KALMAN FILTERS FOR NON-UNIFORMLY SAMPLED MULTIRATE SYSTEMS

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Abstract: This paper proposes Kalman filter algorithms, including one-step prediction and filtering, for non-uniformly sampled multirate systems. The stability and convergence of the algorithms are analyzed, and their application to fault detection as well as state estimation in the framework of irregularly sampled data is investigated. Numerical examples are provided to demonstrate the applicability of the newly proposed algorithms.

Keywords: Kalman filters, non-uniformly sampled multirate systems, one-step prediction, filtering, fault detection

1. INTRODUCTION

Originally developed (Kalman, 1960) in the 1960s, Kalman filters have demonstrated their significant power in state estimation, system identification, adaptive control, signal processing (Haykin, 1996) and found many industrial applications (Sorenson, 1970). In a chemical engineering process, Kalman filters are frequently used to estimate unmeasured variables based on available measurements of other process variables. There have been numerous variants of the discrete-time (DT) Kalman filtering algorithms (Sorenson, 1985). However, most of them are for single rate systems.

In many industrial processes, variables are sampled at more than one rate, i.e. *multiple rates*. Take a polymer reactor as an example, where the manipulated variables can be adjusted at relatively fast rates (Gudi *et al.*, 1994), while the measurements of quality variables, e.g. the composition, and density are typically obtained after several minutes of analysis. Furthermore, the sampling is termed as *non-uniform*, if the sampling intervals for each variable are *non-equally* spaced, as is typically the case when manual samples are taken for laboratory analysis.

This paper attempts to develop Kalman filters for non-uniformly sampled multirate (NUSM) systems. The development is conducted in a generic framework: each variable in a physical system is non-uniformly sampled with multiple rates. This represents a very general starting point. All other multirate sampling scenarios are sub-sets of this case.

This paper considers a physical system with multi-inputs and multi-outputs represented by a continuous-time (CT) state space model. Moreover, the non-uniformly sampling technique proposed by Sheng *et al.*, (2002) is utilized to *lift* such a system. *Lifting* discretizes a CT system with variables sampled at different rates, and converts the resulting time-varying multirate system into a time-invariant single rate system.

2. PROBLEM FORMULATION

Consider the following CT state space system:

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$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\tilde{\mathbf{u}}(t) + \boldsymbol{\phi}(t)$$
$$\tilde{\mathbf{y}}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\tilde{\mathbf{u}}(t) \quad (1)$$

where (i) $\tilde{\mathbf{u}}(t) \in \Re^l$ and $\tilde{\mathbf{y}}(t) \in \Re^m$ are *noise-free* inputs and outputs, respectively; (ii) $\mathbf{x}(t) \in \Re^n$ is the state; (iii) $\boldsymbol{\phi}(t)$ is a Gaussian distributed white noise vector with covariance $\mathbf{R}_\phi$, i.e. $\boldsymbol{\phi}(t) \sim \aleph(\mathbf{0}, \mathbf{R}_\phi)$; and (iv) $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are known system matrices.

2.1 The lifted model for a NUSM system

For $t \in [kT, kT+T)$, where T is a *frame period*, we collect data non-uniformly from Eqn. 1 as follows: (Sheng *et al.*, 2002)

- The inputs $\tilde{\mathbf{u}}(t)$ are sampled g times at time instants: $\{kT+t_1, kT+t_2, kT+t_3, \cdots, kT+t_g\}$, where $0 = t_1 < t_2 < \cdots < t_g < T$.
- The outputs $\tilde{\mathbf{y}}(t)$ are sampled p times. Moreover, within the time interval $[kT+t_i, kT+t_{i+1})$, for $i = [1, g]$, n_i (≥ 0) output samples are taken at instants: $\{kT+t_i^1, kT+t_i^2, \cdots, kT+t_i^{n_i}\}$ with $t_i \leq t_i^1 < t_i^2 \cdots < t_i^{n_i} < t_{i+1}$ and $t_{g+1} = T$. Note that $p = \sum_{i=1}^{g} n_i$.

For simplicity we assume that (i) the l inputs and the disturbances are sampled synchronously; and (ii) the m outputs may be sampled asynchronously relative to the inputs.

We construct the lifted vectors for inputs and outputs, respectively,

$$\underline{\tilde{\mathbf{u}}}(k) = \begin{bmatrix} \tilde{\mathbf{u}}(kT+t_1) \\ \vdots \\ \tilde{\mathbf{u}}(kT+t_g) \end{bmatrix}, \underline{\tilde{\mathbf{y}}}(k) \equiv \begin{bmatrix} \tilde{\mathbf{y}}(kT+t_1^1) \\ \vdots \\ \tilde{\mathbf{y}}(kT+t_1^{n_1}) \\ \vdots \\ \tilde{\mathbf{y}}(kT+t_g^1) \\ \vdots \\ \tilde{\mathbf{y}}(kT+t_g^{n_g}) \end{bmatrix}.$$

In addition, the lifted vector for the disturbance, $\underline{\boldsymbol{\phi}}(k)$, is structurally identical to $\underline{\tilde{\mathbf{u}}}(k)$.

The lifted model of Eqn. 1 can be derived as follows (Sheng *et al.*, 2002):

$$\mathbf{x}(k+1) = \underline{\mathbf{A}}\,\mathbf{x}(k) + \underline{\mathbf{B}}\,\underline{\tilde{\mathbf{u}}}(k) + \underline{\mathbf{W}}\,\underline{\boldsymbol{\phi}}(k)$$
$$\underline{\tilde{\mathbf{y}}}(k) = \underline{\mathbf{C}}\,\mathbf{x}(k) + \underline{\mathbf{D}}\,\underline{\tilde{\mathbf{u}}}(k) + \underline{\mathbf{J}}\,\underline{\boldsymbol{\phi}}(k) \quad (2)$$

where $\underline{\mathbf{A}}$, $\underline{\mathbf{B}}$, $\underline{\mathbf{C}}$, $\underline{\mathbf{D}}$, $\underline{\mathbf{J}}$, and $\underline{\mathbf{W}}$ are functions of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, t_i, and $t_i^{n_i}$, $\forall\, i = 1, \cdots, g$. Eqn. 2 preserves the causality, controllability and observability of Eqn. 1, if the frame period T is non-pathological relative to matrix $\mathbf{A}$ (Sheng *et al.*, 2002).

2.2 Statement of Kalman filtering problems

At the time instant $kT+t_i^j$, for $j = [1, n_i]$, the sampled outputs are

$$\mathbf{y}(kT+t_i^j) = \tilde{\mathbf{y}}(kT+t_i^j) + \mathbf{o}(kT+t_i^j) \quad (3)$$

where $\mathbf{o}(\ ) \sim \aleph(\mathbf{0}, \mathbf{R}_o)$ is the measurement error, and independent of the initial state, $\mathbf{x}(0)$. However, at instant $kT+t_i$ for $i = [1, g]$, $\mathbf{u}(kT+t_i) = \tilde{\mathbf{u}}(kT+t_i)$, because the inputs to a plant in a closed loop system are the controller outputs, which can be known exactly.

It follows from Eqn. 3 that $\underline{\mathbf{y}}(k) = \underline{\tilde{\mathbf{y}}}(k) + \underline{\mathbf{o}}(k)$, where $\underline{\mathbf{y}}(k)$ and $\underline{\mathbf{o}}(k)$ have the identical structure to $\underline{\tilde{\mathbf{y}}}(k)$. Consequently, Eqn. 2 can be rewritten as

$$\mathbf{x}(k+1) = \underline{\mathbf{A}}\,\mathbf{x}(k) + \underline{\mathbf{B}}\,\underline{\mathbf{u}}(k) + \underline{\mathbf{W}}\,\underline{\boldsymbol{\phi}}(k) \quad (4)$$
$$\underline{\mathbf{y}}(k) = \underline{\mathbf{C}}\,\mathbf{x}(k) + \underline{\mathbf{D}}\,\underline{\mathbf{u}}(k) + \underline{\mathbf{J}}\,\underline{\boldsymbol{\phi}}(k) + \underline{\mathbf{o}}(k)$$

where, $\underline{\mathbf{o}}(k) \sim \aleph(\mathbf{0}, \underline{\mathbf{R}}_0)$, and $\underline{\mathbf{R}}_0 = \mathbf{I}_p \otimes \mathbf{R}_0$ with $\mathbf{I}_p$ being a $p \times p$ identity matrix and $\otimes$ standing for the Kronecker tensor product.

Given $\{\underline{\mathbf{A}}, \underline{\mathbf{B}}, \underline{\mathbf{C}}, \underline{\mathbf{D}}, \underline{\mathbf{J}}, \underline{\mathbf{W}}, \underline{\mathbf{R}}_\phi, \underline{\mathbf{R}}_o\}$ and $\{\underline{\mathbf{u}}(i), \underline{\mathbf{y}}(i)\}$, for $i \in [1, k]$, the Kalman filters for NUSM systems should estimate $\mathbf{x}(k)$ such that the covariance of $\mathbf{x}(k) - \hat{\mathbf{x}}(k|i)$ is minimized for $i = k-1, k$, where $\hat{\mathbf{x}}(k|i)$ is the estimate of $\mathbf{x}(k)$ from data $\{\underline{\mathbf{u}}(1), \underline{\mathbf{y}}(1), \cdots, \underline{\mathbf{u}}(i), \underline{\mathbf{y}}(i)\}$. The algorithms serve as *one-step predictor* if $i = k-1$ or *filters* if $i = k$.

3. KALMAN FILTERS FOR NUSM SYSTEMS

3.1 Algorithms for one-step prediction

Let the Kalman filters for one-step prediction have the following form (Åstrom, 1970; Åstrom and Wittenmark, 1997):

$$\hat{\mathbf{x}}(k+1|k) = \underline{\mathbf{A}}\,\hat{\mathbf{x}}(k|k-1) + \underline{\mathbf{B}}\,\underline{\mathbf{u}}(k) + \quad (5)$$
$$\underline{\mathbf{L}}(k)\left[\underline{\mathbf{y}}(k) - \underline{\mathbf{C}}\,\hat{\mathbf{x}}(k|k-1) - \underline{\mathbf{D}}\,\underline{\mathbf{u}}(k)\right]$$

where, $\underline{\mathbf{L}}(k)$ is the Kalman gain. Define $\bar{\mathbf{x}}(i|i-1) \equiv \mathbf{x}(i) - \hat{\mathbf{x}}(i|i-1)$ as the estimation error vector, for $i = k$ or $k+1$. Then, we can derive from Eqns. 4 and 5 that

$$\bar{\mathbf{x}}(k+1|k) = [\underline{\mathbf{A}} - \underline{\mathbf{L}}(k)\underline{\mathbf{C}}]\,\bar{\mathbf{x}}(k|k-1) + \quad (6)$$
$$[\underline{\mathbf{W}} - \underline{\mathbf{L}}(k)\underline{\mathbf{J}}]\,\underline{\boldsymbol{\phi}}(k) - \underline{\mathbf{L}}(k)\underline{\mathbf{o}}(k)$$

By choosing $\hat{\mathbf{x}}(0|-1) = \mathbf{E}\mathbf{x}(0)$, we arrive at $\mathbf{E}\left[\bar{\mathbf{x}}(k|k-1)\right] = \mathbf{0}, \forall\, k$, where $\mathbf{E}(\ )$ is the expectation of the argument.

To develop the Kalman filters, the covariance, $\underline{\mathbf{M}}(k) = \mathbf{E}\left[\bar{\mathbf{x}}(k|k-1)\bar{\mathbf{x}}'(k|k-1)\right]$, of $\bar{\mathbf{x}}(k|k-1)$ must be minimized, where $(\ )'$ stands for the transpose. Eqn. 6 gives

$$\underline{\mathbf{M}}(k+1) = \mathbf{E}[\bar{\mathbf{x}}(k+1|k)\bar{\mathbf{x}}'(k+1|k)] \quad (7)$$
$$= [\underline{\mathbf{A}} - \underline{\mathbf{L}}(k)\underline{\mathbf{C}}]\,\underline{\mathbf{M}}(k)\,[\underline{\mathbf{A}} - \underline{\mathbf{L}}(k)\underline{\mathbf{C}}]'$$
$$+ [\underline{\mathbf{W}} - \underline{\mathbf{L}}(k)\underline{\mathbf{J}}]\,\underline{\mathbf{R}}_\phi\,[\underline{\mathbf{W}} - \underline{\mathbf{L}}(k)\underline{\mathbf{J}}]'$$
$$+ \underline{\mathbf{L}}(k)\underline{\mathbf{R}}_o\underline{\mathbf{L}}'(k)$$

where the independency among $\bar{\mathbf{x}}(k|k-1)$, $\underline{\phi}(k)$, and $\underline{\mathbf{o}}(k)$ has been considered. It can be proved that $\underline{\mathbf{M}}(k+1)$ is at least positive semidefinite because $\underline{\mathbf{M}}(k)$, $\underline{\mathbf{R}}_o$, and $\underline{\mathbf{R}}_\phi$ are covariance matrices (Haykin, 1996).

Eqn. 7 is the algebraic Ricatti difference equation (ARDE), and can be further manipulated into

$$\begin{aligned}\underline{\mathbf{M}}(k+1) = {} & \underline{\mathbf{A}}\,\underline{\mathbf{M}}(k)\underline{\mathbf{A}}' + \underline{\mathbf{W}}\,\underline{\mathbf{R}}_\phi\underline{\mathbf{W}}' - \\ & \left(\underline{\mathbf{A}}\,\underline{\mathbf{M}}(k)\underline{\mathbf{C}}' + \underline{\mathbf{W}}\,\underline{\mathbf{R}}_\phi\,\underline{\mathbf{J}}'\right)\underline{\mathbf{H}}^{-1}(k)\bullet \\ & \left(\underline{\mathbf{A}}\,\underline{\mathbf{M}}(k)\underline{\mathbf{C}}' + \underline{\mathbf{W}}\,\underline{\mathbf{R}}_\phi\,\underline{\mathbf{J}}'\right)' + \\ & [\underline{\mathbf{L}}(k) - \left(\underline{\mathbf{A}}\,\underline{\mathbf{M}}(k)\underline{\mathbf{C}}' + \underline{\mathbf{W}}\,\underline{\mathbf{R}}_\phi\,\underline{\mathbf{J}}'\right)\bullet \\ & \underline{\mathbf{H}}^{-1}(k)]\underline{\mathbf{H}}(k)\bullet[\underline{\mathbf{L}}(k) - \\ & \left(\underline{\mathbf{A}}\,\underline{\mathbf{M}}(k)\underline{\mathbf{C}}' + \underline{\mathbf{W}}\,\underline{\mathbf{R}}_\phi\,\underline{\mathbf{J}}'\right)\underline{\mathbf{H}}^{-1}(k)]\end{aligned} \quad (8)$$

where, $\underline{\mathbf{H}}(k) = \underline{\mathbf{C}}\,\underline{\mathbf{M}}(k)\underline{\mathbf{C}}' + \underline{\mathbf{J}}\,\underline{\mathbf{R}}_\phi\,\underline{\mathbf{J}}' + \underline{\mathbf{R}}_0$ is positive definite. Eqn. 8 achieves its minimum

$$\begin{aligned}\underline{\mathbf{M}}(k+1) = {} & \underline{\mathbf{A}}\,\underline{\mathbf{M}}(k)\underline{\mathbf{A}}' + \underline{\mathbf{W}}\,\underline{\mathbf{R}}_\phi\underline{\mathbf{W}}' - \\ & \left(\underline{\mathbf{A}}\,\underline{\mathbf{M}}(k)\underline{\mathbf{C}}' + \underline{\mathbf{W}}\,\underline{\mathbf{R}}_\phi\,\underline{\mathbf{J}}'\right)\underline{\mathbf{H}}^{-1}(k)\bullet \\ & \left(\underline{\mathbf{A}}\,\underline{\mathbf{M}}(k)\underline{\mathbf{C}}' + \underline{\mathbf{W}}\,\underline{\mathbf{R}}_\phi\,\underline{\mathbf{J}}'\right)'\end{aligned} \quad (9)$$

if and only if (Åstrom, 1970)

$$\underline{\mathbf{L}}(k) = \left(\underline{\mathbf{A}}\,\underline{\mathbf{M}}(k)\underline{\mathbf{C}}' + \underline{\mathbf{W}}\,\underline{\mathbf{R}}_\phi\,\underline{\mathbf{J}}'\right)\underline{\mathbf{H}}^{-1}(k) \quad (10)$$

At last, we define

$$\bar{\underline{\mathbf{y}}}(k|k-1) \equiv \underline{\mathbf{y}}(k) - \hat{\underline{\mathbf{y}}}(k|k-1) \quad (11)$$

as the innovation vector, where $\hat{\underline{\mathbf{y}}}(k|k-1) = \underline{\mathbf{C}}\hat{\underline{\mathbf{x}}}(k|k-1) + \underline{\mathbf{D}}\,\underline{\mathbf{u}}(k)$ is the prediction of $\underline{\mathbf{y}}(k)$. The innovation is a white noise vector as proved, e.g. in Haykin (1996), with covariance $\mathbf{Cov}[\bar{\underline{\mathbf{y}}}(k|k-1)] = \underline{\mathbf{H}}(k)$.

Eqns. 5, 9, 10, and 11 construct the Kalman filers for one-step prediction given a system described by Eqn. 4 and initial conditions $\hat{\mathbf{x}}(0|-1) = \mathbf{E}[\mathbf{x}(0)]$. The estimated states $\hat{\mathbf{x}}(k|k-1)$ are unbiased.

3.2 Stability and convergence analysis

Manipulating Eqn. 7 yields

$$\begin{aligned}\underline{\mathbf{M}}(k+1) = {} & [\underline{\mathbf{A}} - \underline{\mathbf{L}}(k)\underline{\mathbf{C}}]\,\underline{\mathbf{M}}(k)\,[\underline{\mathbf{A}} - \underline{\mathbf{L}}(k)\underline{\mathbf{C}}]' + \\ & \underline{\mathbf{W}}_\phi(k)\underline{\mathbf{R}}_\phi\underline{\mathbf{W}}_\phi'(k) + \underline{\mathbf{L}}(k)\underline{\mathbf{R}}_o\underline{\mathbf{L}}'(k)\end{aligned}$$

where $\underline{\mathbf{W}}_\phi(k) = \underline{\mathbf{W}}(k) - \underline{\mathbf{L}}(k)\underline{\mathbf{J}}$. With $\underline{\mathbf{M}}(0) > \mathbf{0}$, assume that (i) the pair $\{\underline{\mathbf{A}}, \underline{\mathbf{C}}\}$ is detectable; and (ii) there exists no unreachable mode of $\{\underline{\mathbf{A}}, \underline{\mathbf{W}}_\phi(k)\underline{\mathbf{R}}_\phi^{1/2}\}$ ($\underline{\mathbf{R}}_\phi^{1/2}\underline{\mathbf{R}}_\phi^{1/2} = \underline{\mathbf{R}}_\phi$) on the unit circle. By extending the stability analysis in (Souza *et al.*, 1986), it can be proved that the ARDE has a unique stabilizing solution $\underline{\mathbf{M}}(k)$ (poles of $\underline{\mathbf{A}} - \underline{\mathbf{L}}(k)\underline{\mathbf{C}}$ are within the stability boundary), and $\underline{\mathbf{M}}(k)$ converges exponentially to $\underline{\mathbf{M}}(\infty)$.

3.3 Algorithms for filtering

In a plant, the state variables usually represent the controlled variables (CVs). To ensure that the CVs are manipulated around their desired values in a closed-loop control system, precise measurements of the CVs are needed. However the CVs are either not always measurable or noisy in practice. If this is the case, the Kalman filtering algorithms can be used to provide accurate estimates of the CVs, i.e. $\hat{\mathbf{x}}(k|k)$ from $\{\underline{\mathbf{u}}(1), \underline{\mathbf{y}}(1), \cdots, \underline{\mathbf{u}}(k), \underline{\mathbf{y}}(k)\}$.

$\hat{\mathbf{x}}(k|k)$ is the least mean-square (LMS) projection of $\mathbf{x}(k)$ onto the space spanned by data matrix (Haykin, 1996), $\underline{\mathbf{Z}}_{1:k} \equiv [\underline{\mathbf{z}}(1)\cdots\underline{\mathbf{z}}(k)]$, i.e. $\hat{\mathbf{x}}(k|k) = \hat{\mathbf{x}}(k|_{\underline{\mathbf{Z}}_{1:k}})$, where $\underline{\mathbf{z}}(i) = \underline{\mathbf{y}}(i) - \underline{\mathbf{D}}\;\underline{\mathbf{u}}(i)$, for $i = 1, \cdots, k$. According to (Haykin, 1996), $\forall\ k$, we can have the decomposition, $\underline{\mathbf{Z}}_{1:k} = \underline{\mathbf{Z}}_{1:k-1} \oplus \bar{\underline{\mathbf{y}}}(k|k-1)$, where $\underline{\mathbf{Z}}_{1:k-1}$ is similar to $\underline{\mathbf{Z}}_{1:k}$, and $\oplus$ indicates the *direct sum* of two spaces. As a result,

$$\begin{aligned}\hat{\mathbf{x}}(k|k) &= \hat{\mathbf{x}}(k|_{\underline{\mathbf{Z}}_{1:k-1}}) + \hat{\mathbf{x}}(k|_{\bar{\underline{\mathbf{y}}}(k|k-1)}) \\ &= \hat{\mathbf{x}}(k|k-1) + \underline{\mathbf{N}}(k)\bar{\underline{\mathbf{y}}}(k|k-1)\end{aligned} \quad (12)$$

where $\hat{\mathbf{x}}(k|k-1) = \hat{\mathbf{x}}(k|_{\underline{\mathbf{Z}}_{1:k-1}})$, $\hat{\mathbf{x}}(k|_{\bar{\underline{\mathbf{y}}}(k|k-1)}) = \underline{\mathbf{N}}(k)\;\bar{\underline{\mathbf{y}}}(k|k-1)$ is the LMS projection of $\mathbf{x}(k)$ onto $\bar{\underline{\mathbf{y}}}(k|k-1)$, and $\underline{\mathbf{N}}(k)$ is a gain matrix.

The substitution of Eqn. 11 into Eqn. 12 leads to $\hat{\mathbf{x}}(k|k) = \hat{\mathbf{x}}(k|k-1) + \underline{\mathbf{N}}(k)\{\underline{\mathbf{C}}\,[\underline{\mathbf{x}}(k) - \hat{\underline{\mathbf{x}}}(k|k-1)] + \underline{\mathbf{J}}\;\underline{\phi}(k) + \underline{\mathbf{o}}(k)\}$, from which the estimation error can be obtained as

$$\begin{aligned}\bar{\mathbf{x}}(k|k) = {} & [\mathbf{I} - \underline{\mathbf{N}}(k)\underline{\mathbf{C}}]\,\bar{\mathbf{x}}(k|k-1) - \\ & \underline{\mathbf{N}}(k)\left[\underline{\mathbf{J}}\underline{\phi}(k) + \underline{\mathbf{o}}(k)\right]\end{aligned} \quad (13)$$

Note that $\mathbf{E}[\bar{\mathbf{x}}(k|k)] = \mathbf{0}$. Thus the covariance of $\bar{\mathbf{x}}(k|k)$ is $\mathbf{Cov}[\bar{\mathbf{x}}(k|k)] = \mathbf{E}[\bar{\mathbf{x}}(k|k)\bar{\mathbf{x}}'(k|k)] = [\mathbf{I} - \underline{\mathbf{N}}(k)\underline{\mathbf{C}}]\,\underline{\mathbf{M}}(k)\,[\mathbf{I} - \underline{\mathbf{N}}(k)\underline{\mathbf{C}}]' + \underline{\mathbf{N}}(k)(\underline{\mathbf{J}}\,\underline{\mathbf{R}}_\phi\underline{\mathbf{J}}' + \underline{\mathbf{R}}_0)\underline{\mathbf{N}}'(k)$. Minimizing $\mathbf{Cov}[\bar{\mathbf{x}}(k|k)]$ leads to

$$\underline{\mathbf{N}}(k) = \underline{\mathbf{M}}(k)\underline{\mathbf{C}}'\;\underline{\mathbf{H}}^{-1}(k) \quad (14)$$

which in turn results in $\mathbf{Cov}\,[\bar{\mathbf{x}}(k|k)] = \underline{\mathbf{M}}(k) - \underline{\mathbf{M}}(k)\underline{\mathbf{C}}'\underline{\mathbf{H}}^{-1}(k)\underline{\mathbf{C}}\,\underline{\mathbf{M}}(k)$. Finally, with $\hat{\mathbf{x}}(k|k)$,

$$\hat{\underline{\mathbf{y}}}(k|k) = \underline{\mathbf{C}}\hat{\mathbf{x}}(k|k) + \underline{\mathbf{D}}\;\underline{\mathbf{u}}(k) \quad (15)$$

The one-step prediction algorithms plus Eqns. 12, 14, and 15 constitute the Kalman filtering algorithms.

4. APPLICATION OF KALMAN FILTERS FOR FAULT DETECTION

Since the pioneering work of Mehra and Peschon (1971), Kalman filters have been applied to fault detection and isolation (FDI) in single rate systems. A survey of this area has been provided by Frank (1990) and the most recent work has been reported by Keller (1999).

Recently, research attention has been diverted to FDI in uniformly sampled multirate systems (Fadali and Shabaik, 2002; Zhang *et al.*, 2002). In addition, FDI in NUSM systems has also been considered (Li and Shah, 2004; Li *et al.*, 2005) by extending the Chow-Willsky scheme (Chow and Willsky, 1984). We next investigate the use of Kalman filters for FDI in NUSM systems. For simplicity, we only consider the detection of faults in output sensors. Nevertheless, the scheme of fault detection to be proposed later can be readily extended to actuator and additive process faults.

The measured outputs with sensor faults, for $j = [1, n_i]$, can be represented by

$$\mathbf{y}(kT + t_i^j) = \mathbf{y}^*(kT + t_i^j) + \mathbf{f}_y(kT + t_i^j) \quad (16)$$

where $\mathbf{y}^*(kT+t_i^j)$ is the fault-free value, and $\mathbf{f}_y(kT+t_i^j)$ is the fault magnitude vector with zero and non-zero elements.

Given data: $\{\mathbf{u}(kT + t_i)\}$ and $\{\mathbf{y}(kT + t_i^j)\}$, for $i = [1, g]$, $j = [1, n_i]$, and $k = [1, 2, \cdots)$, the purpose of sensor fault detection is to indicate if $\mathbf{f}_y(kT+t_i^j)$ is non-zero.

We define a lifted vector:

$$\underline{\mathbf{y}}(k) = \underline{\mathbf{y}}^*(k) + \underline{\mathbf{f}}_y(k) \quad (17)$$

where $\underline{\mathbf{y}}(k)$ and $\underline{\mathbf{f}}_y(k)$ are structurally similar to $\tilde{\underline{\mathbf{y}}}(k)$. Substituting Eqn. 17 into Eqn. 4 produces

$$\begin{aligned}\mathbf{x}(k+1) &= \underline{\mathbf{A}}\,\mathbf{x}(k) + \underline{\mathbf{B}}\,\underline{\mathbf{u}}(k) + \underline{\mathbf{W}}\,\underline{\boldsymbol{\phi}}(k)\\ \underline{\mathbf{y}}(k) &= \underline{\mathbf{C}}\,\mathbf{x}(k) + \underline{\mathbf{D}}\,\underline{\mathbf{u}}(k) + \underline{\mathbf{J}}\,\underline{\boldsymbol{\phi}}(k) + \underline{\mathbf{o}}(k) + \\ &\quad \underline{\mathbf{f}}_y(k)\end{aligned} \quad (18)$$

We use the developed one-step prediction algorithms, i.e.,

$$\begin{aligned}\hat{\mathbf{x}}(k+1|k) &= \underline{\mathbf{A}}\,\hat{\mathbf{x}}(k|k-1) + \underline{\mathbf{B}}\,\underline{\mathbf{u}}(k) + \\ &\quad \underline{\mathbf{L}}\,\bar{\underline{\mathbf{y}}}(k|k-1)\\ \hat{\underline{\mathbf{y}}}(k|k-1) &= \underline{\mathbf{C}}\hat{\mathbf{x}}(k|k-1) + \underline{\mathbf{D}}\,\underline{\mathbf{u}}(k)\\ \bar{\underline{\mathbf{y}}}(k|k-1) &\equiv \underline{\mathbf{y}}(k) - \hat{\underline{\mathbf{y}}}(k|k-1)\end{aligned} \quad (19)$$

to generate a primary residual vector (PRV) for fault detection, where $\underline{\mathbf{L}}$ is the steady value of $\underline{\mathbf{L}}(k)$. Combining Eqn. 18 and Eqn. 19 further results in

$$\begin{aligned}\hat{\mathbf{x}}(k+1|k) &= \underline{\mathbf{A}}\,\hat{\mathbf{x}}(k|k-1) + \underline{\mathbf{B}}\,\underline{\mathbf{u}}(k) + \\ &\quad \underline{\mathbf{L}}\bar{\mathbf{y}}(k|k-1)\\ \underline{\bar{\mathbf{y}}}(k|k-1) &= \underline{\mathbf{C}}\bar{\mathbf{x}}(k|k-1) + \underline{\mathbf{J}}\,\phi(k) + \underline{\mathbf{o}}(k) + \\ &\quad \underline{\mathbf{f}}_y(k)\end{aligned} \quad (20)$$

Subtracting Eqn. 18 by Eqn. 20 gives

$$\begin{aligned}\bar{\mathbf{x}}(k+1|k) &= (\underline{\mathbf{A}} - \underline{\mathbf{L}}\,\underline{\mathbf{C}})\;\bar{\mathbf{x}}(k|k-1) - \underline{\mathbf{L}}\,\underline{\mathbf{f}}_y(k) + \\ &\quad (\underline{\mathbf{W}} - \underline{\mathbf{L}}\,\underline{\mathbf{J}})\,\underline{\boldsymbol{\phi}}(k) - \underline{\mathbf{L}}\,\underline{\mathbf{o}}(k)\end{aligned} \quad (21)$$

It follows from Eqn. 21 that

$$\begin{aligned}\bar{\mathbf{x}}(k|k-1) &= \bar{\mathbf{x}}^*(k|k-1) - \\ &\quad \sum_{i=0}^{k-1} (\underline{\mathbf{A}} - \underline{\mathbf{L}}\,\underline{\mathbf{C}})^{k-1-i}\,\underline{\mathbf{L}}\,\underline{\mathbf{f}}_y(i)\end{aligned} \quad (22)$$

where $\bar{\mathbf{x}}^*(k|k-1) = (\underline{\mathbf{A}} - \underline{\mathbf{L}}\,\underline{\mathbf{C}})^k\,\bar{\mathbf{x}}(0|-1) + \sum_{i=0}^{k-1} (\underline{\mathbf{A}} - \underline{\mathbf{L}}\,\underline{\mathbf{C}})^{k-1-i}\left[(\underline{\mathbf{W}} - \underline{\mathbf{L}}\,\underline{\mathbf{J}})\,\underline{\boldsymbol{\phi}}(i) - \underline{\mathbf{L}}\,\underline{\mathbf{o}}(i)\right]$ is the fault-free value of $\bar{\mathbf{x}}(k|k-1)$, while the second term on the RHS is the fault-contributed value.

If we substitute Eqn. 22 into Eqn. 20 and then define $\underline{\mathbf{e}}(k) \equiv \bar{\underline{\mathbf{y}}}(k|k-1)$ as the PRV for fault detection, we are led to $\underline{\mathbf{e}}(k) = \underline{\mathbf{e}}^*(k) + \underline{\mathbf{e}}^f(k)$. In $\underline{\mathbf{e}}(k)$, $\underline{\mathbf{e}}^*(k) = \underline{\mathbf{C}}\bar{\mathbf{x}}^*(k|k-1) + \underline{\mathbf{J}}\;\underline{\boldsymbol{\phi}}(k) + \underline{\mathbf{o}}(k)$ is the fault-free value. It is a Gaussian distributed white noise vector with covariance $\mathbf{R}_e(k) = \underline{\mathbf{H}}(k)$, while $\underline{\mathbf{e}}^f(k) = -\underline{\mathbf{C}}\sum_{i=0}^{k-1} (\underline{\mathbf{A}} - \underline{\mathbf{L}}\,\underline{\mathbf{C}})^{k-1-i}\;\underline{\mathbf{f}}_y(i) + \underline{\mathbf{f}}_y(k)$ is the fault-contributed value.

In the absence of sensor fault, $\underline{\mathbf{e}}(k) = \underline{\mathbf{e}}^*(k) \sim \aleph\,[\mathbf{0}, \mathbf{R}_e(k)]$. However, in the presence of any faults, $\underline{\mathbf{e}}(k) \sim \aleph\left[\underline{\mathbf{e}}^f(k), \mathbf{R}_e(k)\right]$. Therefore, the main objective in fault detection is to test if the PRV is zero-mean. One can define a scalar $f_d(k) = \underline{\mathbf{e}}'(k)\mathbf{R}_e^{-1}(k)\underline{\mathbf{e}}(k)$, which follows a (non-central) chi-square distribution with mp degrees of freedom in the normal (faulty) case (Basseville and Nikiforov, 1993). Given a threshold, $\chi^2_\beta(mp)$, for $f_d(k)$, where β is a level of significance; $f_d(k) < \chi^2_\beta(mp)$ indicates the absence of fault and $f_d(k) \geq \chi^2_\beta(mp)$ triggers sensor fault alarms.

5. NUMERICAL EXAMPLES

A quadruple tank system described in (Ge and Fang, 1988) is used as a test bed to justify the correctness and effectiveness of the proposed Kalman filters in a NUSM scenario. In Example 1, the Kalman filter-based scheme is applied for sensor fault detection. In Example 2, the Kalman filtering algorithms are applied for estimation of the state variables in the system. Physically the state variables are levels of the tanks. The tank system is depicted in Figure 1, where four identical tanks are serially connected by outlets that have identical cross sectional areas. The model of the tank system, linearized at a steady operating point, can be described by (Ge and Fang, 1988):

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\tilde{u}(t) + \boldsymbol{\phi}(t)\\ \tilde{\mathbf{y}}(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned} \quad (23)$$

where the input, $\tilde{u}(t)$, is the controlled water flowing into Tank 1; $\mathbf{x}(t)$ is the state variable vector whose i^{th} element, $x_i(t)$, represents the level of the i^{th} tank,

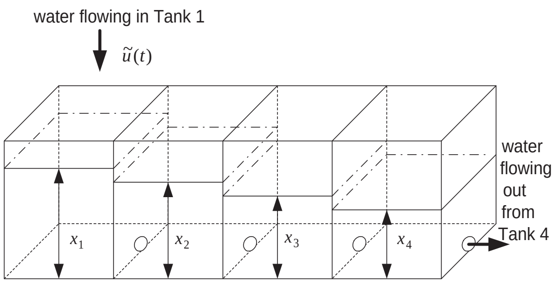

Fig. 1. Schematic of a quadruple tank system

for $i = 1 \cdots 4$; $\tilde{\mathbf{y}}(t)$ is the output vector; and $\boldsymbol{\phi}(t)$ accounts for the linearization and modelling errors. It is assumed that $\boldsymbol{\phi}(t)$ is a Gaussian-distributed white noise vector with covariance $\mathbf{R}_\phi = 0.1^2\mathbf{I}_4$. The values of $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ are not reproduced due to space limitation.

The noise-free input to the tank system is simulated by (Ge and Fang, 1988) $\tilde{u}(t) = \frac{50000}{3}[1 + 0.36\sin t]$ $\text{cm}^3/\text{minute}$., where t is in the unit of minute. A frame period $T = 0.5$ minute is selected. For $k = [0, 1, \cdots)$, within the period $[kT, kT + T]$, we sample $\tilde{u}(t)$ at $t = kT$ and $t = kT + 0.2$; and $\tilde{\mathbf{y}}(t)$ at $t = kT$ and $t = kT + 0.3$, respectively. Thus, the lifted input and output vectors are

$$\underline{\tilde{u}}(k) = \left[\, \tilde{u}'(kT)\ \tilde{u}'(kT + 0.2) \,\right]' \in \Re^2$$
$$\underline{\tilde{\mathbf{y}}}(k) = \left[\, \tilde{\mathbf{y}}'(kT)\ \tilde{\mathbf{y}}'(kT + 0.3) \,\right]' \in \Re^8.$$

The lifted model of Eqn. 23 can be represented by Eqn. 2.

A white noise, $\mathbf{o}(\ ) \sim \aleph(\mathbf{0}, \mathbf{R}_0)$ with $\mathbf{R}_0 = 4\mathbf{I}_4$, is introduced at the outputs, $\tilde{\mathbf{y}}(\ )$. However, the exact value of $\underline{\tilde{\mathbf{u}}}(k)$ is supposed to be known, i.e. $\underline{\mathbf{u}}(k) = \underline{\tilde{\mathbf{u}}}(k)$. From known $\underline{\mathbf{A}}$, $\underline{\mathbf{B}}$, $\underline{\mathbf{C}}$, $\underline{\mathbf{D}}$, $\underline{\mathbf{W}}$, $\underline{\mathbf{J}}$, $\underline{\mathbf{R}}_\phi$ and $\underline{\mathbf{R}}_0$, which are not presented due to the lack of space in this paper, we calculate steady-state values of $\underline{\mathbf{L}}(k)$ by means of the one step prediction algorithms after 100 iterations. In addition, the covariance, $\mathbf{R}_e(k) = \underline{\mathbf{C}}\ \underline{\mathbf{M}}(k)\ \underline{\mathbf{C}}'$, of $\underline{\bar{\mathbf{y}}}(k|k-1)$ with $k = 100$ is also calculated.

In the fault-free case, given $\beta = 0.01$, the confidence limit for $f_d(k)$ is $\chi^2_{0.01}(8) = 20.090$. It must be noted that in the fault detection results to be shown later, all the fault detection indices are scaled to have a unit confidence limit.

5.1 Example 1, sensor fault detection

Fault detection results in two cases are presented next. In each case, a fault is introduced in one of the four output sensors at any time.

Case 1. The CT function, $0.01(t - t_f)$ with $t > t_f$, is employed to simulate an incipient fault, which is then sampled in the same way as $\tilde{\mathbf{y}}(t)$ is. Assume that the first output sensor begins to be faulty at $t_f = 473 * T = 236.5$ minutes, we construct $\underline{\mathbf{f}}_y(k)$ and have $\underline{\mathbf{y}}(k) = \underline{\mathbf{y}}^*(k) + \underline{\mathbf{f}}_y(k)$. The fault detection results are depicted in the first subplot of Figure 2, where in the x-axis each sample represents one frame period of 0.5 minute. In addition, Fd is the scaled fault detection index. It can be seen that Fd is beyond its confidence limit, 1, after the occurrence of the fault, indicating successful fault detection.

Case 2. A bias fault with magnitude 10 is introduced in one output sensor at $t_f = 801 * 0.5 = 400.5$ minutes. The fault detection results are displayed in the second subplot of Figure 2, where the fault is detected promptly after it occurs.

We define $r_{f/s} = \frac{\sum_{k=t_f/T}^{N_0} \|\underline{\mathbf{f}}_y(k)\|}{\sum_{k=t_f/T}^{N_0} \|\underline{\mathbf{y}}(k)\|}\%$, as the fault-to-signal ratio, to measure the sensitivity of the proposed fault detection scheme. Note that $\|\ \|$ stands for the norm of a vector, and N_0 $(= 2000)$ is the total frame period of used data. In Cases 1 and 2, $r_{f/s} = 2.62\%$ and $r_{f/s} = 4.81\%$, respectively. This demonstrates that the sensitivity of the proposed fault detection scheme is satisfactory with respect to any sensor faults.

5.2 Example 2, estimation of the state variables

We use Eqn. 4 to generate simulation data with initial states $\mathbf{x}(0) = [1\ 1\ 1\ 1]'$. A set of data within 2000 frame periods is collected for states estimation. The exact states (the first column), the estimate (the second column), and the estimation errors (the third column) are depicted in Figure 3, where the i^{th} subplot in each column corresponds to the i^{th} element of the shown vector.

We define $r_{n/S} = \sum_{k=1}^{N_0} \frac{\|\underline{\mathbf{o}}(k)\|}{\|\underline{\tilde{\mathbf{y}}}(k)\|}\%$ to quantify the corruption of noise in the lifted output vectors, and $\rho_{x/\bar{x}} = \frac{\sum_{k=1}^{N_0} \|\bar{\mathbf{x}}(k|k)\|}{\sum_{k=1}^{N_0} \|\mathbf{x}(k)\|}\%$ to quantify the estimation errors of the states. In this example, with $N_0 = 2000$, we obtain $r_{n/s} = 14.1\%$ and $\rho_{x/\bar{x}} = 0.52\%$, indicating that the developed Kalman filters work well with noisy data.

6. CONCLUSION

Kalman filters for NUSM systems have been proposed and applied to two numerical examples. Example 1 justifies the effectiveness of the one-step prediction algorithms in fault detection. Furthermore, Example 2 shows the power of the filtering algorithms in estimating the state variables from highly noisy measurements.

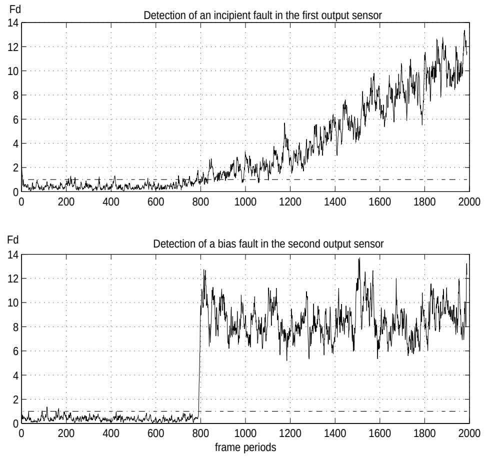

Fig. 2. Fault detection results in Example 1

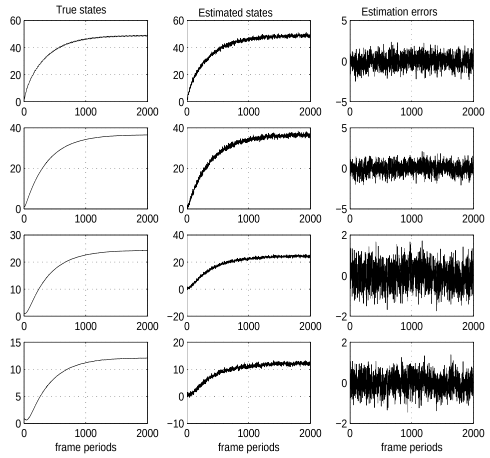

Fig. 3. State estimation results in Example 2

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