Solvability of some multivariate interpolation problems

Dedicated to Professor G.G. Lorentz with esteem and affection

By Rong-Qing Jia at Eugene and A. Sharma at Edmonton

Following G.G. Lorentz and R.A. Lorentz, we say that an interpolation problem is regular if it is uniquely solvable for all selections of distinct nodes and all data. While in the univariate case, Lagrange and Hermite interpolation are regular, in the multivariate case even Lagrange interpolation is not regular. Here we study some solvable interpolation problems in the multivariate case.

1. Introduction

Let $\mathbb{N}^s$ be the set of all $s$-tuples $x = (x_1, \ldots, x_s)$ with $x_j \in \mathbb{N}$, $1 \leq j \leq s$, where $\mathbb{N}$ denotes the set of all non-negative integers. By $x \leq \beta$ we mean $x_j \leq \beta_j$ for $1 \leq j \leq s$, and the factorial of $\alpha$ is defined by

$$\alpha! := \alpha_1! \cdots \alpha_s!.$$

Let $\mathbb{R}^s$ be the $s$-dimensional Euclidean space. We denote by $X^s = X_1^{x_1} \cdots X_s^{x_s}$ the function on $\mathbb{R}^s$ given by

$$x \mapsto x^s := x_1^{x_1} \cdots x_s^{x_s}, \quad x = (x_1, \ldots, x_s) \in \mathbb{R}^s.$$

Such a function is called a monomial. By a polynomial we mean a linear combination of monomials over the field $\mathbb{C}$ of complex numbers. The collection of all polynomials on $\mathbb{R}^s$ will be denoted by $\mathcal{P}$.

Let $D_j$ denote the partial derivative with respect to the $j$th coordinate, $1 \leq j \leq s$. We set

$$D^s := D_1^{x_1} \cdots D_s^{x_s}, \quad \alpha = (x_1, \ldots, x_s) \in \mathbb{N}^s.$$

For a polynomial $p = \sum_a a_s X^s$, we denote by $p(D)$ the corresponding constant coefficient.
differential operator: \( p(D) = \sum_{x} a_{x} D^{x} \). For \( \theta \in \mathbb{R}^{s} \), we denote by \([\theta]\) the point-evaluation functional at \( \theta \):

\[
[\theta] f := f(\theta).
\]

More generally, for \( \theta \in \mathbb{R}^{s} \) and \( p \in \Pi \), \([\theta] p(D)\) is defined to be the linear functional given by

\[
f \mapsto p(D) f(\theta).
\]

A linear subspace \( P \) of \( \Pi \) is called \( D \)-invariant if for any \( p \in P \), all its partial derivatives \( D_{j} p \), \( 1 \leq j \leq s \), are also in \( P \). A linear subspace \( P \) of \( \Pi \) is called scale-invariant, if for any \( p \in P \) and \( \lambda \in \mathbb{R} \), \( p(\lambda \cdot) \) is also in \( P \). It is easily seen that \( P \) is scale-invariant if and only if \( P \) has a basis of homogeneous polynomials.

In a recent interesting paper [2], de Boor and Ron considered multivariate polynomial interpolation in a very general setting. They formulated the interpolation problem as follows: Given a finite set \( \Theta \subset \mathbb{R}^{s} \) and corresponding finite dimensional spaces \( P_{\theta} \) of polynomials for each \( \theta \in \Theta \), interpolate from a given space of polynomials \( \mathcal{Q} \) with the interpolation conditions given by

\[
A := \sum_{\theta} \{[\theta] p(D) : p \in P_{\theta}\}.
\]

If all the spaces \( P_{\theta} \) are scale invariant, then the interpolation is called Hermite-Birkhoff interpolation. If, in addition, all the spaces \( P_{\theta} \) are \( D \)-invariant, then it is called Hermite interpolation.

In [5] and a series of papers thereafter, G.G. Lorentz and R.A. Lorentz investigated bivariate and multivariate interpolation. The present paper is strongly influenced by their work and we use their ideas and methods freely in the following. We shall now reformulate their problems in the terminology of de Boor and Ron in [2].

Given a subset \( A \) of \( \mathbb{N}^{s} \), we set

\[
\Pi_{A} := \text{span}\{X^{\alpha} : \alpha \in A\}.
\]

Let \( S \) be a finite subset of \( \mathbb{N}^{s} \) and let \( A_{1}, \ldots, A_{m} \) be subsets of \( S \). Let \( Q := \Pi_{S} \) and \( P_{j} := \Pi_{A_{j}} \) \( (j = 1, \ldots, m) \). Following G.G. Lorentz and R.A. Lorentz, we say that the interpolation scheme \((Q; P_{1}, \ldots, P_{m})\) is regular if for all possible choices of \( \theta_{1}, \ldots, \theta_{m} \) in \( \mathbb{R}^{s} \), the problem of interpolation from the space \( Q \) with the interpolation conditions

\[
A(\theta_{j}; P_{j}; j = 1, \ldots, m) := \sum_{j=1}^{m} \{[\theta_{j}] p(D) : p \in P_{j}\}
\]

is uniquely solvable. In their work, \( S \) is always assumed to be a lower set, that is, \( S \) satisfies the following conditions:

\[
\alpha \in \mathbb{N}^{s}, \beta \in S \text{ and } \alpha \leq \beta \Rightarrow \alpha \in S.
\]
For $s = 1$, there exist some simple necessary and some simple sufficient conditions for an interpolation scheme to be regular (see [6]). For $s = 2$, G. G. Lorentz and R. A. Lorentz proved the surprising result that the interpolation scheme is regular if and only if $S$ is a disjoint union of $A_1, \ldots, A_m$. This result was extended for $s > 2$ by G. G. Lorentz in [4].

2. A sufficient condition for regularity

In this section we consider regularity of an interpolation scheme $(Q; P_1, \ldots, P_m)$, where $Q$ and the $P_j$'s are finite-dimensional scale-invariant spaces of polynomials on $\mathbb{R}^s$ and $Q \supseteq \sum_{j=1}^m P_j$. In particular, $Q$ and the $P_j$'s are not necessarily spanned by monomials. Obviously,

$$\dim(Q) = \sum_{j=1}^m \dim(P_j)$$

is a necessary condition for regularity. The following theorem gives a sufficient condition for the interpolation scheme $(Q; P_1, \ldots, P_m)$ to be regular.

**Theorem 1.** If $Q = \bigoplus_{1 \leq j \leq m} P_j$, then the interpolation scheme $(Q; P_1, \ldots, P_m)$ is regular.

**Proof.** Let $\Theta = \{\theta_1, \ldots, \theta_m\} \subset \mathbb{R}^s$ be any choice of nodes of interpolation and let $A = A(\theta_j, P_j; j = 1, \ldots, m)$ be as given in (1). We shall prove that the interpolation problem $(Q, A)$ is uniquely solvable. For this purpose, since

$$\dim(A) = \sum_{j=1}^m \dim(P_j) = \dim(Q),$$

it is sufficient to show that for any $q \in Q$, $q \neq 0$, there exists some $\lambda \in A$ such that $\lambda(q) \neq 0$ (see e.g., [1]). For any $q \in Q$, we write

$$q = q_0 + q_1 + \ldots + q_n, \quad q_n \neq 0$$

in terms of its homogeneous components $q_j$ of degree $j$, $j = 0, \ldots, n$. Since $Q$ and the $P_j$'s are scale invariant, $q_n$ also belongs to $Q$, so that $q_n = \sum_{j=1}^m p_j$, where $p_j$ is a homogeneous polynomial in $P_j$ of degree $n$ for each $j$. Set

$$\lambda := \sum_{j=1}^m [\theta_j] p_j(D).$$

Then for $k < n$, $\deg(q_k) < n$, so that $[\theta_j] p_j(D) q_k = 0, j = 1, \ldots, m$. Moreover $q_n$ and $p_j$'s
are homogeneous polynomials of degree \( n \), hence \( p_j(D) q_n \) are constants, \( j = 1, \ldots, m \). Thus

\[
\lambda(q) = \left( \sum_{j=1}^{m} [\theta_j] p_j(D) \right) \left( \sum_{v=0}^{n} q_v \right) = \left( \sum_{j=1}^{m} [\theta_j] p_j(D) \right) q_n
\]

\[
= \sum_{j=1}^{m} p_j(D) q_n = q_n(D) q_n + 0,
\]
as desired. \( \square \)

Let us consider a special case of Theorem 1, in which each \( P_j \) is the space of homogeneous polynomials of degree \( j - 1 \), \( j = 1, \ldots, m \). Then \( Q = \bigoplus_{1 \leq j \leq m} P_j \) is the space of all polynomials of (total) degree \( < m \). When \( s = 1 \), such an interpolation scheme \( (Q; P_1, \ldots, P_m) \) is known as Abel-Goncharov interpolation. This was extended by Cavaretta, Micchelli and Sharma [3] to the case \( s > 1 \). Thus Theorem 1 is a further extension of their result. Motivated by this, we call an interpolation scheme \( (Q; P_1, \ldots, P_m) \) Abel-Goncharov interpolation if \( Q \) and the \( P_j \)'s are scale-invariant spaces of polynomials and \( Q = \bigoplus_{1 \leq j \leq m} P_j \). Now Theorem 1 can be restated as follows: Any Abel-Goncharov interpolation scheme is regular. The following examples illustrate Theorem 1.

**Example 1.** Consider the interpolation scheme \( (Q; P_1, P_2, P_3) \) in \( \mathbb{R}^2 \), where

\[
P_1 \coloneqq \text{span} \{1\}, \quad P_2 \coloneqq \text{span} \{X_1^2, X_1, X_2\}, \quad P_3 \coloneqq \text{span} \{X_1^2 X_2, X_1 X_2^2, X_2^3\}
\]

and \( Q = \Pi_S \) with

\[
S = \{(0,0), (2,0), (1,1), (2,1), (1,2), (0,3)\}.
\]

Then \( S \) is not a lower set. But \( Q = P_1 \oplus P_2 \oplus P_3 \), hence by Theorem 1 the interpolation scheme \( (Q; P_1, P_2, P_3) \) is regular. Indeed, for any given set \( \Theta = \{\theta_1, \theta_2, \theta_3\} \) and the corresponding interpolation conditions

\[
\lambda_1 = [\theta_1], \quad \lambda_2 = [\theta_2] D_1^2, \quad \lambda_3 = [\theta_3] D_1 D_2, \\
\lambda_4 = [\theta_3] D_1^2 D_2, \quad \lambda_5 = [\theta_3] D_1 D_2^2, \quad \lambda_6 = [\theta_3] D_2^3,
\]

we can easily find the fundamental polynomials \( f_k \in Q \) \( (k = 1, \ldots, 6) \) such that \( \lambda_j(f_k) = \delta_{jk} \) \( (j, k = 1, \ldots, 6) \).

**Example 2.** Let

\[
P_1 \coloneqq \text{span} \{X_1 + X_2, X_1^2 + X_2^2\}, \quad P_2 \coloneqq \text{span} \{X_1 X_2, X_2^3 - X_1^3\} \quad \text{and} \quad Q = P_1 \oplus P_2.
\]

Then the interpolation scheme \( (Q; P_1, P_2) \) in \( \mathbb{R}^2 \) is regular by Theorem 1, although neither \( P_1 \) nor \( P_2 \) is spanned by monomials.

The converse of Theorem 1 is not true for \( s = 1 \) (see [6]). However, for the case \( s \geq 2 \), G.G. Lorentz and R.A. Lorentz in [5] and [4] have given a nice result showing that the
The converse of Theorem 1 is true if the $P_j$’s are spanned by monomials and $Q = \Pi_s$ with $S$ being a lower set. This motivates us to make the following conjecture:

**Conjecture.** Let $Q$ and $P_1, \ldots, P_m$ be scale-invariant spaces of polynomials on $\mathbb{R}^d$ such that $Q \supseteq \sum_{j=1}^m P_j$. Then for $s \geq 2$, the interpolation scheme $(Q; P_1, \ldots, P_m)$ is regular if and only if $Q = \bigoplus_{1 \leq j \leq m} P_j$.

**Example 3.** Consider the interpolation scheme $(Q; P_1, P_2, P_3)$ in $\mathbb{R}^2$, where

$$P_1 := \text{span} \{1, X_1\}, \quad P_2 := \text{span} \{X_2, X_2^2\}, \quad P_3 := \text{span} \{X_1 + X_2, X_1^3\}$$

and

$$Q := \text{span} \{1, X_1, X_2, X_2^2, X_3, X_1^3\}.$$ 

Then $Q \supseteq P_1 + P_2 + P_3$, but $(P_1 \oplus P_2) \cap P_3 \neq \{0\}$. We shall show that this interpolation scheme is not regular. To see this, let $\theta_1 = (0, 0), \theta_2 = (1, 0)$ and $\theta_3 = (0, 1)$ and let $f := X_1^3$. Then $\lambda(f) = 0$ for all $\lambda \in \lambda^2(\theta_j, P_j; j = 1, 2, 3)$. This example supports the conjecture.

We shall confirm the conjecture when $Q$ and the $P_j$’s are spanned by monomials. Our proof requires a lemma.

3. A Lemma

Let $S$ be a finite subset of $\mathbb{N}^2$. A subset $L$ of $S$ is called a lower subset relative to $S$, or simply a lower subset of $S$ if

$$\alpha \in L, \beta \in S \text{ and } \beta \leq \alpha \quad \Rightarrow \quad \beta \in L.$$ 

A subset $U$ is called an upper subset of $S$ if

$$\alpha \in U, \beta \in S \text{ and } \beta \geq \alpha \quad \Rightarrow \quad \beta \in U.$$ 

Thus a lower set as defined in Section 1 is a lower subset of $\mathbb{N}^2$. Note that $S$ is not necessarily a lower set, but $S$ is always a lower subset of itself.

For a fixed $i \in \mathbb{N}$, set

$$S(i, \cdot) := \{(i, j) : (i, j) \in S\}.$$ 

Similarly, for a fixed $j \in \mathbb{N}$, set

$$S(\cdot, j) := \{(i, j) : (i, j) \in S\}.$$ 

Thus $S(i, \cdot)$ is the section of $S$ by the line $x = i$, and $S(\cdot, j)$ is the section of $S$ by the line $y = j$. 
Lemma 1. Let $\mu$ and $v$ be two additive measures on $S$ (a finite subset of $\mathbb{N}^2$). Then $\mu$ agrees with $v$ on $S$ if the following three conditions are satisfied:

(i) $\mu(S(i, \cdot)) = v(S(i, \cdot))$ for any fixed $i \in \mathbb{N};$

(ii) $\mu(S(\cdot, j)) = v(S(\cdot, j))$ for any fixed $j \in \mathbb{N};$

(iii) $\mu(L) \leq v(L)$ for any lower subset $L$ of $S.$

Proof. First we note that (i) or (ii) implies that $\mu(S) = v(S).$ Moreover, $\mu(S) = v(S)$ and (iii) imply that

(iv) $\mu(U) \geq v(U)$ for any upper subset $U$ of $S,$

because $S \setminus U$ is a lower subset of $S.$

The proof proceeds by induction on $\# S$, the number of elements of $S.$ The case $\# S = 1$ is trivial. Let $S$ be given and let $\mu$ and $v$ be two additive measures on $S$ which satisfy (i), (ii) and (iii). Suppose inductively that the lemma is true for any $S'$ with $\# S' < \# S.$

Let $(i, j)$ be an extreme point of $S,$ i.e., for any $(i', j') \in \mathbb{N}^2$ with $i' > i$ or $j' > j,$ $(i', j') \notin S.$ Then

$$\{(i, j)\} = S \setminus (S(<i, \cdot) \cup S(\cdot, <j)),$$

where

$$S(<i, \cdot) := \bigcup_{k < i} S(k, \cdot) \quad \text{and} \quad S(\cdot, <j) := \bigcup_{k < j} S(\cdot, k).$$

It follows that

$$\mu(\{(i, j)\}) = \mu(S) - \mu(S(<i, \cdot)) - \mu(S(\cdot, <j)) + \mu(L),$$

and that

$$v(\{(i, j)\}) = v(S) - v(S(<i, \cdot)) - v(S(\cdot, <j)) + v(L),$$

with the intersection $L := S(<i, \cdot) \cap S(\cdot, <j)$ being a lower subset of $S.$ By the hypotheses (i), (ii) and (iii), it then follows that

$$\mu(\{(i, j)\}) \leq v(\{(i, j)\}).$$

On the other hand, $\{(i, j)\}$ is an upper subset of $S,$ so that by (iv),

$$\mu(\{(i, j)\}) \geq v(\{(i, j)\}).$$

This shows that

$$\mu(\{(i, j)\}) = v(\{(i, j)\}).$$

Let $S' := S \setminus \{(i, j)\},$ and let $\mu'$ and $v'$ be the restrictions of $\mu$ and $v$ on $S',$ respectively. Then $\mu'$ and $v'$ and $S'$ also satisfy the conditions (i), (ii) and (iii), hence $\mu'$ and $v'$ agree on $S'$ by the induction hypothesis. Therefore $\mu$ and $v$ agree on $S.$ \qed
4. Proof of the Conjecture in a special case

**Theorem 2.** Let \( Q = \Pi_S \) and \( P_j = \Pi_{A_j} \) \((j = 1, \ldots, m)\), where \( S \) is an arbitrary finite subset of \( \mathbb{N}^s \), \( s > 1 \), and \( A_j \) are subsets of \( S \). Then the interpolation scheme \((Q; P_1, \ldots, P_m)\) is regular if and only if \( S \) is the disjoint union of \( A_1, \ldots, A_m \), or equivalently, if and only if

\[
Q = \bigoplus_{1 \leq j \leq m} P_j.
\]

**Proof.** In view of Theorem 1, we only need to prove the "only if" part. Suppose that the interpolation scheme \((Q; P_1, \ldots, P_m)\) is regular. We shall show that \( S \) is a disjoint union of \( A_1, \ldots, A_m \).

Following G. G. Lorentz [4], for a subset \( A \) of \( S \), we set

\[
\mu(A) := \# A \quad \text{and} \quad v(A) := \sum_{j=1}^{m} \mu(A \cap A_j).
\]

Then \( \mu \) and \( v \) are additive measures on \( S \). To show that \( S \) is a disjoint union of \( A_j \)'s, it is enough to show that \( \mu = v \). Indeed, if \( \mu = v \), then for any \( \alpha \in S \),

\[
1 = \mu(\{\alpha\}) = \sum_{j=1}^{m} \mu(\{\alpha\} \cap A_j),
\]

hence any \( \alpha \in S \) lies in exactly one \( A_j \).

For a subset \( \Gamma \) of \( \{1, \ldots, s\} \) and an element \( \gamma \in S \), define

\[
S(\gamma, \Gamma) := \{\alpha \in S : \alpha_i = \gamma_i \text{ for } i \in \Gamma\}.
\]

In particular, if \( \Gamma \) is the empty set, then \( S(\gamma, \Gamma) = S \); if \( \Gamma = \{1, \ldots, s\} \), then \( S(\gamma, \Gamma) = \{\gamma\} \).

We claim that for any proper subset \( \Gamma \) of \( \{1, \ldots, s\} \), an element \( \gamma = (\gamma_1, \ldots, \gamma_s) \in S \) and a lower subset \( T \) of \( S(\gamma, \Gamma) \), we have

\[
(2) \quad \mu(T) \leq v(T).
\]

To prove (2), we may assume without loss of generality that \( \Gamma = \{1, \ldots, r\} \), \( 0 \leq r < s \). (If \( r = 0 \), then \( \Gamma = \emptyset \) and \( S(\gamma, \Gamma) = S \).) Choose a subset \( \Theta \) of \( \Theta^s \) such that \( \# \Theta = m \) and

\[
\theta_1 = \ldots = \theta_r = 0, \quad \text{for all} \quad \theta = (\theta_1, \ldots, \theta_s) \in \Theta.
\]

There is a one-to-one correspondence between \( \Theta \) and the set \( \{P_1, \ldots, P_m\} \). If \( P_j \) corresponds to \( \theta \), then we write \( P_\theta \) for \( P_j \). Set

\[
A_1 := \sum_{\theta \in \Theta} \{[\theta] p(D) : p \in P_\theta \cap \Pi_T\}, \quad A_2 := \sum_{\theta \in \Theta} \{[\theta] p(D) : p \in P_\theta \cap \Pi_{S(\gamma, \Gamma)}\},
\]

and \( A := A_1 + A_2 \).
Since the interpolation problem \((Q, A)\) is uniquely solvable, we have
\[
\mu(S) = \dim(Q) = \dim(A) = v(S).
\]
Suppose the contrary of (2), i.e., \(\mu(T) > v(T)\). Then
\[
\dim(A_i) \leq \sum_{j=1}^{m} \mu(A_j \cap T) = v(T) < \mu(T).
\]
Hence there exists \(q \in \Pi_T \setminus \{0\}\) such that
\[
(\lambda_1(q) = 0 \text{ for all } \lambda_1 \in A_1).
\]
We claim that for this \(q\) we also have
\[
\lambda_2(q) = 0 \text{ for all } \lambda_2 \in A_2.
\]
To prove (4), it suffices to show that for \(\theta \in \Theta\),
\[
[\theta] D^\beta(\chi^x) = 0 \text{ for any } \alpha \in T \text{ and } \beta \in S \setminus T.
\]
Note that for \(\theta = (\theta_1, \ldots, \theta_s)\),
\[
[\theta] D^\beta(\chi^x) = \prod_{k=1}^{s} \left[ \frac{\alpha_k!}{(\alpha_k - \beta_k)!} \theta_k^{\alpha_k - \beta_k} \right],
\]
where we follow the convention that \(\theta_k^{\alpha_k - \beta_k} = 0\) if \(\alpha_k < \beta_k\).

There are two possibilities to be considered: Either \((\beta_1, \ldots, \beta_r) = (\gamma_1, \ldots, \gamma_r)\) or \((\beta_1, \ldots, \beta_r) \neq (\gamma_1, \ldots, \gamma_r)\). In the former case, since \(\alpha \in T \subseteq S(\gamma, \Gamma)\), we have
\[
(\alpha_1, \ldots, \alpha_r) = (\gamma_1, \ldots, \gamma_r) = (\beta_1, \ldots, \beta_r),
\]
hence there exists \(i \in \{r+1, \ldots, s\}\) such that \(\beta_i > \alpha_i\), for otherwise \(\beta \in T\), contradicting the fact that \(\beta \in S \setminus T\). With \(\beta_i > \alpha_i\), (5) gives
\[
[\theta] D^\beta(\chi^x) = 0.
\]
In the latter case, \((\beta_1, \ldots, \beta_r) \neq (\gamma_1, \ldots, \gamma_r)\). But by the choice of \(\Theta\), \(\theta_1 = \ldots = \theta_r = 0\); hence (6) also holds.

This proves the assertion (4). Now (3) and (4) together imply that
\[
\lambda(q) = 0 \text{ for all } \lambda \in A.
\]
This contradicts the hypothesis that the interpolation scheme \((Q; P_1, \ldots, P_m)\) is regular. Therefore (2) is true. In particular, for any proper subset \(\Gamma\) of \(\{1, \ldots, s\}\),
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\[(7) \quad \mu(S(\gamma, \Gamma)) \leq v(S(\gamma, \Gamma)).\]

Let now \(\Gamma\) be a fixed proper subset of \(\{1, \ldots, s\}\) and let \(\gamma\) be a fixed element of \(S\). We can set \(\gamma_1 = \gamma\) and find elements \(\gamma_2, \ldots, \gamma_n \in S\) such that \(S\) is the disjoint union of

\(S(\gamma_1, \Gamma), \ldots, S(\gamma_n, \Gamma).\)

Hence

\[(8) \quad \mu(S) = \sum_{k=1}^{n} \mu(S(\gamma_k, \Gamma)) \leq \sum_{k=1}^{n} v(S(\gamma_k, \Gamma)) = v(S) = \mu(S).\]

It follows from (7) and (8) that

\[\mu(S(\gamma, \Gamma)) = v(S(\gamma, \Gamma)).\]

If \(s = 2\), then by what has been proved above, \(\mu\) and \(v\) satisfy all the conditions of Lemma 1, hence \(\mu\) and \(v\) agree on \(S\).

When \(s > 2\), let \(\Gamma\) be a subset of \(\{1, \ldots, s\}\) with \(\# \Gamma = s - 2\). Pick any \(\gamma \in S\). Then \(S(\gamma, \Gamma)\) and the restrictions of \(\mu\) and \(v\) on \(S(\gamma, \Gamma)\) satisfy all the conditions of Lemma 1. Hence \(\mu\) and \(v\) agree on \(S(\gamma, \Gamma)\), so that \(\mu(\{\gamma\}) = v(\{\gamma\})\) for every \(\gamma \in S\). This completes the proof of Theorem 2. \(\Box\)

Remark. There seems to be an oversight in the proof given in [4]. Lemma 3 in that paper is only valid for \(\Gamma \subseteq \{1, \ldots, s\}\) with \(\# \Gamma < s\). However, to derive (13) from formula (12) in that paper, the author implicitly used (11) for \(\Gamma = (1, \ldots, s)\), which had not been proved.

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Department of Mathematics, University of Oregon, Eugene, OR 97403, U.S.A.
Department of Mathematics, University of Alberta, Edmonton, Canada T6G 2G1

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