} t_\gamma x_\gamma \right) dt,$$

where ψ is taken from a suitable space of test functions. It is known from [12] that the collection $\{B_h(\Gamma | \cdot)\}_{h>0}$ satisfies conditions (i), (ii), and (iii).

Let $\mathcal{D}'(\mathbb{R}^n)$ denote the space of all n -dimensional complex-valued distributions. For $K \subseteq \Gamma$, let $p_K(D)$ be the differential operator induced by the polynomial

$$p_K(x) := \prod_{\gamma \in K} (x \cdot x_\gamma - \lambda_\gamma).$$

Let

$$K(\Gamma) := \{K \subseteq \Gamma : \text{span}\{x_\gamma\}_{\gamma \in \Gamma \setminus K} \neq \mathbb{R}^n\},$$

and

$$k(X) := \min\{\#K : K \in K(\Gamma)\}.$$

Define

$$H := \{f \in \mathcal{D}'(\mathbb{R}^n) : p_K(D)f = 0 \text{ for all } K \in K(\Gamma)\}.$$

Then H is a finite-dimensional D -invariant space of exponentials (see, e.g., [1]). Let S_h be the range of $B_h(\Gamma | \cdot) *'_h$. Then S_h is $h\mathbb{Z}^n$ -translation invariant, $S_h \supset H$, and $S_h|_{(0,h)^n}$ consists of piecewise H -functions. It is also known from [12] that with $k = k(X)$, $H_\downarrow \supseteq \Pi_{<k}$ but $H_\downarrow \not\supseteq \Pi_k$. Dyn and Ron [12] proved that k is a lower bound for the L_∞ -approximation order of $\{S_h\}_{h>0}$. Now that $\{S_h\}_{h>0}$ and H satisfy the conditions of Theorems 2.1 and 3.1, we have the following concluding result.

THEOREM 3.2. *Let S_h be the range of $B_h(\Gamma | \cdot) *'_h$. Then the exact L_p -approximation order of $\{S_h\}_{h>0}$ is $k(X)$ ($1 \leq p \leq \infty$).*

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